Torque on Current Loop Immersed in Constant Magnetic Field

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Setup Consider a closed loop of current I lying on a plane, immersed within a constant magnetic field $\vec{B}$. We will now discuss why the total magnetic force on this loop is zero; but the total torque $\vec{\tau}$ is non-zero and in fact is the cross product between the current loop's magnetic moment $\vec{\mu}$ and the magnetic field:

$$\vec{\tau} = \vec{\mu} \times \vec{B}. \tag{0.0.1}$$

The magnetic moment is, in turn, defined as the directed area of the loop times its current:

$$\vec{\mu} = I\vec{A} = IA\widehat{n}, \tag{0.0.2}$$

where $\widehat{n}$ is the unit vector perpendicular to the loop given by the 'right hand rule'. Namely, if the $\widehat{n}$ is pointed at you while you are looking down at the loop, the current needs to appear flowing in the counter-clockwise direction.

Total Force Remember the force on an infinitesimal segment of a current carrying wire is

$$\mathrm{d}\vec{F} = I\mathrm{d}\vec{\ell} \times \vec{B}. \tag{0.0.3}$$

The total force is therefore

$$\begin{aligned}
\vec{F} &= I\left(\oint \mathrm{d}\vec{\ell}\right) \times \vec{B} &(0.0.4)\\
&= I\left(\oint \mathrm{d}x\widehat{x} + \oint \mathrm{d}y\widehat{y} + \oint \mathrm{d}z\widehat{z}\right) \times \vec{B} &(0.0.5)\\
&= I\left[x\widehat{x} + y\widehat{y} + z\widehat{z}\right]^{\text{final position on loop}}_{\text{initial position on loop}} \times \vec{B} = \vec{0}. &(0.0.6)
\end{aligned}$$

The initial and final positions are the same – we are integrating around a closed loop – and hence the integral is zero.

Area Before tackling the torque we first note that the directed area $\vec{A}$ can itself be expressed as

$$\vec{A} = A\widehat{n} = \frac{1}{2}\oint \vec{r} \times \mathrm{d}\vec{\ell}. \tag{0.0.7}$$

The $\vec{r} = (x, y, z)$ is the position vector joining the (arbitrary) origin of the coordinate system to the point (x, y, z) on the current loop; whereas $\mathrm{d}\vec{\ell} = \widehat{x}\mathrm{d}x + \widehat{y}\mathrm{d}y + \widehat{z}\mathrm{d}z$ in turn is the infinitesimal displacement on the loop along the direction of the current. To understand eq. (0.0.7), we note that the infinitesimal triangle formed by the sides $\vec{r}$, $\vec{r}+\mathrm{d}\vec{\ell}$ and $\mathrm{d}\vec{\ell}$ has area

$$\mathrm{d}A = \frac{1}{2}|\vec{r}+\mathrm{d}\vec{\ell}||\mathrm{d}\vec{\ell}|\sin\psi \approx \frac{1}{2}|\vec{r}||\mathrm{d}\vec{\ell}|\sin\psi, \tag{0.0.8}$$

where ψ is the angle between $\mathrm{d}\vec{\ell}$ and either $\vec{r}$ or $\vec{r}+\mathrm{d}\vec{\ell}$ (the difference between the two is infinitesimal). But remember $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$. We have arrived at

$$\mathrm{d}A = \frac{1}{2}|\vec{r}\times\mathrm{d}\vec{\ell}|. \tag{0.0.9}$$

The directed area is therefore eq. (0.0.7). Finally, the magnetic moment of a single current loop is

$$\vec{\mu} = I\vec{A} = \frac{I}{2}\oint \vec{r}\times\mathrm{d}\vec{\ell}. \tag{0.0.10}$$

Torque The torque exerted upon an infinitesimal segment of current carrying wire is

$$\mathrm{d}\vec{\tau} = \vec{r}\times\mathrm{d}\vec{F} = \vec{r}\times\left(I\mathrm{d}\vec{\ell}\times\vec{B}\right), \tag{0.0.11}$$

where we have used eq. (0.0.3). Employing the identity

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} \tag{0.0.12}$$

then yields

$$\vec{\tau} = I\oint\left((\vec{r}\cdot\vec{B})\mathrm{d}\vec{\ell} - (\vec{r}\cdot\mathrm{d}\vec{\ell})\vec{B}\right). \tag{0.0.13}$$

Note that the second term involves

$$\oint \vec{r}\cdot\mathrm{d}\vec{\ell} = \oint (x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z) = \oint \frac{1}{2}\left(\mathrm{d}(x^2)+\mathrm{d}(y^2)+\mathrm{d}(z^2)\right) \tag{0.0.14}$$

$$= \frac{1}{2}\left[x^2+y^2+z^2\right]_{\text{initial point}}^{\text{final point}} = 0. \tag{0.0.15}$$

Therefore, the total torque is

$$\vec{\tau} = I\oint(\vec{r}\cdot\vec{B})\mathrm{d}\vec{\ell}. \tag{0.0.16}$$

Note that the definition of total torque is actually independent of the choice of origin. That is, suppose we choose a different origin, so $\vec{r}' \equiv \vec{r}+\vec{d}$ (for constant $\vec{d}$):

$$\vec{\tau} = \oint\left(\vec{r}+\vec{d}\right)\times\left(I\mathrm{d}\vec{\ell}\times\vec{B}\right) \tag{0.0.17}$$

$$= \oint\left(\vec{r}\times\left(I\mathrm{d}\vec{\ell}\times\vec{B}\right) + \vec{d}\times\left(I\mathrm{d}\vec{\ell}\times\vec{B}\right)\right). \tag{0.0.18}$$

In the second term, only the $\mathrm{d}\vec{\ell}$ is involved in the integration, since $\vec{d}$ and $\vec{B}$ are constant; but $\oint \mathrm{d}\vec{\ell} = [x\widehat{x} + y\widehat{y} + z\widehat{z}]_{\text{initial}}^{\text{final}} = \vec{0}$ and we have

$$\vec{\tau} = \oint \vec{r}' \times \left(I\mathrm{d}\vec{\ell} \times \vec{B}\right) = \oint \vec{r} \times \left(I\mathrm{d}\vec{\ell} \times \vec{B}\right). \tag{0.0.19}$$

We turn to

$$\vec{\mu} \times \vec{B} = \frac{I}{2}\left(\oint \vec{r} \times \mathrm{d}\vec{\ell}\right) \times \vec{B} = -\frac{I}{2}\vec{B} \times \left(\oint \vec{r} \times \mathrm{d}\vec{\ell}\right) \tag{0.0.20}$$

$$= -\frac{I}{2}\left(\oint \vec{r}(\vec{B} \cdot \mathrm{d}\vec{\ell}) - \oint (\vec{B} \cdot \vec{r})\mathrm{d}\vec{\ell}\right). \tag{0.0.21}$$

Let us examine the first term:

$$\oint \vec{r}(\vec{B} \cdot \mathrm{d}\vec{\ell}) = \oint (x, y, z)(B_x \mathrm{d}x + B_y \mathrm{d}y + B_z \mathrm{d}z). \tag{0.0.22}$$

We may integrate by parts – remember $\int_a^b f\mathrm{d}g = [fg]_a^b - \int_a^b g\mathrm{d}f$, and since we have a closed loop the 'surface terms' $[\dots]_a^b$ actually vanish – to deduce

$$\oint \vec{r}(\vec{B} \cdot \mathrm{d}\vec{\ell}) = -\oint (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)(B_x x + B_y y + B_z z) \tag{0.0.23}$$

$$= -\oint (\vec{B} \cdot \vec{r})\mathrm{d}\vec{\ell}. \tag{0.0.24}$$

In other words, the two terms in eq. (0.0.21) are equal and we have

$$\vec{\mu} \times \vec{B} = I \oint (\vec{B} \cdot \vec{r})\mathrm{d}\vec{\ell}. \tag{0.0.25}$$

Comparing equations (0.0.25) and (0.0.16), we have arrived at eq. (0.0.1).