QFT HW 4

1. Consider a real scalar field φ with Lagrangian density $\mathcal{L}(\varphi, \partial\varphi)$. Show that the conserved angular momentum current $J^{\mu\alpha\beta}$ generated by a small Lorentz transformation

$$x^\mu \to x^\mu + \Omega^\mu{}_\nu x^\nu \tag{0.0.1}$$

   is given by

$$J^{\mu\alpha\beta} = T^{\mu[\alpha} x^{\beta]} = T^{\mu\alpha} x^\beta - T^{\mu\beta} x^\alpha; \tag{0.0.2}$$

   where $T^{\mu\nu}$ is the "canonical" stress tensor

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial^\nu \varphi - \eta^{\mu\nu} \mathcal{L}. \tag{0.0.3}$$

   (Hint: Remember the generator $\Omega^{\mu\nu}$ is anti-symmetric.)

   From the conservation of the current, namely $\partial_\mu J^{\mu\alpha\beta} = 0$ explain why the stress tensor is symmetric: $T^{\alpha\beta} = T^{\beta\alpha}$. For higher rank tensors, the angular momentum current would have additional terms that can be interpreted to arise from their internal angular momentum/spin; this will turn out to spoil this symmetry property.[1] □

2. Peskin & Schroeder 2.1

3. Peskin & Schroeder 2.2

4. Peskin & Schroeder 2.3

5. An alternate means to canonical quantization that emphasizes the Lorentz covariance nature of quantum fields is to demand that their (anti-)commutators yield the difference between the classical retarded and advanced Green's functions. For a massless scalar field with $\mathcal{L} = (1/2)(\partial\varphi)^2$, for instance, we have

$$i\left[\varphi(x), \varphi(x')\right] = G^+(x - x') - G^-(x - x'), \tag{0.0.4}$$

$$G^\pm(z) = -\int_{-\infty \pm i0^+}^{+\infty \pm i0^+} \frac{\mathrm{d}\omega}{2\pi} \int_{\mathbb{R}^{d-1}} \frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{e^{-i\omega z^0} e^{i\vec{k}\cdot\vec{z}}}{\omega^2 - \vec{k}^2}. \tag{0.0.5}$$

[1] It is physically important for the stress energy tensor to be symmetric. For it is otherwise possible to exert a finite torque on an infinitesimal volume; see Schutz's "First Course in General Relativity" for a discussion.

Note that:

$$\partial^2 G^{\pm}(x) = \delta^{(4)}(x). \tag{0.0.6}$$

Show that this leads to the canonical equal time commutation relations

$$[\varphi(t,\vec{x}), \Pi(t,\vec{x}')] = [\varphi(t,\vec{x}), \dot{\varphi}(t,\vec{x}')] = i\delta^{(d-1)}(\vec{x}-\vec{x}'). \tag{0.0.7}$$

Hint: You do not need to evaluate the entire Fourier integral, but you need to apply the residue theorem to the ω integral before computing the equal time commutator. □

6. **Vacuum Wave Functional** Argue that the vacuum wavefunctional of the massive scalar QFT is proportional to a Gaussian:

$$\langle\varphi|\,0\rangle = C\exp\left(-\frac{1}{2}\int_{\mathbb{R}^{d-1}} \varphi(\vec{k})^* \sqrt{\vec{k}^2+m^2}\varphi(\vec{k})\frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}}\right). \tag{0.0.8}$$

The C is a constant. Hint: Focus on a given $\vec{k}$-mode. What is its ground state wave function? The wave functional is the product of all $\vec{k}-$wave functions.

7. **Massless Spin-1 Photons and Spin-2 Gravitons** Consider a scalar-vector decomposition of the photon vector potential A_i and electric current J_i:

$$A_i = \partial_i\alpha + \alpha_i, \qquad \partial_i\alpha_i = 0, \tag{0.0.9}$$

$$J_i = \partial_i\Gamma + \Gamma_i, \qquad \partial_i\Gamma_i = 0. \tag{0.0.10}$$

Explain why the conservation of current $\partial^\mu J_\mu = 0$ implies

$$\dot{J}^0 = \partial_i\partial_i\Gamma. \tag{0.0.11}$$

Next, recall the gauge-invariant vector potential variables

$$\Phi \equiv A_0 - \dot{\alpha} \qquad \text{and} \qquad A_i^{\mathrm{T}} \equiv \alpha_i. \tag{0.0.12}$$

Show that the electromagnetic action

$$S_{\mathrm{EM}} \equiv \int \mathrm{d}^d x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu\right) \tag{0.0.13}$$

can be written in the manifestly gauge-invariant form

$$S_{\mathrm{EM}} \equiv \int \mathrm{d}^d x \left(\frac{1}{2}\partial_\alpha\alpha_j\partial^\alpha\alpha_j + \alpha_i\Gamma_i + \frac{1}{2}\partial_i\Phi\partial_i\Phi - \Phi J^0\right). \tag{0.0.14}$$

(Assume it is alright to integrate-by-parts freely.) Now, let us quantize the dynamical massless spin-1 photon α_i in the (3+1)D vacuum, i.e., where $J_\mu = 0$.

$$\alpha_i(x) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{2k}}\sum_{s=1}^{2}\left\{a_{\vec{k}}^s\epsilon_i^s e^{-ik\cdot x} + (a_{\vec{k}}^s)^\dagger(\epsilon_i^s)^* e^{+ik\cdot x}\right\}, \qquad k \equiv |\vec{k}|. \tag{0.0.15}$$

The $\{\epsilon^s_i | s = 1, 2\}$ are the two transverse orthonormal polarization vectors of the photon: $k^i \epsilon^s_i = 0$. Whereas $k_\mu k^\mu = 0$ and the ladder operators obey the simple harmonic algebra

$$\left[a^r_{\vec{k}}, (a^s_{\vec{k}'})^\dagger\right] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{k} - \vec{k}'). \tag{0.0.16}$$

Show that, for $x^\mu \equiv (t, \vec{x})$ and $x'^\mu \equiv (t', \vec{x}')$,

$$i\left[\alpha_i(x), \alpha_j(x')\right] = \text{sgn}(t - t') \int \frac{\text{d}^3\vec{k}}{(2\pi)^3} P_{ij}(\vec{k}) \frac{\sin(k(t - t'))}{k} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}, \tag{0.0.17}$$

$$P_{ij}(\vec{k}) \equiv \delta_{ij} - \frac{k_i k_j}{\vec{k}^2}. \tag{0.0.18}$$

Next, we turn to the massless spin-2 graviton h^{TT}_{ij},[2] which is transverse and traceless:

$$\partial_i h^{\text{TT}}_{ij} = 0 = \delta^{ij} h^{\text{TT}}_{ij}. \tag{0.0.19}$$

Its quantum operator admits the expansion, in (3+1)D,

$$h^{\text{TT}}_{ij}(x) = \int \frac{\text{d}^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2k}} \sum_{s=1}^{2} \left\{ a^s_{\vec{k}} \epsilon^s_{ij} e^{-ik\cdot x} + (a^s_{\vec{k}})^\dagger (\epsilon^s_{ij})^* e^{+ik\cdot x} \right\}, \qquad k \equiv |\vec{k}|. \tag{0.0.20}$$

The transverse traceless conditions in Fourier space reads

$$k^i \epsilon^s_{ij} = 0 = \delta^{ij} \epsilon^s_{ij}. \tag{0.0.21}$$

Imposing the simple harmonic algebra

$$\left[a^r_{\vec{k}}, (a^s_{\vec{k}'})^\dagger\right] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{k} - \vec{k}'). \tag{0.0.22}$$

demonstrate that

$$i\left[h^{\text{TT}}_{ij}(x), h^{\text{TT}}_{mn}(x')\right] = \text{sgn}(t - t') \int \frac{\text{d}^3\vec{k}}{(2\pi)^3} P_{ijmn}(\vec{k}) \frac{\sin(k(t - t'))}{k} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}, \tag{0.0.23}$$

$$P_{ijmn}(\vec{k}) \equiv \frac{1}{2}\left(P_{im}(\vec{k})P_{jn}(\vec{k}) + P_{in}(\vec{k})P_{jm}(\vec{k}) - P_{ij}(\vec{k})P_{mn}(\vec{k})\right). \tag{0.0.24}$$

Hints: Recall that the (Fourier space) transverse polarization vector can be obtained by projection:

$$\varepsilon^{\text{T}}_i = P_{ij}\varepsilon_j; \tag{0.0.25}$$

and likewise for the (Fourier space) TT gravitation polarization tensor:

$$\varepsilon^{\text{TT}}_{ij} = \frac{1}{2}\left(P_{im}(\vec{k})P_{jn}(\vec{k}) + P_{in}(\vec{k})P_{jm}(\vec{k}) - P_{ij}(\vec{k})P_{mn}(\vec{k})\right)\varepsilon_{mn}. \tag{0.0.26}$$

[2]In a weakly curved spacetime $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $|h_{\mu\nu}| \ll 1$, the h^{TT}_{ij} is the transverse-traceless part of $h_{\mu\nu}$.

This implies, when one encounters a sum over orthonormal polarization vectors such as

$$\sum_s \epsilon_i^{(s)\mathrm{T}} (\varepsilon_j^{(s)\mathrm{T}})^* = P_{ia} P_{jb} \sum_s \epsilon_a^{(s)} (\varepsilon_b^{(s)})^*; \tag{0.0.27}$$

even though the sum only runs over the two orthonormal polarizations perpendicular to k_i, it can be extended to include $\epsilon_i^{(3)} = \widehat{k}_i$ since it is annihilated by P_{ij} anyway. This observation may then be invoked to then apply the completeness relation. Similar remarks apply for the graviton case.

8. **Gaussian & Grassmannian Integrals** Compute the integrals involving real variables x^i and symmetric matrix A_{ij}:

$$\langle x^{i_1} x^{i_2} \rangle \equiv \int_{\mathbb{R}} \exp\left(i x^i A_{ij} x^j\right) x^{i_1} x^{i_2} \mathrm{d}^D \vec{x}, \tag{0.0.28}$$

$$\langle x^{i_1} x^{i_2} x^{i_3} \rangle \equiv \int_{\mathbb{R}} \exp\left(i x^i A_{ij} x^j\right) x^{i_1} x^{i_2} x^{i_3} \mathrm{d}^D \vec{x}, \tag{0.0.29}$$

$$\langle x^{i_1} x^{i_2} x^{i_3} x^{i_4} \rangle \equiv \int_{\mathbb{R}} \exp\left(i x^i A_{ij} x^j\right) x^{i_1} x^{i_2} x^{i_3} x^{i_4} \mathrm{d}^D \vec{x}. \tag{0.0.30}$$

Explain how to obtain the general even-point function $\langle x^{i_1} \dots x^{i_{2n}} \rangle$ and odd-point function $\langle x^{i_1} \dots x^{i_{2n+1}} \rangle$.

Next, compute the integrals involving complex Grassmann variables θ^i and Hermitian matrix H_{ij}:

$$\langle \theta^{i_1} \overline{\theta^{i_2}} \rangle \equiv \int_{\mathbb{R}} \exp\left(i \overline{\theta^i} H_{ij} \theta^j\right) \theta^{i_1} \overline{\theta^{i_2}} \mathrm{d}^D \vec{\theta} \mathrm{d}^D \vec{\theta}^*, \tag{0.0.31}$$

$$\langle \theta^{i_1} \overline{\theta^{i_2}} \theta^{i_3} \overline{\theta^{i_4}} \rangle \equiv \int_{\mathbb{R}} \exp\left(i \overline{\theta^i} H_{ij} \theta^j\right) \theta^{i_1} \overline{\theta^{i_2}} \theta^{i_3} \overline{\theta^{i_4}} \mathrm{d}^D \vec{\theta} \mathrm{d}^D \vec{\theta}^*. \tag{0.0.32}$$

Also evaluate the integrals involving only θ or only its complex conjugate $\overline{\theta} \equiv \theta^*$, namely $\langle \theta^{i_1} \dots \theta^{i_N} \rangle$ and $\langle \overline{\theta^{i_1}} \dots \overline{\theta^{i_N}} \rangle$. Can you explain how to get the general function $\langle \theta^{i_1} \overline{\theta^{j_1}} \dots \theta^{i_n} \overline{\theta^{j_n}} \rangle$? This and the above real variable case is the discrete analog of □