Translations, Rotations, Parity Flips in Euclidean (Flat) Space

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1 Introduction

The distance $\mathrm{d}\ell$ between $\vec{x}$ and $\vec{x}+\mathrm{d}\vec{x}$ in $D-$dimensional flat space is governed by the Pythagorean theorem

$$\mathrm{d}\ell^2 = \mathrm{d}\vec{x} \cdot \mathrm{d}\vec{x} = \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j. \tag{1.0.1}$$

In this context, the δ_{ij} is the flat metric, describing the geometry of space written in Cartesian coordinates.

Translations Translations – displacements in space – preserve length. Upon the replacement

$$x^i \to x^i + a^i \tag{1.0.2}$$

for constant $\vec{a}$, we have

$$\mathrm{d}x^i \to \mathrm{d}x^i; \tag{1.0.3}$$

which turn says

$$\delta_{ij} \mathrm{d}x^i \mathrm{d}x^j \to \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j. \tag{1.0.4}$$

This yields *translation symmetry*, that $\mathrm{d}\ell$ at $\vec{x}$ is the same as that at $\vec{x}+\vec{a}$.

Rotations and Parity Flips Rotations are defined to be the linear transformations acting upon flat space that leaves lengths invariant. Specifically, if we replace

$$\mathrm{d}x^i \to O^i{}_j \mathrm{d}x^j \tag{1.0.5}$$

the length-squared $\mathrm{d}\ell^2$ is replaced in the following manner:

$$\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \to \left(\delta_{ij}O^i{}_a O^j{}_b\right)\mathrm{d}x^a\mathrm{d}x^b. \tag{1.0.6}$$

Thus, if we wish to obtain the same answer $\delta_{ab}\mathrm{d}x^a\mathrm{d}x^b$ for arbitrary infinitesimal displacements $\{\mathrm{d}x^i\}$, we need to leave the flat metric invariant:

$$\delta_{ij}O^i{}_a O^j{}_b = \delta_{ab}, \tag{1.0.7}$$

$$\Leftrightarrow \qquad O^T O = \mathbb{I}. \tag{1.0.8}$$

Problem 1.1. $O^i{}_a$ refers to the ith row and a column of the matrix O. Explain why eq. (1.0.7) implies (1.0.8), provided $\mathbb{I}$ denotes the unit matrix. □

Actually, eq. (1.0.7) is not quite accurate enough a definition for rotations. Reversing the direction of one of the Cartesian axis would also leave lengths invariant. Specifically, if we define the ith *parity flip* to be

$$_{(i)}P = \mathrm{diag}\left[1,\dots,1,\underbrace{-1}_{i\text{ th slot from the left}},1,\dots,1\right], \tag{1.0.9}$$

where "diag" simply means a matrix whose only non-zero entries are given by the diagonal ones (from upper left to lower right) provided by the row within the $[\dots]$; then a direct calculation tells us

$$\left({}_{(i)}P\right)^T {}_{(i)}P = \mathbb{I}. \tag{1.0.10}$$

This $_{(i)}P$ acting on $\mathrm{d}\vec{x}$ would leave all components except the ith one invariant:

$$\mathrm{d}x^a \to \left({}_{(i)}P\right)^a{}_b\,\mathrm{d}x^b = \left(\mathrm{d}x^1,\dots,\mathrm{d}x^{i-1},-\mathrm{d}x^i,\mathrm{d}x^{i+1},\dots,\mathrm{d}x^D\right)^T. \tag{1.0.11}$$

Our intuitive notion of rotations involve continuous angles that parametrize the size and orientation of rotations themselves. In particular, this continuity implies it must be possible to choose angles such that we recover the identity matrix. A better definition of the rotation group is therefore –

> *Rotation group:* The set of all linear transformations $\{O(\vec{\theta})\}$ that obeys eq. (1.0.7), which also depend on a continuous set of parameters $\{\theta^{\mathrm{I}}|\mathrm{I} = 1,\dots\}$ such that they can be tuned to obtain $O = \mathbb{I}$, forms the rotation group.

One distinguishing feature between the rotation matrices and the parity flip is their determinant. We may take the determinant on both sides of eq. (1.0.8); recalling $\det[AB] = \det A \det B$ and $\det A^T = \det A$,

$$\det O^T \det O = 1 \qquad \Rightarrow \qquad (\det O)^2 = 1 \qquad \Rightarrow \qquad \det O = \pm 1. \tag{1.0.12}$$

This includes rotations. However, we should be able to tune the rotation angles $\vec{\theta}$ to obtain $R = \mathbb{I}$, where

$$\det O = 1. \tag{1.0.13}$$

Since $\vec{\theta}$ is a continuous parameter, it cannot be that – upon tuning O away from the identity – that its determinant will jump abruptly from $+1$ to -1 (cf. (1.0.12)); hence eq. (1.0.13) should hold for all rotation angles. On the other hand, the determinant of any parity flip matrix (in eq. (1.0.9)) is -1.

Some jargon. The full group of matrices obeying eq. (1.0.7) is known as $O(D)$; whereas, if the additional constraint in eq. (1.0.13) is also imposed, i.e., $\det O \neq -1$, then this restricted group is known as SO(D). The 'S' here is 'special', denoting the det $= 1$ condition.

Problem 1.2. From eq. (1.0.7), show that

$$\delta^{ab} O^i{}_a O^j{}_b = \delta^{ij}. \tag{1.0.14}$$

The Levi-Civita symbol in $D-$dimensional flat space $\epsilon^{i_1 i_2 \dots i_D} = \epsilon_{i_1 i_2 \dots i_D}$ is defined as the object that is fully antisymmetric in its indices, and $\epsilon^{123\dots D} = \epsilon_{123\dots D} \equiv 1$. For O in SO(D), prove that

$$O^{i_1}{}_{j_1} O^{i_2}{}_{j_2} \dots O^{i_D}{}_{j_D} \epsilon_{i_1 \dots i_D} = \epsilon_{j_1 \dots j_D}, \tag{1.0.15}$$

$$O^{i_1}{}_{j_1} O^{i_2}{}_{j_2} \dots O^{i_D}{}_{j_D} \epsilon^{j_1 \dots j_D} = \epsilon^{i_1 \dots i_D}. \tag{1.0.16}$$

See §3.2 of Analytical Methods in Physics. □

Problem 1.3. We have asserted that the set of matrices $\{O\}$ that obeys eq. (1.0.7) forms a group. Can you prove it? □

2 Two Dimensions

Let us now denote rotations as R, instead of the O of the previous section. In 2D, let us now rotate the coordinate (x, y) counter-clockwise by θ. To achieve this, we simply need to understand how to perform such an operation on the basis vectors $(\widehat{x}, \widehat{y})$:[1]

$$R(\theta)\widehat{x} = \cos(\theta)\widehat{x} + \sin(\theta)\widehat{y}, \tag{2.0.1}$$

$$R(\theta)\widehat{y} = -\sin(\theta)\widehat{x} + \cos(\theta)\widehat{y}. \tag{2.0.2}$$

The components of the rotation matrix may then be read off these relations. For example, since $\widehat{x}^i = \delta^i_1$ and $\widehat{y}^i = \delta^i_2$, we have

$$\begin{aligned} \widehat{x} \cdot (R(\theta)\widehat{x}) &= \widehat{x}^i R(\theta)^i{}_j \widehat{x}^j && (2.0.3) \\ &= R(\theta)^1{}_1 && (2.0.4) \\ \widehat{x} \cdot (\cos(\theta)\widehat{x} + \sin(\theta)\widehat{y}) &= \cos\theta && (2.0.5) \end{aligned}$$

and

$$\begin{aligned} \widehat{y} \cdot (R(\theta)\widehat{x}) &= R(\theta)^2{}_1 && (2.0.6) \\ \widehat{y} \cdot (\cos(\theta)\widehat{x} + \sin(\theta)\widehat{y}) &= \sin\theta. && (2.0.7) \end{aligned}$$

[1]Drawing a picture here helps.

The most general 2D rotation matrix is therefore

$$R(\theta)=\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}. \tag{2.0.8}$$

We may check directly that $\det R(\theta)=\cos(\theta)^2-(-\sin(\theta)^2)=1$.

Problem 2.1. Verify that $R(\theta)R(\phi)=R(\theta+\phi)$. Explain this in words too. □

Group generator We may also readily check that

$$R(\theta=0)=\mathbb{I}_{2\times 2}. \tag{2.0.9}$$

As it turns out, any group element that is continuously connected to the identity can always be written in an exponential form:

$$R(\theta)=e^{-i\theta J}, \tag{2.0.10}$$

where J is known as the *generator* of the group elements. In this 2D case, there is only one generator because there is only one way you can rotate a 2D plane. The exponential of a matrix (or linear operator) is defined through its Taylor series, namely

$$\exp X=\mathbb{I}+X+\frac{X^2}{2!}+\frac{X^3}{3!}+\dots=\sum_{\ell=0}^{\infty}\frac{X^\ell}{\ell!}. \tag{2.0.11}$$

If eq. (2.0.10) is true we may in fact figure out what J is by Taylor expanding

$$R(\theta)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\theta\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}+\mathcal{O}\left(\theta^2\right). \tag{2.0.12}$$

By comparing this with the Taylor expansion of $\exp(-i\theta J)$ in eq. (2.0.10),

$$R(\theta)=\mathbb{I}+\theta(-iJ)+\mathcal{O}\left(\theta^2\right); \tag{2.0.13}$$

we immediately identify

$$J=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}. \tag{2.0.14}$$

Problem 2.2. Use the results in Problem 3.9 of Analytical Methods in Physics to identify J. Specifically, what linear combination of the Pauli matrices would give the right J, so that $\exp(-i\theta J)$ equals the 2D rotation matrix $R(\theta)$ in eq. (2.0.8)? □

Under a spatial translations, $\vec{x}\to\vec{x}+\vec{a}$, a (scalar) function f would transform as

$$f(\vec{x})\to f(\vec{x}+\vec{a})=f(\vec{x})+\vec{a}\cdot\vec{\nabla}f(\vec{x})+\dots \tag{2.0.15}$$

$$=\exp\left(a^j\partial_j\right)f(\vec{x})=\exp\left(ia^j(-i\partial_j)\right)f(\vec{x}). \tag{2.0.16}$$

Of course, this is just Taylor expansion of a function; but in quantum mechanics we would identify $-i\partial_j$ as the generator-of-translations, namely, the momentum operator.

Let us see how a parallel discussion would hold for 2D rotations. According to eq. (2.0.12), under a small rotation, we have

$$x^i \to \left(\delta^i_j - \theta\epsilon^{ij}\right) x^j, \tag{2.0.17}$$

where we have identified the matrix coefficient of $-\theta$ as the Levi-Civita tensor in 2D:

$$\epsilon^{ij} \doteq \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{2.0.18}$$

Performing such a rotation would amount to replacing

$$f(x^i) \to f\left(x^i - \theta\epsilon^{ij}x^j + \dots\right) \tag{2.0.19}$$

$$= f(\vec{x}) - \theta\epsilon^{ij}x^j\partial_i f(\vec{x}) + \mathcal{O}\left(\theta^2\right). \tag{2.0.20}$$

In differential geometry, directional derivatives $v^i\partial_i$ are viewed as tangent vectors spanning a vector space at a given point in space. Their geometric meaning is they 'generate' flow along the direction v^i, not unlike how J in eq. (2.0.10) generates rotations on the coordinate vector $(x^1, x^2)^T$.

Problem 2.3. Re-express the first order term $-\theta\epsilon^{ij}x^j\partial_i f$ in eq. (2.0.20) using polar coordinates, namely the (r, ϕ) in

$$\mathrm{d}\ell^2 = \mathrm{d}r^2 + r^2\mathrm{d}\phi^2. \tag{2.0.21}$$

Explain your results in words. □

3 Higher Dimensions

We may now borrow what we have learned in 2D to 3 and higher dimensions. Most of us have significant intuition for rotations in two and three dimensional space, but we would need mathematics to guide us in 4 and higher. One such issue is that, we are often told that rotations in 3D can be specified by a unit-length rotation axis $\widehat{n}$ together with the rotation angle θ. This is, in fact, a luxury and does not generalize to higher dimensions. What does generalize to all dimensions is that rotation is really a 'mixing' of 2 coordinates; i.e., rotation always involves a 2D plane – and that is why we spent some time understanding it in the previous section. In 3D, specifying an axis of rotation is the same as specifying the 2D plane perpendicular to it; but in $D > 3$ dimensions, the space orthogonal to some unit vector $\widehat{n}$ is $(D-1) > 2$ dimensional and one would need more parameters to describe exactly what sort of rotation is being performed.

Problem 3.1. To understand these statements, let us note the anti-symmetric nature of $J^i{}_j = -i\epsilon^{ij} = -J^j{}_i$ in eq. (2.0.14) is not a coincidence. Assuming any rotation matrix continuously connected to the identity can be written as

$$R = e^X, \tag{3.0.1}$$

use eq. (1.0.8) to show that $X^T = -X$. Hint: Taylor expand $(e^X)^T e^X$ up to first order in X. □

Since the first-order-in-rotation angles piece (denoted here as X) of any rotation matrix has to be anti-symmetric, we may search for its basis $\{J^a\}$ and witness how each of the basis vectors are really generators of rotations of a 2D plane. Now, for $X^T = -X$, all its diagonal components must be zero; since $X^a{}_a = -X^a{}_a$ for some fixed a (i.e., with no sum over a). Whereas $X^a{}_b = -X^b{}_a$ for $a \neq b$. Thus, an anti-symmetric $D \times D$ matrix must have $(D^2 - D)/2$ non-zero components. The $(1 \leq i \leq D(D-1)/2)$-th basis vector $-iJ^i$ is simply the matrix, for $m < n$, with the (m,n) component equal to -1; the (n,m) component $+1$; and the rest of its entries zero. For technical convenience, we will now give our generators two indices $-iJ^{mn}$ instead of the original one $(-iJ^a)$ because now the (m,n) would refer directly to its non-zero components.

$$i\left(J^{mn}\right)^a{}_b = \delta^m_a \delta^n_b - \delta^m_b \delta^n_a. \tag{3.0.2}$$

A rotation generated by J^{mn} is given by

$$R(\theta)^a{}_b = \delta^a{}_b + \theta(-iJ^{mn})^a{}_b + \mathcal{O}\left(\theta^2\right). \tag{3.0.3}$$

(Compare with eq. (2.0.12).) If we let $\widehat{e}_i$ be the unit vector pointing along the ith Cartesian axis, and employ eq. (3.0.2),

$$\begin{aligned}(R(\theta)\widehat{e}_i)^a = R(\theta)^a{}_b \widehat{e}^b_i &= \widehat{e}^a_i + \theta(-iJ^{mn})^a{}_b \widehat{e}^b_i + \mathcal{O}\left(\theta^2\right) &(3.0.4)\\ &= \widehat{e}^a_i - \theta\left(\delta^m_a \delta^n_i - \delta^m_i \delta^n_a\right) + \mathcal{O}\left(\theta^2\right), &(3.0.5)\end{aligned}$$

where we have exploited the unit vector character of

$$\widehat{e}^a_i = \delta^a_i. \tag{3.0.6}$$

When neither m nor n equals to i, notice only the first term survives – this means we get back the original $\widehat{e}_i$ and no rotation occurs. (Remember pure rotations on a 2D plane are generated by the first order term; so if it is zero then higher order terms do not matter.) When m equals i – remember $m \neq n$ by anti-symmetry – we have from equations (3.0.5) and (3.0.6)

$$\begin{aligned}(R(\theta)\widehat{e}_m)^a &= \widehat{e}^a_m + \theta \widehat{e}^a_n + \mathcal{O}\left(\theta^2\right), &(3.0.7)\\ R(\theta)\widehat{e}_m &= \widehat{e}_m + \theta \widehat{e}_n + \mathcal{O}\left(\theta^2\right). &(3.0.8)\end{aligned}$$

When n equals i instead, we have from equations (3.0.5) and (3.0.6)

$$\begin{aligned}(R(\theta)\widehat{e}_n)^a &= \widehat{e}^a_n - \theta \widehat{e}^a_m + \mathcal{O}\left(\theta^2\right), &(3.0.9)\\ R(\theta)\widehat{e}_n &= \widehat{e}_n - \theta \widehat{e}_m + \mathcal{O}\left(\theta^2\right). &(3.0.10)\end{aligned}$$

At this point, let us recall the 2D case we worked out previously. Setting $\widehat{x} \to \widehat{e}_1$ and $\widehat{y} \to \widehat{e}_2$, equations (2.0.1) and (2.0.2), for small angle θ, reads

$$\begin{aligned}R(\theta)\widehat{e}_1 &\approx \widehat{e}_1 + \theta \widehat{e}_2, &(3.0.11)\\ R(\theta)\widehat{e}_2 &\approx \widehat{e}_2 - \theta \widehat{e}_1. &(3.0.12)\end{aligned}$$

We see that eq. (3.0.8) corresponds to eq. (3.0.11); and eq. (3.0.10) to eq. (3.0.12). Once again, if we recall that from eq. (3.0.1) that the action of any rotation operator is determined by its first order term, we see that the $-iJ^{mn}$ generates rotation on the (m,n) plane. To this end, we may therefore re-express eq. (3.0.1) as

$$R(\vec{\theta}) = \exp\left(-\frac{i}{2}\theta_{ab}J^{ab}\right), \qquad J^{ab} = -J^{ba}; \tag{3.0.13}$$

where if it were the only non-zero rotation parameter, θ_{ab} would denote the angle of rotation on the (a,b) plane.

Problem 3.2. 3-dimensions The generators of rotation in 3D are given by

$$-iJ^{12} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad -iJ^{13} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad -iJ^{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{3.0.14}$$

(a) Show that these generators obey

$$\frac{i}{2}\left(J^{ab}\right)^{i}{}_{j}\,\epsilon_{abc} = \epsilon_{ijc}, \tag{3.0.15}$$

where ϵ_{abc} is the Levi-Civita symbol with $\epsilon_{123} \equiv 1$. Hint: Use eq. (3.0.2).
(b) From the discussion above, write down the explicit forms of the following rotations:

$$R_3(\theta_{12}) = e^{-i\theta_{12}J^{12}}, \qquad R_2(\theta_{13}) = e^{-i\theta_{13}J^{13}}, \qquad R_1(\theta_{23}) = e^{-i\theta_{23}J^{23}}. \tag{3.0.16}$$

Hint: How do you check your answers?
(c) Define the 'dual' of the generators:

$$L_i \equiv \frac{1}{2}\epsilon_{iab}J^{ab}. \tag{3.0.17}$$

Verify that they satisfy the angular momentum algebra

$$[L_i, L_j] = i\epsilon_{ijk}L_k. \tag{3.0.18}$$

Can you re-express the rotations in eq. (3.0.16) in terms of these $\{L_i\}$? Hint: You may need to know $\epsilon_{ijl}\epsilon_{abl} = \delta^i_a\delta^j_b - \delta^j_a\delta^i_b$.
(d) Do the rotations in eq. (3.0.16) commute? (Hint: You could either use the full expressions for the rotations and do a brute force calculation; or, you may exploit eq. (3.0.18) and work out $[R_1, R_2]$ up to second order in their angles.) Why does that imply, for example,

$$R_3(\theta_{12})R_2(\theta_{13}) \neq e^{-i(\theta_{12}J^{12}+\theta_{13}J^{13})}, \tag{3.0.19}$$

i.e., the exponents do not just add when you multiply the two rotations? $\square$

Non-commutativity