QFT: Problems for Final Presentation

Problem 1. QFT in a "Square Well" & Casimir Effect Consider the massless $m = 0$ free scalar field in (1+1)-dimensional Minkowski, but within a finite spatial domain of length L:

$$x^\mu \equiv (t \in \mathbb{R}, 0 \leq x \leq L). \tag{1}$$

Demand that the field vanishes at $x = 0, L$. Compute the vacuum expectation value of the energy density ρ, defined as:

$$\rho \equiv \left\langle \mathrm{vac} \middle| : T^{00} : \middle| \mathrm{vac} \right\rangle, \tag{2}$$

$$T^{00} \equiv \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\partial_x \varphi)^2. \tag{3}$$

What is the total energy E inside the square well, as a function of L? What is $\partial E/\partial L$? The latter can be interpreted as an effective force experienced by the walls of the square well as a result of quantum fluctuations. (Bonus: What are the pressure and momentum densities?) $\square$

Problem 2. de Sitter Power Spectrum de Sitter spacetime in 'flat slicing' coordinates is described by the metric

$$\mathrm{d}s^2 = a(\eta)^2 \left(\mathrm{d}\eta^2 - \mathrm{d}\vec{x}^2\right), \tag{4}$$

$$a[\eta] \equiv -(H\eta)^{-1}, \qquad \eta \in (-\infty, 0). \tag{5}$$

In this problem, compute in this geometry the one point function

$$\left\langle \mathrm{vac} \middle| \widetilde{\varphi}(\eta, \vec{k}) \middle| \mathrm{vac} \right\rangle \tag{6}$$

and the power spectrum of a massless scalar field

$$\left\langle \mathrm{vac} \middle| \widetilde{\varphi}(\eta, \vec{k}) \widetilde{\varphi}(\eta, \vec{k}') \middle| \mathrm{vac} \right\rangle. \tag{7}$$

Consider their late time limits $\eta \to 0$.

Hints: If $\sqrt{|g|}$ is the square root of the determinant of the metric $g_{\mu\nu} = \mathrm{diag}[1, -1, -1, -1]/(H\eta)^2$ and $g^{\mu\nu} = \mathrm{diag}[1, -1, -1, -1](H\eta)^2$ is its inverse, the scalar field obeys the wave equation

$$\Box\varphi = \frac{\partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu \varphi\right)}{\sqrt{|g|}} = 0. \tag{8}$$

The spatial translation invariance means we may Fourier decompose our scalar field in plane waves. Specifically, if we first re-scale

$$\varphi(\eta, \vec{x}) = \frac{\phi(\eta, \vec{x})}{a(\eta)}, \tag{9}$$

followed by examining a single Fourier mode

$$\phi(\eta, \vec{x}) = f(\xi) e^{i\vec{k}\cdot\vec{x}}, \tag{10}$$

where $\xi \equiv k\eta$ and $k \equiv |\vec{k}|$; show that f obeys

$$f''(\xi) + \left(1 - \frac{2}{\xi^2}\right) f(\xi) = 0. \tag{11}$$

You should find the two linearly independent solutions to be

$$f_\pm(\xi) \equiv \frac{e^{\pm i\xi}}{\sqrt{2}} \left(\pm i + \frac{1}{\xi}\right). \tag{12}$$

Choose your mode functions for φ such that, as $\eta \to -\infty$, the positive energy solutions approach those of Minkowski spacetime. Moreover, to quantize such a system, you should be able to verify its associated Lagrangian is

$$L_f \equiv \frac{1}{2} f'(\xi)^2 + \frac{f(\xi) f'(\xi)}{\xi} + \frac{1}{2} f(\xi)^2 \left(\frac{1}{\xi^2} - 1\right). \tag{13}$$

These vacuum fluctuations – if inflationary cosmologists are right – may be responsible for generating inhomogeneities in the very early universe, from which all of cosmic structure (galaxy clusters, etc.) were produced. Comment on how the calculations apply to primordial gravitational waves in particular. □

Problem 3. Fractional Fermion Number on Scalar Kinks in (1+1)D In this problem we will explore, in (1+1)D with Cartesian coordinates $x^\mu \equiv (t, x)$, how the number operator of a fermion ψ can acquire a fractional eigenvalue in the presence of a background static 'kink' solution to the scalar field ϕ. We start with the total Lagrangian

$$\mathcal{L} \equiv \mathcal{L}_\phi + \mathcal{L}_\psi, \tag{14}$$

$$\mathcal{L}_\phi \equiv \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\lambda}{4} \left(\phi^2 - \eta^2\right)^2, \tag{15}$$

$$\mathcal{L}_\psi \equiv \overline{\psi} \left(i\partial\!\!\!/ - g\phi\right) \psi. \tag{16}$$

Note that the Dirac spinor is a 2−component object in $(1 + 1)D$. Explain why the following choice is a valid one for the γ^μ matrices. Denoting the Pauli matrices by $\{\sigma^i | i = 1, 2, 3\}$, define

$$\gamma^0 \equiv \sigma^3 \qquad \text{and} \qquad \gamma^1 \equiv i\sigma^1. \tag{17}$$

Scalar Solutions First focus on the scalar sector. Verify that the following are static solutions. You may be able to actually *derive* them; try plotting the potential.

$$\phi_{\mathrm{V}} = \pm\eta \tag{18}$$

$$\phi_{\mathrm{K}} = \pm\eta \tanh\left(\eta\sqrt{\lambda/2}\cdot x\right). \tag{19}$$

Fermion Solutions Next, solve the Dirac equation

$$\left(i\partial\!\!\!/ - g\phi_{\mathrm{V}}\right)\psi = 0. \tag{20}$$

And, derive the *zero energy* solutions to

$$\left(i\partial\!\!\!/ - g\phi_{\mathrm{K}}\right)\psi = 0. \tag{21}$$

Fractional Fermion Number Quantize the Dirac fermion in eq. (21) and explain why the existence of zero energy solutions on a kink background ϕ_{K} leads to half-integer eigenvalues for the fermion number operator. □