Physics In Curved Spacetimes

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Contents

1 Differential Geometry of Curved Spaces

1.1 Preliminaries, Tangent Vectors, Metric, and Curvature

[1]Being fluent in the mathematics of differential geometry is mandatory if you wish to understand Einstein's General Relativity, humanity's current theory of gravity. But it also gives you a coherent framework to understand the multi-variable calculus you have learned, and will allow you to generalize it readily to dimensions other than the 3 spatial ones you are familiar with. In this section I will provide a practical introduction to differential geometry, and will show you how to recover from it what you have encountered in 2D/3D vector calculus. My goal here is that you will understand the subject well enough to perform concrete calculations, without worrying too much about the more abstract notions like, for e.g., what a manifold is.

I will assume you have an intuitive sense of what space means – after all, we live in it! Spacetime is simply space with an extra time dimension appended to it, although the notion of 'distance' in spacetime is a bit more subtle than that in space alone. To specify the (local) geometry of a space or spacetime means we need to understand how to express distances in terms of the coordinates we are using. For example, in Cartesian coordinates (x, y, z) and by invoking Pythagoras' theorem, the square of the distance $(\mathrm{d}\ell)^2$ between (x, y, z) and $(x+\mathrm{d}x, y+\mathrm{d}y, z+\mathrm{d}z)$ in flat (aka Euclidean) space is

$$(\mathrm{d}\ell)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2. \tag{1.1.1}$$

[2]A significant amount of machinery in differential geometry involves understanding how to employ arbitrary coordinate systems – and switching between different ones. For instance, we may convert the Cartesian coordinates flat space of eq. (1.1.1) into spherical coordinates,

$$(x, y, z) \equiv r\left(\sin\theta \cdot \cos\phi, \sin\theta \cdot \sin\phi, \cos\theta\right), \tag{1.1.2}$$

and find

$$(\mathrm{d}\ell)^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin(\theta)^2\mathrm{d}\phi^2). \tag{1.1.3}$$

The geometries in eq. (1.1.1) and eq. (1.1.3) are exactly the same. All we have done is to express them in different coordinate systems.

Conventions This is a good place to (re-)introduce the Einstein summation convention and the index convection. First, instead of (x, y, z), we can instead use $x^i \equiv (x^1, x^2, x^3)$; here, the superscript does not mean we are raising x to the first, second and third powers. A derivative with respect to the ith coordinate is $\partial_i \equiv \partial/\partial x^i$. The advantage of such a notation is its

[1]This and the next Chapter were directly gotten from their corresponding ones in Analytical Methods in Physics, so that the current set of notes would be a self-contained one.

[2]In 4-dimensional flat space*time*, with time t in addition to the three spatial coordinates $\{x, y, z\}$, the infinitesimal distance is given by a modified form of Pythagoras' theorem: $\mathrm{d}s^2 \equiv (\mathrm{d}t)^2 - (\mathrm{d}x)^2 - (\mathrm{d}y)^2 - (\mathrm{d}z)^2$. (The opposite sign convention, i.e., $\mathrm{d}s^2 \equiv -(\mathrm{d}t)^2 + (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2$, is also equally valid.) Why the "time" part of the distance differs in sign from the "space" part of the metric would lead us to a discussion of the underlying Lorentz symmetry. Because I wish to postpone the latter for the moment, I will develop differential geometry for curved spaces, not curved spacetimes. Despite this restriction, rest assured most of the subsequent formulas do carry over to curved spacetimes by simply replacing Latin/English alphabets with Greek ones – see the "Conventions" paragraph below.

compactness: we can say we are using coordinates $\{x^i\}$, where $i \in \{1,2,3\}$.[3] Not only that, we can employ Einstein's summation convention, which says all repeated indices are automatically summed over their relevant range. For example, eq. (1.1.1) now reads:

$$(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 = \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \equiv \sum_{1\leq i,j\leq 3} \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j. \tag{1.1.4}$$

(We say the indices of the $\{\mathrm{d}x^i\}$ are being contracted with that of δ_{ij}.) The symbol δ_{ij} is known as the Kronecker delta, defined as

$$\delta_{ij} = 1, \qquad i = j, \tag{1.1.5}$$
$$= 0, \qquad i \neq j. \tag{1.1.6}$$

Of course, δ_{ij} is simply the ij component of the identity matrix. Already, we can see δ_{ij} can be readily defined in an arbitrary D dimensional space, by allowing i, j to run from 1 through D. With these conventions, we can re-express the change of variables from eq. (1.1.1) and eq. (1.1.3) as follows. First write

$$\xi^i \equiv (r \geq 0, 0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi). \tag{1.1.7}$$

Then (1.1.1) becomes

$$\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j = \delta_{ab}\frac{\partial x^a}{\partial \xi^i}\frac{\partial x^b}{\partial \xi^j}\mathrm{d}\xi^i\mathrm{d}\xi^j = \frac{\partial \vec{x}}{\partial \xi^i}\cdot\frac{\partial \vec{x}}{\partial \xi^j}\mathrm{d}\xi^i\mathrm{d}\xi^j, \tag{1.1.8}$$

where in the second equality we have, for convenience, expressed the contraction with the Kronecker delta as an ordinary (vector calculus) dot product. At this point, let us notice, if we call the coefficients of the quadratic form g_{ij}; for example, $\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \equiv g_{ij}\mathrm{d}x^i\mathrm{d}x^j$, we have

$$g_{i'j'}(\vec{\xi}) = \frac{\partial \vec{x}}{\partial \xi^i}\cdot\frac{\partial \vec{x}}{\partial \xi^j}, \tag{1.1.9}$$

where the primes on the indices are there to remind us this is not $g_{ij}(\vec{x}) = \delta_{ij}$, the components written in the Cartesian coordinates, but rather the ones written in spherical coordinates. In fact, what we are finding in eq. (1.1.8) is

$$g_{i'j'}(\vec{\xi}) = g_{ab}(\vec{x})\frac{\partial x^a}{\partial \xi^i}\frac{\partial x^b}{\partial \xi^j}. \tag{1.1.10}$$

Let's proceed to work out the above dot products out. Firstly,

$$\frac{\partial \vec{x}}{\partial r} = (\sin\theta\cdot\cos\phi, \sin\theta\cdot\sin\phi, \cos\theta), \tag{1.1.11}$$
$$\frac{\partial \vec{x}}{\partial \theta} = r(\cos\theta\cdot\cos\phi, \cos\theta\cdot\sin\phi, -\sin\theta), \tag{1.1.12}$$
$$\frac{\partial \vec{x}}{\partial \phi} = r(-\sin\theta\cdot\sin\phi, \sin\theta\cdot\cos\phi, 0). \tag{1.1.13}$$

[3] It is common to use the English alphabets to denote space coordinates and Greek letters to denote spacetime ones. We will adopt this convention in these notes, but note that it is not a universal one; so be sure to check the notation of the book you are reading.

A direct calculation should return the results

$$g_{r\theta} = g_{\theta r} = \frac{\partial \vec{x}}{\partial r} \cdot \frac{\partial \vec{x}}{\partial \theta} = 0, \qquad g_{r\phi} = g_{\phi r} = \frac{\partial \vec{x}}{\partial r} \cdot \frac{\partial \vec{x}}{\partial \phi} = 0, \qquad g_{\theta\phi} = g_{\phi\theta} = \frac{\partial \vec{x}}{\partial \theta} \cdot \frac{\partial \vec{x}}{\partial \phi} = 0; \tag{1.1.14}$$

and

$$g_{rr} = \frac{\partial \vec{x}}{\partial r} \cdot \frac{\partial \vec{x}}{\partial r} \equiv \left(\frac{\partial \vec{x}}{\partial r}\right)^2 = 1, \tag{1.1.15}$$

$$g_{\theta\theta} = \left(\frac{\partial \vec{x}}{\partial \theta}\right)^2 = r^2, \tag{1.1.16}$$

$$g_{\phi\phi} = \left(\frac{\partial \vec{x}}{\partial \phi}\right)^2 = r^2 \sin^2(\theta). \tag{1.1.17}$$

Altogether, these yield eq. (1.1.3).

Tangent vectors In Euclidean space, we may define vectors by drawing a directed straight line between one point to another. In curved space, the notion of a 'straight line' is not straightforward, and as such we no longer try to implement such a definition of a vector. Instead, the notion of tangent vectors, and their higher rank tensor generalizations, now play central roles in curved spacetime geometry and physics. Imagine, for instance, a thin layer of water flowing over an undulating 2D surface – an example of a tangent vector on a curved space is provided by the velocity of an infinitesimal volume within the flow.

More generally, let $\vec{x}(\lambda)$ denote the trajectory swept out by an infinitesimal volume of fluid as a function of (fictitious) time λ, transversing through a $(D \geq 2)-$dimensional space. (The $\vec{x}$ need not be Cartesian coordinates.) We may then define the tangent vector $v^i(\lambda) \equiv \mathrm{d}\vec{x}(\lambda)/\mathrm{d}\lambda$. Conversely, given a vector field $v^i(\vec{x})$ – a $(D \geq 2)-$component object defined at every point in space – we may find a trajectory $\vec{x}(\lambda)$ such that $\mathrm{d}\vec{x}/\mathrm{d}\lambda = v^i(\vec{x}(\lambda))$. (This amounts to integrating an ODE, and in this context is why $\vec{x}(\lambda)$ is called the *integral curve of* v^i.) In other words, tangent vectors do fit the mental picture that the name suggests, as 'little arrows' based at each point in space, describing the local 'velocity' of some (perhaps fictitious) flow.

You may readily check that tangent vectors at a given point p in space do indeed form a vector space. However, we have written the components v^i but did not explain what their basis vectors were. Geometrically speaking, v tells us in what direction and how quickly to move away from the point p. This can be formalized by recognizing that the number of independent directions that one can move away from p corresponds to the number of independent partial derivatives on some arbitrary (scalar) function defined on the curved space; namely $\partial_i f(\vec{x})$ for $i = 1, 2, \dots, D$, where $\{x^i\}$ are the coordinates used. Furthermore, the set of $\{\partial_i\}$ do span a vector space, based at p. We would thus say that any tangent vector v is a superposition of partial derivatives:

$$v = v^i(\vec{x}) \frac{\partial}{\partial x^i} \equiv v^i(x^1, x^2, \dots, x^D) \frac{\partial}{\partial x^i} \equiv v^i \partial_i. \tag{1.1.18}$$

As already alluded to, given these components $\{v^i\}$, the vector v can be thought of as the velocity with respect to some (fictitious) time λ by solving the ordinary differential equation

$v^i = \mathrm{d}x^i(\lambda)/\mathrm{d}\lambda$. We may now see this more explicitly; $v^i\partial_i f(\vec{x})$ is the time derivative of f along the integral curve of $\vec{v}$ because

$$v^i\partial_i f\left(\vec{x}(\lambda)\right) = \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\partial_i f(\vec{x}) = \frac{\mathrm{d}f(\lambda)}{\mathrm{d}\lambda}. \tag{1.1.19}$$

To sum: the $\{\partial_i\}$ are the basis kets based at a given point p in the curved space, allowing us to enumerate all the independent directions along which we may compute the 'time derivative' of f at the same point p.

General spatial metric In a generic curved space, the square of the infinitesimal distance between the neighboring points $\vec{x}$ and $\vec{x}+\mathrm{d}\vec{x}$, which we will continue to denote as $(\mathrm{d}\ell)^2$, is no longer given by eq. (1.1.1) – because we cannot expect Pythagoras' theorem to apply. But by scaling arguments it should still be quadratic in the infinitesimal distances $\{\mathrm{d}x^i\}$. The most general of such expression is

$$(\mathrm{d}\ell)^2 = g_{ij}(\vec{x})\mathrm{d}x^i\mathrm{d}x^j. \tag{1.1.20}$$

Since it measures distances, g_{ij} needs to be real. It is also symmetric, since any antisymmetric portion would drop out of the summation in eq. (1.1.20) anyway. (Why?) Finally, because we are discussing curved spaces for now, g_{ij} needs to have strictly positive eigenvalues.

Additionally, given g_{ij}, we can proceed to define the inverse metric g^{ij} in any coordinate system, as the matrix inverse of g_{ij}:

$$g^{ij}g_{jl} \equiv \delta^i_{\ l}. \tag{1.1.21}$$

Everything else in a differential geometric calculation follows from the curved metric in eq. (1.1.20), once it is specified for a given setup:[4] the ensuing Christoffel symbols, Riemann/Ricci tensors, covariant derivatives/curl/divergence; what defines straight lines; parallel transportation; etc.

Distances If you are given a path $\vec{x}(\lambda_1 \leq \lambda \leq \lambda_2)$ between the points $\vec{x}(\lambda_1) = \vec{x}_1$ and $\vec{x}(\lambda_2) = \vec{x}_2$, then $\mathrm{d}\vec{x} = (\mathrm{d}\vec{x}/\mathrm{d}\lambda)\mathrm{d}\lambda$, and the distance swept out by this path is given by the integral

$$\ell = \int_{\vec{x}(\lambda_1 \leq \lambda \leq \lambda_2)} \sqrt{g_{ij}\left(\vec{x}(\lambda)\right)\mathrm{d}x^i\mathrm{d}x^j} = \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \sqrt{g_{ij}\left(\vec{x}(\lambda)\right)\frac{\mathrm{d}x^i(\lambda)}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j(\lambda)}{\mathrm{d}\lambda}}. \tag{1.1.22}$$

Problem 1.1. 1D Coordinate Invariance Show that this definition of distance is invariant under change of the parameter λ, as long as the transformation is orientation preserving. That is, suppose we replace $\lambda \to \lambda(\lambda')$ and thus $\mathrm{d}\lambda = (\mathrm{d}\lambda/\mathrm{d}\lambda')\mathrm{d}\lambda'$ – then as long as $\mathrm{d}\lambda/\mathrm{d}\lambda' > 0$, we have

$$\ell = \int_{\lambda'_1}^{\lambda'_2} \mathrm{d}\lambda' \sqrt{g_{ij}\left(\vec{x}(\lambda')\right)\frac{\mathrm{d}x^i(\lambda')}{\mathrm{d}\lambda'}\frac{\mathrm{d}x^j(\lambda')}{\mathrm{d}\lambda'}}, \tag{1.1.23}$$

where $\lambda(\lambda'_{1,2}) = \lambda_{1,2}$.

[4] As with most physics texts on differential geometry, we will ignore torsion.

Affine Parametrization Why can we always choose λ such that

$$\sqrt{g_{ij}\left(\vec{x}(\lambda)\right)\frac{\mathrm{d}x^i(\lambda)}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j(\lambda)}{\mathrm{d}\lambda}} = \text{constant}, \tag{1.1.24}$$

i.e., the square root factor can be made constant along the entire path linking $\vec{x}_1$ to $\vec{x}_2$?

Up to a re-scaling and 1D translation, this affine parametrization amounts to using the path length itself as the parameter λ.

Kets and Bras Earlier, while discussing tangent vectors, we stated that the $\{\partial_i\}$ are the ket's, the basis tangent vectors at a given point in space. The infinitesimal distances $\{\mathrm{d}x^i\}$ can now, in turn, be thought of as the basis dual vectors (the bra's) – through the definition

$$\left\langle \mathrm{d}x^i \middle| \partial_j \right\rangle = \delta^i{}_j. \tag{1.1.25}$$

Why this is a useful perspective is due to the following. Let us consider an infinitesimal variation of our arbitrary function at $\vec{x}$:

$$\mathrm{d}f = \partial_i f(\vec{x})\mathrm{d}x^i. \tag{1.1.26}$$

Then, given a vector field v, we can employ eq. (1.1.25) to construct the derivative of the latter along the former, at some point $\vec{x}$, by

$$\langle \mathrm{d}f | v\rangle = v^j \partial_i f(\vec{x}) \left\langle \mathrm{d}x^i \middle| \partial_j \right\rangle = v^i \partial_i f(\vec{x}). \tag{1.1.27}$$

What about the inner products $\langle \mathrm{d}x^i | \mathrm{d}x^j\rangle$ and $\langle \partial_i | \partial_j \rangle$? They are

$$\left\langle \mathrm{d}x^i \middle| \mathrm{d}x^j \right\rangle = g^{ij} \qquad \text{and} \qquad \langle \partial_i | \partial_j \rangle = g_{ij}. \tag{1.1.28}$$

This is because

$$g_{ij} \left|\mathrm{d}x^j\right\rangle \equiv |\partial_i\rangle \qquad \Leftrightarrow \qquad g_{ij}\left\langle \mathrm{d}x^j\right| \equiv \langle \partial_i | ; \tag{1.1.29}$$

or, equivalently,

$$\left|\mathrm{d}x^j\right\rangle \equiv g^{ij} |\partial_i\rangle \qquad \Leftrightarrow \qquad \left\langle \mathrm{d}x^j\right| \equiv g^{ij} \langle \partial_i | . \tag{1.1.30}$$

In other words,

> At a given point in a curved space, one may define two different vector spaces – one spanned by the basis tangent vectors $\{|\partial_i\rangle\}$ and another by its dual 'bras' $\{|\mathrm{d}x^i\rangle\}$. Moreover, these two vector spaces are connected through the metric g_{ij} and its inverse.

Parallel transport and Curvature Roughly speaking, a curved space is one where the usual rules of Euclidean (flat) space no longer apply. For example, Pythagoras' theorem does not hold; and the sum of the angles of an extended triangle is not π.

The quantitative criteria to distinguish a curved space from a flat one, is to parallel transport a tangent vector $v^i(\vec{x})$ around a closed loop on a coordinate grid. If, upon bringing it back to the same location $\vec{x}$, the tangent vector is the same one we started with – *for all possible coordinate loops* – then the space is flat. Otherwise the space is curved. In particular, if you parallel transport a vector around an infinitesimal closed loop formed by two pairs of coordinate lines, starting from any one of its corners, and if the resulting vector is compared with original one, you would find that the difference is proportional to the Riemann curvature tensor $R^i{}_{jkl}$. More specifically, suppose v^i is parallel transported along a parallelogram, from $\vec{x}$ to $\vec{x} + \mathrm{d}\vec{y}$; then to $\vec{x} + \mathrm{d}\vec{y} + \mathrm{d}\vec{z}$; then to $\vec{x} + \mathrm{d}\vec{z}$; then back to $\vec{x}$. Then, denoting the end result as v'^i, we would find that

$$v'^i - v^i \propto R^i{}_{jkl} v^j \mathrm{d}y^k \mathrm{d}z^l. \tag{1.1.31}$$

Therefore, whether or not a geometry is locally curved is determined by this tensor. Of course, we have not defined what parallel transport actually is; to do so requires knowing the covariant derivative – but let us first turn to a simple example where our intuition still holds.

2−sphere as an example A common textbook example of a curved space is that of a 2−sphere of some fixed radius, sitting in 3D flat space, parametrized by the usual spherical coordinates $(0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi)$.[5] Start at the north pole with the tangent vector $v = \partial_\theta$ pointing towards the equator with azimuthal direction $\phi = \phi_0$. Let us parallel transport v along itself, i.e., with $\phi = \phi_0$ fixed, until we reach the equator itself. At this point, the vector is perpendicular to the equator, pointing towards the South pole. Next, we parallel transport v along the equator from $\phi = \phi_0$ to some other longitude $\phi = \phi_0'$; here, v is still perpendicular to the equator, and still pointing towards the South pole. Finally, we parallel transport it back to the North pole, along the $\phi = \phi_0'$ line. Back at the North pole, v now points along the $\phi = \phi_0'$ longitude line and no longer along the original $\phi = \phi_0$ line. Therefore, v does not return to itself after parallel transport around a closed loop: the 2−sphere is a curved surface. This same example also provides us a triangle whose sum of its internal angles is $\pi + |\phi_0 - \phi_0'| > \pi$.[6] Finally, notice in this 2-sphere example, the question of what a straight line means – let alone using it to define a vector, as one might do in flat space – does not produce a clear answer.

Comparing tangent vectors at different places That tangent vectors cannot, in general, be parallel transported in a curved space also tells us comparing tangent vectors based at different locations is not a straightforward procedure, especially compared to the situation in flat Euclidean space. This is because, if $\vec{v}(\vec{x})$ is to be compared to $\vec{w}(\vec{x}')$ by parallel transporting $\vec{v}(\vec{x})$ to $\vec{x}'$; different results will be obtained by simply choosing different paths to get from $\vec{x}$ to $\vec{x}'$.

Intrinsic vs extrinsic curvature A 2D cylinder (embedded in 3D flat space) formed by rolling up a flat rectangular piece of paper has a surface that is *intrinsically* flat – the Riemann tensor is zero everywhere because the intrinsic geometry of the surface is the same flat metric before the paper was rolled up. However, the paper as viewed by an ambient 3D observer does have an *extrinsic* curvature due to its cylindrical shape. To characterize extrinsic curvature mathematically, one would erect a vector perpendicular to the surface in question and parallel transport it along this same surface: the latter is flat if the vector remains parallel; otherwise it

[5] Any curved space can in fact always be viewed as a curved surface residing in a higher dimensional flat space.

[6] The 2−sphere has positive curvature; whereas a saddle has negative curvature, and would support a triangle whose angles add up to less than π. In a very similar spirit, the Cosmic Microwave Background (CMB) sky contains hot and cold spots, whose angular size provide evidence that we reside in a spatially flat universe. See the Wilkinson Microwave Anisotropy Probe (WMAP) pages here and here.

is curved. In curved spacetimes, when this vector refers to the flow of time and is perpendicular to some spatial surface, the extrinsic curvature also describes its time evolution.

1.2 Locally Flat Coordinates & Symmetries, Infinitesimal Volumes, General Tensors, Orthonormal Basis

Locally flat coordinates[7] and symmetries It is a mathematical fact that, given some fixed point y_0^i on the curved space, one can find coordinates y^i such that locally the metric does become flat:

$$\lim_{\vec{y}\to\vec{y}_0} g_{ij}(\vec{y}) = \delta_{ij} + g_2 \cdot R_{ikjl}(\vec{y}_0)\ (y-y_0)^k (y-y_0)^l + \dots, \tag{1.2.1}$$

with a similar result for curved spacetimes. In this "locally flat" coordinate system, the first corrections to the flat Euclidean metric is quadratic in the displacement vector $\vec{y}-\vec{y}_0$, and $R_{ikjl}(\vec{y}_0)$ is the Riemann tensor – which is the chief measure of curvature – evaluated at $\vec{y}_0$. (The g_2 is just a numerical constant, whose precise value is not important for our discussion.) In a curved space*time*, that geometry can always be viewed as locally flat is why the mathematics you are encountering here is the appropriate framework for reconciling gravity as a force, Einstein's equivalence principle, and the Lorentz symmetry of Special Relativity.

Note that under spatial rotations $\{\widehat{R}^i{}_j\}$, which obeys $\widehat{R}^a{}_i \widehat{R}^b{}_j \delta_{ab} = \delta_{ij}$, if we define in Euclidean space the following change-of-Cartesian coordinates (from $\vec{x}$ to $\vec{x}'$)

$$x^i \equiv \widehat{R}^i{}_j x'^j; \tag{1.2.2}$$

the flat metric would retain the same form

$$\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j = \delta_{ab}\widehat{R}^a{}_i \widehat{R}^b{}_j \mathrm{d}x'^i\mathrm{d}x'^j = \delta_{ij}\mathrm{d}x'^i\mathrm{d}x'^j. \tag{1.2.3}$$

A similar calculation would tell us flat Euclidean space is invariant under parity flips, i.e., $x'^k \equiv -x^k$ for some fixed k, as well as spatial translations $\vec{x}' \equiv \vec{x}+\vec{a}$, for constant $\vec{a}$. To sum:

> At a given point in a curved space, it is always possible to find a coordinate system – i.e., a geometric viewpoint/'frame' – such that the space is flat up to distances of $\mathcal{O}(1/|\max R_{ijlk}(\vec{y}_0)|^{1/2})$, and hence 'locally' invariant under rotations, translations, and reflections.

This is why it took a while before humanity came to recognize we live on the curved surface of the (approximately spherical) Earth: locally, the Earth's surface looks flat!

Coordinate-transforming the metric Note that, in the context of eq. (1.1.20), $\vec{x}$ is not a vector in Euclidean space, but rather another way of denoting x^a without introducing too many dummy indices $\{a, b, \dots, i, j, \dots\}$. Also, x^i in eq. (1.1.20) are not necessary Cartesian coordinates, but can be completely arbitrary. The metric $g_{ij}(\vec{x})$ can viewed as a 3×3 (or $D\times D$, in D dimensions) matrix of functions of $\vec{x}$, telling us how the notion of distance vary as one moves about in the space. Just as we were able to translate from Cartesian coordinates to spherical

[7] Also known as Riemann normal coordinates.

ones in Euclidean 3-space, in this generic curved space, we can change from $\vec{x}$ to $\vec{\xi}$, i.e., one arbitrary coordinate system to another, so that

$$g_{ij}\left(\vec{x}\right)\mathrm{d}x^i\mathrm{d}x^j = g_{ij}\left(\vec{x}(\vec{\xi})\right)\frac{\partial x^i(\vec{\xi})}{\partial \xi^a}\frac{\partial x^j(\vec{\xi})}{\partial \xi^b}\mathrm{d}\xi^a\mathrm{d}\xi^b \equiv g_{ab}(\vec{\xi})\mathrm{d}\xi^a\mathrm{d}\xi^b. \tag{1.2.4}$$

We can attribute all the coordinate transformation to how it affects the components of the metric:

$$g_{ab}(\vec{\xi}) = g_{ij}\left(\vec{x}(\vec{\xi})\right)\frac{\partial x^i(\vec{\xi})}{\partial \xi^a}\frac{\partial x^j(\vec{\xi})}{\partial \xi^b}. \tag{1.2.5}$$

The left hand side are the metric components in $\vec{\xi}$ coordinates. The right hand side consists of the Jacobians $\partial x/\partial \xi$ contracted with the metric components in $\vec{x}$ coordinates – but now with the $\vec{x}$ replaced with $\vec{x}(\vec{\xi})$, their corresponding expressions in terms of $\vec{\xi}$.

Inverse metric Previously, we defined g^{ij} to be the matrix inverse of the metric tensor g_{ij}. We can also view g^{ij} as components of the tensor

$$g^{ij}(\vec{x})\partial_i \otimes \partial_j, \tag{1.2.6}$$

where we have now used $\otimes$ to indicate we are taking the tensor product of the partial derivatives ∂_i and ∂_j. In $g_{ij}\left(\vec{x}\right)\mathrm{d}x^i\mathrm{d}x^j$ we really should also have $\mathrm{d}x^i \otimes \mathrm{d}x^j$, but I prefer to stick with the more intuitive idea that the metric (with lower indices) is the sum of squares of distances. Just as we know how $\mathrm{d}x^i$ transforms under $\vec{x} \to \vec{x}(\vec{\xi})$, we also can work out how the partial derivatives transform.

$$g^{ij}(\vec{x})\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} = g^{ab}\left(\vec{x}(\vec{\xi})\right)\frac{\partial \xi^i}{\partial x^a}\frac{\partial \xi^j}{\partial x^b}\frac{\partial}{\partial \xi^i} \otimes \frac{\partial}{\partial \xi^j} \tag{1.2.7}$$

In terms of its components, we can read off their transformation rules:

$$g^{ij}(\vec{\xi}) = g^{ab}\left(\vec{x}(\vec{\xi})\right)\frac{\partial \xi^i}{\partial x^a}\frac{\partial \xi^j}{\partial x^b}. \tag{1.2.8}$$

The left hand side is the inverse metric written in the $\vec{\xi}$ coordinate system, whereas the right hand side involves the inverse metric written in the $\vec{x}$ coordinate system – contracted with two Jacobian's $\partial \xi/\partial x$ – except all the $\vec{x}$ are replaced with the expressions $\vec{x}(\vec{\xi})$ in terms of $\vec{\xi}$.

A technical point: here and below, the Jacobian $\partial x^a(\vec{\xi})/\partial \xi^j$ can be calculated in terms of $\vec{\xi}$ by direct differentiation if we have defined $\vec{x}$ in terms of $\vec{\xi}$, namely $\vec{x}(\vec{\xi})$. But the Jacobian $(\partial \xi^i/\partial x^a)$ in terms of $\vec{\xi}$ requires a matrix inversion. For, by the chain rule,

$$\frac{\partial x^i}{\partial \xi^l}\frac{\partial \xi^l}{\partial x^j} = \frac{\partial x^i}{\partial x^j} = \delta^i_j, \qquad \text{and} \qquad \frac{\partial \xi^i}{\partial x^l}\frac{\partial x^l}{\partial \xi^j} = \frac{\partial \xi^i}{\partial \xi^j} = \delta^i_j. \tag{1.2.9}$$

In other words, given $\vec{x} \to \vec{x}(\vec{\xi})$, we can compute $\mathcal{J}^a_{\ i} \equiv \partial x^a/\partial \xi^i$ in terms of $\vec{\xi}$, with a being the row number and i as the column number. Then find the inverse, i.e., $(\mathcal{J}^{-1})^a_{\ i}$ and identify it with $\partial \xi^a/\partial x^i$ in terms of $\vec{\xi}$.

General tensor A *scalar* φ is an object with no indices that transforms as

$$\varphi(\vec{\xi}) = \varphi\left(\vec{x}(\vec{\xi})\right). \tag{1.2.10}$$

That is, take $\varphi(\vec{x})$ and simply replace $\vec{x} \to \vec{x}(\vec{\xi})$ to obtain $\varphi(\vec{\xi})$.

A *vector* $v^i(\vec{x})\partial_i$ transforms as, by the chain rule,

$$v^i(\vec{x})\frac{\partial}{\partial x^i} = v^i(\vec{x}(\vec{\xi}))\frac{\partial \xi^j}{\partial x^i}\frac{\partial}{\partial \xi^j} \equiv v^j(\vec{\xi})\frac{\partial}{\partial \xi^j} \tag{1.2.11}$$

If we attribute all the transformations to the components, the components in the $\vec{x}$-coordinate system $v^i(\vec{x})$ is related to those in the $\vec{y}$-coordinate system $v^i(\vec{\xi})$ through the relation

$$v^i(\vec{\xi}) = v^i(\vec{x}(\vec{\xi}))\frac{\partial \xi^j}{\partial x^i}. \tag{1.2.12}$$

Similarly, a *1-form* $A_i \mathrm{d}x^i$ transforms, by the chain rule,

$$A_i(\vec{x})\mathrm{d}x^i = A_i(\vec{x}(\vec{\xi}))\frac{\partial x^i}{\partial \xi^j}\mathrm{d}\xi^j \equiv A_j(\vec{\xi})\mathrm{d}\xi^j. \tag{1.2.13}$$

If we again attribute all the coordinate transformations to the components; the ones in the $\vec{x}$-system $A_i(\vec{x})$ is related to the ones in the $\vec{\xi}$-system $A_i(\vec{\xi})$ through

$$A_j(\vec{\xi}) = A_i(\vec{x}(\vec{\xi}))\frac{\partial x^i}{\partial \xi^j}. \tag{1.2.14}$$

By taking tensor products of $\{|\partial_i\rangle\}$ and $\{\langle \mathrm{d}x^i|\}$, we may define a *rank* $\binom{N}{M}$ *tensor* T as an object with N "upper indices" and M "lower indices" that transforms as

$$T^{i_1 i_2 \dots i_N}{}_{j_i j_2 \dots j_M}(\vec{\xi}) = T^{a_1 a_2 \dots a_N}{}_{b_i b_2 \dots b_M}\left(\vec{x}(\vec{\xi})\right)\frac{\partial \xi^{i_1}}{\partial x^{a_1}} \dots \frac{\partial \xi^{i_N}}{\partial x^{a_N}}\frac{\partial x^{b_1}}{\partial \xi^{j_1}} \dots \frac{\partial x^{b_M}}{\partial \xi^{j_M}}. \tag{1.2.15}$$

The left hand side are the tensor components in $\vec{\xi}$ coordinates and the right hand side are the Jacobians $\partial x/\partial \xi$ and $\partial \xi/\partial x$ contracted with the tensor components in $\vec{x}$ coordinates – but now with the $\vec{x}$ replaced with $\vec{x}(\vec{\xi})$, their corresponding expressions in terms of $\vec{\xi}$. This multi-indexed object should be viewed as the components of

$$T^{i_1 i_2 \dots i_N}{}_{j_i j_2 \dots j_M}(\vec{x})\left|\frac{\partial}{\partial x^{i_1}}\right\rangle \otimes \dots \otimes \left|\frac{\partial}{\partial x^{i_N}}\right\rangle \otimes \left\langle \mathrm{d}x^{j_1}\right| \otimes \dots \otimes \left\langle \mathrm{d}x^{j_M}\right|. \tag{1.2.16}$$

[8]Above, we only considered T with all upper indices followed by all lower indices. Suppose we

[8]Strictly speaking, when discussing the metric and its inverse above, we should also have respectively expressed them as $g_{ij}\left\langle \mathrm{d}x^i\right| \otimes \left\langle \mathrm{d}x^j\right|$ and $g^{ij}\left|\partial_i\right\rangle \otimes \left|\partial_j\right\rangle$, with the appropriate bras and kets enveloping the $\{\mathrm{d}x^i\}$ and $\{\partial_i\}$. We did not do so because we wanted to highlight the geometric interpretation of $g_{ij}\mathrm{d}x^i\mathrm{d}x^j$ as the square of the distance between $\vec{x}$ and $\vec{x}+\mathrm{d}\vec{x}$, where the notion of $\mathrm{d}x^i$ as (a component of) an infinitesimal 'vector' – as opposed to being a 1-form – is, in our opinion, more useful for building the reader's geometric intuition.

It may help the physicist reader to think of a scalar field in eq. (1.2.10) as an observable, such as the temperature $T(\vec{x})$ of the 2D undulating surface mentioned above. If you were provided such an expression for $T(\vec{x})$, together with an accompanying definition for the coordinate system $\vec{x}$; then, to convert this same temperature field to a different coordinate system (say, $\vec{\xi}$) one would, in fact, do $T(\vec{\xi}) \equiv T(\vec{x}(\vec{\xi}))$, because you'd want $\vec{\xi}$ to refer to the same point in space as $\vec{x} = \vec{x}(\vec{\xi})$. For a general tensor in eq. (1.2.16), the tensor components $T^{i_1 i_2 \dots i_N}{}_{j_i j_2 \dots j_M}$ may then be regarding as scalars describing some weighted superposition of the tensor product of basis vectors and 1-forms. Its transformation rules in eq. (1.2.15) are really a shorthand for the lazy physicist who does not want to carry the basis vectors/1-forms around in his/her calculations.

had ${T^i_{\ j}}^k$; it is the components of

$$T^{i\ \ k}_{\ j}(\vec{x}) \left|\partial_i\right\rangle \otimes \left\langle \mathrm{d}x^j \right| \otimes \left|\partial_k\right\rangle. \tag{1.2.17}$$

Raising and lowering tensor indices The indices on a tensor are moved – from upper to lower, or vice versa – using the metric tensor. For example,

$$T^{m_1 \dots m_a \ \ n_1 \dots n_b}_{\qquad\ \ i} = g_{ij} T^{m_1 \dots m_a j n_1 \dots n_b}, \tag{1.2.18}$$
$$T_{m_1 \dots m_a \ \ n_1 \dots n_b}^{\qquad\ \ i} = g^{ij} T_{m_1 \dots m_a j n_1 \dots n_b}. \tag{1.2.19}$$

Because upper indices transform oppositely from lower indices – see eq. (1.2.9) – when we contract a upper and lower index, it now transforms as a scalar. For example,

$$\begin{aligned} A^i_{\ l}(\vec{\xi}) B^{lj}(\vec{\xi}) &= \frac{\partial \xi^i}{\partial x^m} A^m_{\ \ a}\left(\vec{x}(\vec{\xi})\right) \frac{\partial x^a}{\partial \xi^l} \frac{\partial \xi^l}{\partial x^c} B^{cn}\left(\vec{x}(\vec{\xi})\right) \frac{\partial \xi^j}{\partial x^n} \\ &= \frac{\partial \xi^i}{\partial x^m} \frac{\partial \xi^j}{\partial x^n} A^m_{\ \ c}\left(\vec{x}(\vec{\xi})\right) B^{cn}\left(\vec{x}(\vec{\xi})\right). \end{aligned} \tag{1.2.20}$$

General covariance Tensors are ubiquitous in physics: the electric and magnetic fields can be packaged into one Faraday tensor $F_{\mu\nu}$; the energy-momentum-shear-stress tensor of matter $T_{\mu\nu}$ is what sources the curved geometry of spacetime in Einstein's theory of General Relativity; etc. The coordinate transformation rules in eq. (1.2.15) that defines a tensor is actually the statement that, the mathematical description of the physical world (the tensors themselves in eq. (1.2.16)) should not depend on the coordinate system employed. Any expression or equation with physical meaning – i.e., it yields quantities that can in principle be measured – must be put in a form that is generally covariant: either a scalar or tensor under coordinate transformations.[9] An example is, it makes no sense to assert that your new-found law of physics depends on g^{11}, the 11 component of the inverse metric – for, in what coordinate system is this law expressed in? What happens when we use a different coordinate system to describe the outcome of some experiment designed to test this law?

Another aspect of general covariance is that, although tensor equations should hold in any coordinate system – if you suspect that two tensors quantities are actually equal, say

$$S^{i_1 i_2 \dots} = T^{i_1 i_2 \dots}, \tag{1.2.21}$$

it suffices to find one coordinate system to prove this equality. It is not necessary to prove this by using abstract indices/coordinates because, as long as the coordinate transformations are invertible, then once we have verified the equality in one system, the proof in any other follows immediately once the required transformations are specified. One common application of this observation is to apply the fact mentioned around eq. (1.2.1), that at any given point in a curved space(time), one can always choose coordinates where the metric there is flat. You will often find this "locally flat" coordinate system simplifies calculations – and perhaps even aids in gaining some intuition about the relevant physics, since the expressions usually reduce to their more familiar counterparts in flat space. A simple but important example of this brings us to

[9]You may also demand your equations/quantities to be tensors/scalars under group transformations.

the next concept: what is the curved analog of the infinitesimal volume, which we would usually write as $\mathrm{d}^D x$ in Cartesian coordinates?

Determinant of metric and the infinitesimal volume The determinant of the metric transforms as

$$\det g_{ij}(\vec{\xi}) = \det \left[g_{ab}\left(\vec{x}(\vec{\xi})\right) \frac{\partial x^a}{\partial \xi^i} \frac{\partial x^b}{\partial \xi^j} \right]. \tag{1.2.22}$$

Using the properties $\det A \cdot B = \det A \det B$ and $\det A^T = \det A$, for any two square matrices A and B,

$$\det g_{ij}(\vec{\xi}) = \left(\det \frac{\partial x^a(\vec{\xi})}{\partial \xi^b} \right)^2 \det g_{ij}\left(\vec{x}(\vec{\xi})\right). \tag{1.2.23}$$

The square root of the determinant of the metric is often denoted as $\sqrt{|g|}$. It transforms as

$$\sqrt{\left|g(\vec{\xi})\right|} = \sqrt{\left|g\left(\vec{x}(\vec{\xi})\right)\right|} \left| \det \frac{\partial x^a(\vec{\xi})}{\partial \xi^b} \right|. \tag{1.2.24}$$

We have previously noted that, given any point $\vec{x}_0$ in the curved space, we can always choose local coordinates $\{\vec{x}\}$ such that the metric there is flat. This means at $\vec{x}_0$ the infinitesimal volume of space is $\mathrm{d}^D\vec{x}$ and $\det g_{ij}(\vec{x}_0) = 1$. Recall from multi-variable calculus that, whenever we transform $\vec{x} \to \vec{x}(\vec{\xi})$, the integration measure would correspondingly transform as

$$\mathrm{d}^D\vec{x} = \mathrm{d}^D\vec{\xi} \left| \det \frac{\partial x^i}{\partial \xi^a} \right|, \tag{1.2.25}$$

where $\partial x^i/\partial \xi^a$ is the Jacobian matrix with row number i and column number a. Comparing this multi-variable calculus result to eq. (1.2.24) specialized to our metric in terms of $\{\vec{x}\}$ but evaluated at $\vec{x}_0$, we see the determinant of the Jacobian *is* in fact the square root of the determinant of the metric in some other coordinates $\vec{\xi}$,

$$\sqrt{\left|g(\vec{\xi})\right|} = \left(\sqrt{\left|g\left(\vec{x}(\vec{\xi})\right)\right|} \left| \det \frac{\partial x^i(\vec{\xi})}{\partial \xi^a} \right| \right)_{\vec{x}=\vec{x}_0} = \left| \det \frac{\partial x^i(\vec{\xi})}{\partial \xi^a} \right|_{\vec{x}=\vec{x}_0}. \tag{1.2.26}$$

In flat space and by employing Cartesian coordinates $\{\vec{x}\}$, the infinitesimal volume (at some location $\vec{x} = \vec{x}_0$) is $\mathrm{d}^D\vec{x}$. What is its curved analog? What we have just shown is that, by going from $\vec{\xi}$ to a locally flat coordinate system $\{\vec{x}\}$,

$$\mathrm{d}^D\vec{x} = \mathrm{d}^D\vec{\xi} \left| \det \frac{\partial x^i(\vec{\xi})}{\partial \xi^a} \right|_{\vec{x}=\vec{x}_0} = \mathrm{d}^D\vec{\xi}\sqrt{|g(\vec{\xi})|}. \tag{1.2.27}$$

However, since $\vec{x}_0$ was an arbitrary point in our curved space, we have argued that, in a general coordinate system $\vec{\xi}$, the infinitesimal volume is given by

$$\mathrm{d}^D\vec{\xi}\sqrt{\left|g(\vec{\xi})\right|} \equiv \mathrm{d}\xi^1 \ldots \mathrm{d}\xi^D \sqrt{\left|g(\vec{\xi})\right|}. \tag{1.2.28}$$

Problem 1.2. Upon an orientation preserving change of coordinates $\vec{y} \to \vec{y}(\vec{\xi})$, where $\det \partial y/\partial \xi > 0$, show that

$$\mathrm{d}^D \vec{y} \sqrt{|g(\vec{y})|} = \mathrm{d}^D \vec{\xi} \sqrt{\left|g(\vec{\xi})\right|}. \tag{1.2.29}$$

Therefore calling $\mathrm{d}^D \vec{x} \sqrt{|g(\vec{x})|}$ an infinitesimal volume is a generally covariant statement.

Note: $g(\vec{y})$ is the determinant of the metric written in the $\vec{y}$ coordinate system; whereas $g(\vec{\xi})$ is that of the metric written in the $\vec{\xi}$ coordinate system. The latter is *not* the same as the determinant of the metric written in the $\vec{y}$-coordinates, with $\vec{y}$ replaced with $\vec{y}(\vec{\xi})$; i.e., be careful that the determinant is not a scalar. □

Volume integrals If $\varphi(\vec{x})$ is some scalar quantity, finding its volume integral within some domain $\mathfrak{D}$ in a generally covariant way can be now carried out using the infinitesimal volume we have uncovered; it reads

$$I \equiv \int_{\mathfrak{D}} \mathrm{d}^D \vec{x} \sqrt{|g(\vec{x})|} \varphi(\vec{x}). \tag{1.2.30}$$

In other words, I is the same result no matter what coordinates we used to compute the integral on the right hand side.

Problem 1.3. Spherical coordinates in D space dimensions. In D space dimensions, we may denote the D-th unit vector as $\widehat{e}_D$; and $\widehat{n}_{D-1}$ as the unit radial vector, parametrized by the angles $\{0 \leq \theta^1 < 2\pi, 0 \leq \theta^2 \leq \pi, \dots, 0 \leq \theta^{D-2} \leq \pi\}$, in the plane perpendicular to $\widehat{e}_D$. Let $r \equiv |\vec{x}|$ and $\widehat{n}_D$ be the unit radial vector in the D space. Any vector $\vec{x}$ in this space can thus be expressed as

$$\vec{x} = r\widehat{n}\left(\vec{\theta}\right) = r\cos(\theta^{D-1})\widehat{e}_D + r\sin(\theta^{D-1})\widehat{n}_{D-1}, \qquad 0 \leq \theta^{D-1} \leq \pi. \tag{1.2.31}$$

(Can you see why this is nothing but the Gram-Schmidt process?) Just like in the 3D case, $r\cos(\theta^{D-1})$ is the projection of $\vec{x}$ along the $\widehat{e}_D$ direction; while $r\sin(\theta^{D-1})$ is that along the radial direction in the plane perpendicular to $\widehat{e}_D$.

First show that the Cartesian metric δ_{ij} in D-space transforms to

$$(\mathrm{d}\ell)^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\Omega_D^2 = \mathrm{d}r^2 + r^2\left((\mathrm{d}\theta^{D-1})^2 + (\sin\theta^{D-1})^2 \mathrm{d}\Omega_{D-1}^2\right), \tag{1.2.32}$$

where $\mathrm{d}\Omega_N^2$ is the square of the infinitesimal solid angle in N spatial dimensions, and is given by

$$\mathrm{d}\Omega_N^2 \equiv \sum_{\mathrm{I,J}=1}^{N-1} \Omega_{\mathrm{IJ}}^{(N)} \mathrm{d}\theta^{\mathrm{I}} \mathrm{d}\theta^{\mathrm{J}}, \qquad \Omega_{\mathrm{IJ}}^{(N)} \equiv \sum_{i,j=1}^{N} \delta_{ij} \frac{\partial \widehat{n}_N^i}{\partial \theta^{\mathrm{I}}} \frac{\partial \widehat{n}_N^j}{\partial \theta^{\mathrm{J}}}. \tag{1.2.33}$$

Proceed to argue that the full D-metric in spherical coordinates is

$$\mathrm{d}\ell^2 = \mathrm{d}r^2 + r^2\left((\mathrm{d}\theta^{D-1})^2 + \sum_{\mathrm{I}=2}^{D-1} s_{D-1}^2 \dots s_{D-\mathrm{I}+1}^2 (\mathrm{d}\theta^{D-\mathrm{I}})^2\right), \tag{1.2.34}$$

$$\theta^1 \in [0, 2\pi), \qquad \theta^2, \dots, \theta^{D-1} \in [0, \pi]. \tag{1.2.35}$$

(Here, $s_{\rm I} \equiv \sin\theta^{\rm I}$.) Show that the determinant of the angular metric $\Omega^{(N)}_{\rm IJ}$ obeys a recursion relation

$$\det \Omega^{(N)}_{\rm IJ} = \left(\sin\theta^{N-1}\right)^{2(N-2)} \cdot \det \Omega^{(N-1)}_{\rm IJ}. \tag{1.2.36}$$

Explain why this implies there is a recursion relation between the infinitesimal solid angle in D space and that in $(D-1)$ space. Moreover, show that the integration volume measure $\mathrm{d}^D\vec{x}$ in Cartesian coordinates then becomes, in spherical coordinates,

$$\mathrm{d}^D\vec{x} = \mathrm{d}r \cdot r^{D-1} \cdot \mathrm{d}\theta^1 \dots \mathrm{d}\theta^{D-1} \left(\sin\theta^{D-1}\right)^{D-2} \sqrt{\det \Omega^{(D-1)}_{\rm IJ}}. \tag{1.2.37}$$

□

Problem 1.4. Let x^i be Cartesian coordinates and

$$\xi^i \equiv (r, \theta, \phi) \tag{1.2.38}$$

be the usual spherical coordinates; see eq. (1.1.7). Calculate $\partial\xi^i/\partial x^a$ in terms of $\vec{\xi}$ and thereby, from the flat metric δ^{ij} in Cartesian coordinates, find the inverse metric $g^{ij}(\vec{\xi})$ in the spherical coordinate system. Hint: Compute $\partial x^i/\partial(r,\theta,\phi)^a$. How do you get $\partial(r,\theta,\phi)^a/\partial x^i$ from it? □

Symmetries (aka isometries) and infinitesimal displacements In some Cartesian coordinates $\{x^i\}$ the flat space metric is $\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$. Suppose we chose a different set of axes for new Cartesian coordinates $\{x'^i\}$, the metric will still take the same form, namely $\delta_{ij}\mathrm{d}x'^i\mathrm{d}x'^j$. Likewise, on a 2-sphere the metric is $\mathrm{d}\theta^2 + (\sin\theta)^2\mathrm{d}\phi^2$ with a given choice of axes for the 3D space the sphere is embedded in; upon any rotation to a new axis, so the new angles are now (θ', ϕ'), the 2-sphere metric is still of the same form $\mathrm{d}\theta'^2 + (\sin\theta')^2\mathrm{d}\phi'^2$. All we have to do, in both cases, is swap the symbols $\vec{x} \to \vec{x}'$ and $(\theta,\phi) \to (\theta',\phi')$. The reason why we can simply swap symbols to express the same geometry in different coordinate systems, is because of the symmetries present: for flat space and the 2-sphere, the geometries are respectively indistinguishable under translation/rotation and rotation about its center.

Motivated by this observation that geometries enjoying symmetries (aka isometries) retain their *form* under an active coordinate transformation – one that corresponds to an actual displacement from one location to another[10] – we now consider a infinitesimal coordinate transformation as follows. Starting from $\vec{x}$, we define a new set of coordinates $\vec{x}'$ through an infinitesimal vector $\vec{\xi}(\vec{x})$,

$$\vec{x}' \equiv \vec{x} - \vec{\xi}(\vec{x}). \tag{1.2.39}$$

(The $-$ sign is for technical convenience.) We shall interpret this definition as an active coordinate transformation – given some location $\vec{x}$, we now move to a point $\vec{x}'$ that is displaced infinitesimally far away, with the displacement itself described by $-\vec{\xi}(\vec{x})$. On the other hand, since $\vec{\xi}$ is assumed to be "small," we may replace in the above equation, $\vec{\xi}(\vec{x})$ with $\vec{\xi}(\vec{x}') \equiv \vec{\xi}(\vec{x} \to \vec{x}')$. This is because the error incurred would be of $\mathcal{O}(\xi^2)$.

$$\vec{x} = \vec{x}' + \vec{\xi}(\vec{x}') + \mathcal{O}(\xi^2) \qquad \Rightarrow \qquad \frac{\partial x^i}{\partial x'^a} = \delta^i_a + \partial_{a'}\xi^i(\vec{x}') + \mathcal{O}(\xi\partial\xi) \tag{1.2.40}$$

[10] As opposed to a passive coordinate transformation, which is one where a different set of coordinates are used to describe the same location in the geometry.

How does this change our metric?

$$
\begin{aligned}
g_{ij}(\vec{x})\,\mathrm{d}x^i\mathrm{d}x^j &= g_{ij}\left(\vec{x}' + \vec{\xi}(\vec{x}') + \dots\right)\left(\delta^i_a + \partial_{a'}\xi^i + \dots\right)\left(\delta^j_b + \partial_{b'}\xi^j + \dots\right)\mathrm{d}x'^a\mathrm{d}x'^b \\
&= \left(g_{ij}(\vec{x}') + \xi^c\partial_{c'}g_{ij}(\vec{x}') + \dots\right)\left(\delta^i_a + \partial_{a'}\xi^i + \dots\right)\left(\delta^j_b + \partial_{b'}\xi^j + \dots\right)\mathrm{d}x'^a\mathrm{d}x'^b \\
&= \left(g_{ij}(\vec{x}') + \delta_\xi g_{ij}(\vec{x}') + \mathcal{O}(\xi^2)\right)\mathrm{d}x'^i\mathrm{d}x'^j, \qquad (1.2.41)
\end{aligned}
$$

where

$$
\delta_\xi g_{ij}(\vec{x}') \equiv \xi^c(\vec{x}')\frac{\partial g_{ij}(\vec{x}')}{\partial x'^c} + g_{ia}(\vec{x}')\frac{\partial \xi^a(\vec{x}')}{\partial x'^j} + g_{ja}(\vec{x}')\frac{\partial \xi^a(\vec{x}')}{\partial x'^i}. \qquad (1.2.42)
$$

[11]At this point, we see that if the geometry enjoys a symmetry along the entire curve whose tangent vector is $\vec{\xi}$, then it must retain its form $g_{ij}(\vec{x})\mathrm{d}x^i\mathrm{d}x^j = g_{ij}(\vec{x}')\mathrm{d}x'^i\mathrm{d}x'^j$ and therefore,[12] equations

$$
\delta_\xi g_{ij} = 0, \qquad \text{(isometry along } \vec{\xi}\text{)}. \qquad (1.2.43)
$$

[13]Conversely, if $\delta_\xi g_{ij} = 0$ everywhere in space, then starting from some point $\vec{x}$, we can make incremental displacements along the curve whose tangent vector is $\vec{\xi}$, and therefore find that the metric retain its form along its entirety. Now, a vector $\vec{\xi}$ that satisfies $\delta_\xi g_{ij} = 0$ is called a Killing vector. We may then summarize:

> A geometry enjoys an isometry along $\vec{\xi}$ if and only if $\vec{\xi}$ is a Killing vector satisfying eq. (1.2.43) everywhere in space.

Problem 1.5. Can you justify the statement: "If the metric g_{ij} is independent of one of the coordinates, say x^k, then ∂_k is a Killing vector of the geometry"? □

Orthonormal frame So far, we have been writing tensors in the coordinate basis – the basis vectors of our tensors are formed out of tensor products of $\{\mathrm{d}x^i\}$ and $\{\partial_i\}$. To interpret components of tensors, however, we need them written in an orthonormal basis. This amounts to using a uniform set of measuring sticks on all axes, i.e., a local set of (non-coordinate) Cartesian axes where one "tick mark" on each axis translates to the same length.

As an example, suppose we wish to describe some fluid's velocity $v^x\partial_x + v^y\partial_y$ on a 2 dimensional flat space. In Cartesian coordinates $v^x(x,y)$ and $v^y(x,y)$ describe the velocity at some point $\vec{\xi} = (x,y)$ flowing in the x- and y-directions respectively. Suppose we used polar coordinates, however,

$$
\xi^i = r(\cos\phi, \sin\phi). \qquad (1.2.44)
$$

[11]A point of clarification might be helpful. In eq. (1.2.41), we are not merely asking "What is $\mathrm{d}\ell^2$ at $\vec{x}'$?" The answer to that question would be $(\mathrm{d}\ell)^2_{\vec{x}'} = g_{ij}(\vec{x} - \vec{\xi}(\vec{x}))\mathrm{d}x^i\mathrm{d}x^j$, with no need to transform the $\mathrm{d}x^i$. Rather, here, we are performing a coordinate transformation from $\vec{x}$ to $\vec{x}'$, *induced* by an infinitesimal displacement via $\vec{x}' = \vec{x} - \vec{\xi}(\vec{x}) \Leftrightarrow \vec{x} = \vec{x}' + \vec{\xi}(\vec{x}') + \dots$ – where $\vec{x}$ and $\vec{x}'$ are infinitesimally separated. An elementary example would be to rotate the 2−sphere about the z−axis, so $\theta = \theta'$ but $\phi = \phi' + \epsilon$ for infinitesimal ϵ. Then, $\xi^i\partial_i = \epsilon\partial_\phi$.

[12]We reiterate, by the same *form*, we mean $g_{ij}(\vec{x})$ and $g_{ij}(\vec{x}')$ are the same functions if we treat $\vec{x}$ and $\vec{x}'$ as dummy variables. For example, $g_{33}(r,\theta) = (r\sin\theta)^2$ and $g_{3'3'}(r',\theta') = (r'\sin\theta')^2$ in the 2-sphere metric.

[13]$\delta_\xi g_{ij}$ is known as the Lie derivative of the metric along ξ, and is commonly denoted as $(\pounds_\xi g)_{ij}$.

The metric would read

$$(\mathrm{d}\ell)^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\phi^2. \tag{1.2.45}$$

The velocity now reads $v^r(\vec{\xi})\partial_r + v^\phi(\vec{\xi})\partial_\phi$, where $v^r(\vec{\xi})$ has an interpretation of "rate of flow in the radial direction". However, notice the dimensions of the v^ϕ is not even the same as that of v^r; if v^r were of [Length/Time], then v^ϕ is of [1/Time]. At this point we recall – just as $\mathrm{d}r$ (which is dual to ∂_r) can be interpreted as an infinitesimal length in the radial direction, the arc length $r\mathrm{d}\phi$ (which is dual to $(1/r)\partial_\phi$) is the corresponding one in the perpendicular azimuthal direction. Using these as a guide, we would now express the velocity at $\vec{\xi}$ as

$$v = v^r \frac{\partial}{\partial r} + (r \cdot v^\phi) \left(\frac{1}{r} \frac{\partial}{\partial \phi} \right), \tag{1.2.46}$$

so that now $v^{\widehat{\phi}} \equiv r \cdot v^\phi$ may be interpreted as the velocity in the azimuthal direction.

More formally, given a (real, symmetric) metric g_{ij} we may always find a orthogonal transformation $O^a{}_i$ that diagonalizes it; and by absorbing into this transformation the eigenvalues of the metric, the orthonormal frame fields emerge:

$$\begin{aligned} g_{ij}\mathrm{d}x^i\mathrm{d}x^j &= \sum_{a,b} \left(O^a{}_i \cdot \lambda_a \delta_{ab} \cdot O^b{}_j \right) \mathrm{d}x^i \mathrm{d}x^j \\ &= \sum_{a,b} \left(\sqrt{\lambda_a} O^a{}_i \cdot \delta_{ab} \cdot \sqrt{\lambda_b} O^b{}_j \right) \mathrm{d}x^i \mathrm{d}x^j \\ &= \left(\delta_{ab} \varepsilon^{\widehat{a}}{}_i \varepsilon^{\widehat{b}}{}_j \right) \mathrm{d}x^i \mathrm{d}x^j = \delta_{ab} \left(\varepsilon^{\widehat{a}}{}_i \mathrm{d}x^i \right) \left(\varepsilon^{\widehat{b}}{}_j \mathrm{d}x^j \right), \end{aligned} \tag{1.2.47}$$

$$\varepsilon^{\widehat{a}}{}_i \equiv \sqrt{\lambda_a} O^a{}_i, \qquad \text{(no sum over } a\text{).} \tag{1.2.48}$$

In the first equality, we have exploited the fact that any real symmetric matrix g_{ij} can be diagonalized by an appropriate orthogonal matrix $O^a{}_i$, with real eigenvalues $\{\lambda_a\}$; in the second we have exploited the assumption that we are working in Riemannian spaces, where all eigenvalues of the metric are positive,[14] to take the positive square roots of the eigenvalues; in the third we have defined the orthonormal frame vector fields as $\varepsilon^{\widehat{a}}{}_i = \sqrt{\lambda_a} O^a{}_i$, with no sum over a. Finally, from eq. (1.2.47) and by defining the infinitesimal lengths $\varepsilon^{\widehat{a}} \equiv \varepsilon^{\widehat{a}}{}_i \mathrm{d}x^i$, we arrive at the following curved space parallel to Pythagoras' theorem in flat space:

$$(\mathrm{d}\ell)^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j = \left(\varepsilon^{\widehat{1}} \right)^2 + \left(\varepsilon^{\widehat{2}} \right)^2 + \dots + \left(\varepsilon^{\widehat{D}} \right)^2. \tag{1.2.49}$$

The metric components are now

$$g_{ij} = \delta_{ab} \varepsilon^{\widehat{a}}{}_i \varepsilon^{\widehat{b}}{}_j. \tag{1.2.50}$$

Whereas the metric determinant reads

$$\det g_{ij} = \left(\det \varepsilon^{\widehat{a}}{}_i \right)^2. \tag{1.2.51}$$

[14] As opposed to semi-Riemannian/Lorentzian spaces, where the eigenvalue associated with the 'time' direction has a different sign from the rest.

We say the metric on the right hand side of eq. (1.2.47) is written in an orthonormal frame, because in this basis $\{\varepsilon^{\widehat{a}}{}_i \mathrm{d}x^i | a = 1, 2, \dots, D\}$, the metric components are identical to the flat Cartesian ones. We have put a $\widehat{\cdot}$ over the a-index, to distinguish from the i-index, because the latter transforms as a tensor

$$\varepsilon^{\widehat{a}}{}_i(\vec{\xi}) = \varepsilon^{\widehat{a}}{}_j \left(\vec{x}(\vec{\xi})\right) \frac{\partial x^j(\vec{\xi})}{\partial \xi^i}. \tag{1.2.52}$$

This also implies the i-index can be moved using the metric; for example

$$\varepsilon^{\widehat{a}i}(\vec{x}) \equiv g^{ij}(\vec{x}) \varepsilon^{\widehat{a}}{}_j(\vec{x}). \tag{1.2.53}$$

The $\widehat{a}$ index does not transform under coordinate transformations. But it can be rotated by an orthogonal matrix $R^{\widehat{a}}{}_{\widehat{b}}(\vec{\xi})$, which itself can depend on the space coordinates, while keeping the metric in eq. (1.2.47) the same object. By orthogonal matrix, we mean any R that obeys

$$\widehat{R}^{\widehat{a}}{}_{\widehat{c}} \delta_{ab} \widehat{R}^{\widehat{b}}{}_{\widehat{f}} = \delta_{cf} \tag{1.2.54}$$

$$\widehat{R}^T \widehat{R} = \mathbb{I}. \tag{1.2.55}$$

Upon the replacement

$$\varepsilon^{\widehat{a}}{}_i(\vec{x}) \to \widehat{R}^{\widehat{a}}{}_{\widehat{b}}(\vec{x}) \varepsilon^{\widehat{b}}{}_i(\vec{x}), \tag{1.2.56}$$

we have

$$g_{ij} \mathrm{d}x^i \mathrm{d}x^j \to \left(\delta_{ab} \widehat{R}^{\widehat{a}}{}_{\widehat{c}} \widehat{R}^{\widehat{a}}{}_{\widehat{f}}\right) \varepsilon^{\widehat{c}}{}_i \varepsilon^{\widehat{f}}{}_j \mathrm{d}x^i \mathrm{d}x^j = g_{ij} \mathrm{d}x^i \mathrm{d}x^j. \tag{1.2.57}$$

The interpretation of eq. (1.2.56) is that the choice of local Cartesian-like (non-coordinate) axes are not unique; just as the Cartesian coordinate system in flat space can be redefined through a rotation R obeying $R^T R = \mathbb{I}$, these local axes can also be rotated freely. It is a consequence of this O_D symmetry that upper and lower orthonormal frame indices actually transform the same way. We begin by demanding that rank-1 tensors in an orthonormal frame transform as

$$V^{\widehat{a}'} = \widehat{R}^{\widehat{a}}{}_{\widehat{c}} V^{\widehat{c}}, \qquad V_{\widehat{a}'} = (\widehat{R}^{-1})^{\widehat{f}}{}_{\widehat{a}} V_{\widehat{f}} \tag{1.2.58}$$

so that

$$V^{\widehat{a}'} V_{\widehat{a}'} = V^{\widehat{a}} V_{\widehat{a}}. \tag{1.2.59}$$

But $\widehat{R}^T \widehat{R} = \mathbb{I}$ means $\widehat{R}^{-1} = \widehat{R}^T$ and thus the ath row and cth column of the inverse, namely $(\widehat{R}^{-1})^{\widehat{a}}{}_{\widehat{c}}$, is equal to the cth row and ath column of $\widehat{R}$ itself: $(\widehat{R}^{-1})^{\widehat{a}}{}_{\widehat{c}} = \widehat{R}^{\widehat{c}}{}_{\widehat{a}}$.

$$V_{\widehat{a}'} = \sum_f \widehat{R}^{\widehat{a}}{}_{\widehat{f}} V_{\widehat{f}}. \tag{1.2.60}$$

In other words, $V_{\widehat{a}}$ transforms just like $V^{\widehat{a}}$.

To sum, we have shown that the orthonormal frame index is moved by the Kronecker delta; $V^{\widehat{a}'} = V_{\widehat{a}'}$ for any vector written in an orthonormal frame, and in particular,

$$\varepsilon^{\widehat{a}}{}_i(\vec{x}) = \delta^{ab}\varepsilon_{\widehat{b}i}(\vec{x}) = \varepsilon_{\widehat{a}i}(\vec{x}). \tag{1.2.61}$$

Next, we also demonstrate that these vector fields are indeed of unit length.

$$\varepsilon^{\widehat{f}}{}_j\varepsilon^{\widehat{b}j} = \varepsilon^{\widehat{f}}{}_j\varepsilon^{\widehat{b}}{}_k g^{jk} = \delta^{fb}, \tag{1.2.62}$$

$$\varepsilon_{\widehat{f}}{}^j\varepsilon_{\widehat{b}j} = \varepsilon_{\widehat{f}}{}^j\varepsilon_{\widehat{b}}{}^k g_{jk} = \delta_{fb}. \tag{1.2.63}$$

To understand this we begin with the diagonalization of the metric, $\delta_{cf}\varepsilon^{\widehat{c}}{}_i\varepsilon^{\widehat{f}}{}_j = g_{ij}$. Contracting both sides with the orthonormal frame vector $\varepsilon^{\widehat{b}j}$,

$$\delta_{cf}\varepsilon^{\widehat{c}}{}_i\varepsilon^{\widehat{f}}{}_j\varepsilon^{\widehat{b}j} = \varepsilon^{\widehat{b}}{}_i, \tag{1.2.64}$$

$$(\varepsilon^{\widehat{b}j}\varepsilon_{\widehat{f}j})\varepsilon^{\widehat{f}}{}_i = \varepsilon^{\widehat{b}}{}_i. \tag{1.2.65}$$

If we let M denote the matrix $M^b{}_f \equiv (\varepsilon^{\widehat{b}j}\varepsilon_{\widehat{f}j})$, then we have $i = 1, 2, \dots, D$ matrix equations $M \cdot \varepsilon_i = \varepsilon_i$. As long as the determinant of g_{ab} is non-zero, then $\{\varepsilon_i\}$ are linearly independent vectors spanning $\mathbb{R}^D$ (see eq. (1.2.51)). Since every ε_i is an eigenvector of M with eigenvalue one, that means $M = \mathbb{I}$, and we have proved eq. (1.2.62).

To summarize,

$$g_{ij} = \delta_{ab}\varepsilon^{\widehat{a}}{}_i\varepsilon^{\widehat{b}}{}_j, \qquad g^{ij} = \delta^{ab}\varepsilon_{\widehat{a}}{}^i\varepsilon_{\widehat{b}}{}^j,$$
$$\delta_{ab} = g_{ij}\varepsilon_{\widehat{a}}{}^i\varepsilon_{\widehat{b}}{}^j, \qquad \delta^{ab} = g^{ij}\varepsilon^{\widehat{a}}{}_i\varepsilon^{\widehat{b}}{}_j. \tag{1.2.66}$$

Now, any tensor with written in a coordinate basis can be converted to one in an orthonormal basis by contracting with the orthonormal frame fields $\varepsilon^{\widehat{a}}{}_i$ in eq. (1.2.47). For example, the velocity field in an orthonormal frame is

$$v^{\widehat{a}} = \varepsilon^{\widehat{a}}{}_i v^i. \tag{1.2.67}$$

For the two dimension example above,

$$(\mathrm{d}r)^2 + (r\mathrm{d}\phi)^2 = \delta_{rr}(\mathrm{d}r)^2 + \delta_{\phi\phi}(r\mathrm{d}\phi)^2, \tag{1.2.68}$$

allowing us to read off the only non-zero components of the orthonormal frame fields are

$$\varepsilon^{\widehat{r}}{}_r = 1, \qquad \varepsilon^{\widehat{\phi}}{}_\phi = r; \tag{1.2.69}$$

which in turn implies

$$v^{\widehat{r}} = \varepsilon^{\widehat{r}}{}_r v^r = v^r, \qquad v^{\widehat{\phi}} = \varepsilon^{\widehat{\phi}}{}_\phi v^\phi = r\ v^\phi. \tag{1.2.70}$$

More generally, what we are doing here is really switching from writing the same tensor in coordinates basis $\{\mathrm{d}x^i\}$ and $\{\partial_i\}$ to an orthonormal basis $\{\varepsilon^{\widehat{a}}{}_i \mathrm{d}x^i\}$ and $\{\varepsilon_{\widehat{a}}{}^i \partial_i\}$. For example,

$$T_{ijk}{}^{l} \left\langle \mathrm{d}x^i \right| \otimes \left\langle \mathrm{d}x^j \right| \otimes \left\langle \mathrm{d}x^k \right| \otimes \left| \partial_l \right\rangle = T_{\widehat{i}\widehat{j}\widehat{k}}{}^{\widehat{l}} \left\langle \varepsilon^{\widehat{i}} \right| \otimes \left\langle \varepsilon^{\widehat{j}} \right| \otimes \left\langle \varepsilon^{\widehat{k}} \right| \otimes \left| \varepsilon_{\widehat{l}} \right\rangle \tag{1.2.71}$$

$$\varepsilon^{\widehat{i}} \equiv \varepsilon^{\widehat{i}}{}_a \mathrm{d}x^a \qquad\qquad \varepsilon_{\widehat{i}} \equiv \varepsilon_{\widehat{i}}{}^a \partial_a. \tag{1.2.72}$$

Even though the physical dimension of the whole tensor $[T]$ is necessarily consistent, because the $\{\mathrm{d}x^i\}$ and $\{\partial_i\}$ do not have the same dimensions – compare, for e.g., $\mathrm{d}r$ versus $\mathrm{d}\theta$ in spherical coordinates – the components of tensors in a coordinate basis do not all have the same dimensions, making their interpretation difficult. By using orthonormal frame fields as defined in eq. (1.2.72), we see that

$$\sum_a \left(\varepsilon^{\widehat{a}}\right)^2 = \delta_{ab} \varepsilon^{\widehat{a}}{}_i \varepsilon^{\widehat{b}}{}_j \mathrm{d}x^i \mathrm{d}x^j = g_{ij} \mathrm{d}x^i \mathrm{d}x^j \tag{1.2.73}$$

$$\left[\varepsilon^{\widehat{a}}\right] = \text{Length}; \tag{1.2.74}$$

and

$$\sum_a \left(\varepsilon_{\widehat{a}}\right)^2 = \delta^{ab} \varepsilon_{\widehat{a}}{}^i \varepsilon_{\widehat{b}}{}^j \partial_i \partial_j = g^{ij} \partial_i \partial_j \tag{1.2.75}$$

$$\left[\varepsilon_{\widehat{a}}\right] = 1/\text{Length}; \tag{1.2.76}$$

which in turn implies, for instance, the consistency of the physical dimensions of the orthonormal components $T_{\widehat{i}\widehat{j}\widehat{k}}{}^{\widehat{l}}$ in eq. (1.2.71):

$$[T_{\widehat{i}\widehat{j}\widehat{k}}{}^{\widehat{l}}][\varepsilon^{\widehat{i}}]^3[\varepsilon_{\widehat{l}}] = [T], \tag{1.2.77}$$

$$\left[T_{\widehat{i}\widehat{j}\widehat{k}}{}^{\widehat{l}}\right] = \frac{[T]}{\text{Length}^2}. \tag{1.2.78}$$

Problem 1.6. Find the orthonormal frame fields $\{\varepsilon^{\widehat{a}}{}_i\}$ in 3-dimensional Cartesian, Spherical and Cylindrical coordinate systems. Hint: Just like the 2D case above, by packaging the metric $g_{ij}\mathrm{d}x^i \mathrm{d}x^j$ appropriately, you can read off the frame fields without further work. □

(Curved) Dot Product So far we have viewed the metric $(\mathrm{d}\ell)^2$ as the square of the distance between $\vec{x}$ and $\vec{x}+\mathrm{d}\vec{x}$, generalizing Pythagoras' theorem in flat space. The generalization of the dot product between two (tangent) vectors U and V at some location $\vec{x}$ is

$$U(\vec{x}) \cdot V(\vec{x}) \equiv g_{ij}(\vec{x}) U^i(\vec{x}) V^j(\vec{x}). \tag{1.2.79}$$

That this is in fact the analogy of the dot product in Euclidean space can be readily seen by going to the orthonormal frame:

$$U(\vec{x}) \cdot V(\vec{x}) = \delta_{ij} U^{\widehat{i}}(\vec{x}) V^{\widehat{j}}(\vec{x}). \tag{1.2.80}$$

Line integral The line integral that occurs in 3D vector calculus, is commonly written as $\int \vec{A} \cdot \mathrm{d}\vec{x}$. While the dot product notation is very convenient and oftentimes quite intuitive, there is an implicit assumption that the underlying coordinate system is Cartesian in flat space. The integrand that actually transforms covariantly is the tensor $A_i \mathrm{d}x^i$, where the $\{x^i\}$ are no longer necessarily Cartesian. The line integral itself then consists of integrating this over a prescribed path $\vec{x}(\lambda_1 \leq \lambda \leq \lambda_2)$, namely

$$\int_{\vec{x}(\lambda_1 \leq \lambda \leq \lambda_2)} A_i \mathrm{d}x^i = \int_{\lambda_1}^{\lambda_2} A_i\left(\vec{x}(\lambda)\right) \frac{\mathrm{d}x^i(\lambda)}{\mathrm{d}\lambda} \mathrm{d}\lambda. \tag{1.2.81}$$

1.3 Covariant derivatives, Parallel Transport, Levi-Civita, Hodge Dual

Covariant Derivative How do we take derivatives of tensors in such a way that we get back a tensor in return? To start, let us see that the partial derivative of a tensor is not a tensor. Consider

$$\begin{aligned} \frac{\partial T_j(\vec{\xi})}{\partial \xi^i} &= \frac{\partial x^a}{\partial \xi^i} \frac{\partial}{\partial x^a}\left(T_b\left(\vec{x}(\vec{\xi})\right) \frac{\partial x^b}{\partial \xi^j}\right) \\ &= \frac{\partial x^a}{\partial \xi^i} \frac{\partial x^b}{\partial \xi^j} \frac{\partial T_b\left(\vec{x}(\vec{\xi})\right)}{\partial x^a} + \frac{\partial^2 x^b}{\partial \xi^j \partial \xi^i} T_b\left(\vec{x}(\vec{\xi})\right). \end{aligned} \tag{1.3.1}$$

The second derivative $\partial^2 x^b / \partial \xi^i \partial \xi^j$ term is what spoils the coordinate transformation rule we desire. To fix this, we introduce the concept of the covariant derivative ∇, which is built out of the partial derivative and the Christoffel symbols $\Gamma^i{}_{jk}$, which in turn is built out of the metric tensor,

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{il}\left(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}\right). \tag{1.3.2}$$

Notice the Christoffel symbol is symmetric in its lower indices: $\Gamma^i{}_{jk} = \Gamma^i{}_{kj}$.

For a scalar φ the covariant derivative is just the partial derivative

$$\nabla_i \varphi = \partial_i \varphi. \tag{1.3.3}$$

For a $\binom{0}{1}$ or $\binom{1}{0}$ tensor, its covariant derivative reads

$$\nabla_i T_j = \partial_i T_j - \Gamma^l{}_{ij} T_l, \tag{1.3.4}$$

$$\nabla_i T^j = \partial_i T^j + \Gamma^j{}_{il} T^l. \tag{1.3.5}$$

Under $\vec{x} \to \vec{x}(\vec{\xi})$, we have,

$$\nabla_{\xi^i} \varphi(\vec{\xi}) = \frac{\partial x^a}{\partial \xi^i} \nabla_{x^a} \varphi\left(\vec{x}(\vec{\xi})\right) \tag{1.3.6}$$

$$\nabla_{\xi^i} T_j(\xi) = \frac{\partial x^a}{\partial \xi^i} \frac{\partial x^b}{\partial \xi^j} \nabla_{x^a} T_b\left(\vec{x}(\vec{\xi})\right). \tag{1.3.7}$$

For a general $\binom{N}{M}$ tensor, we have

$$\begin{aligned}\nabla_k T^{i_1 i_2 \dots i_N}{}_{j_i j_2 \dots j_M} &= \partial_k T^{i_1 i_2 \dots i_N}{}_{j_i j_2 \dots j_M} \\ &+ \Gamma^{i_1}{}_{kl} T^{l i_2 \dots i_N}{}_{j_i j_2 \dots j_M} + \Gamma^{i_2}{}_{kl} T^{i_1 l \dots i_N}{}_{j_i j_2 \dots j_M} + \dots + \Gamma^{i_N}{}_{kl} T^{i_1 \dots i_{N-1} l}{}_{j_i j_2 \dots j_M} \\ &- \Gamma^{l}{}_{k j_1} T^{i_1 \dots i_N}{}_{l j_2 \dots j_M} - \Gamma^{l}{}_{k j_2} T^{i_1 \dots i_N}{}_{j_1 l \dots j_M} - \dots - \Gamma^{l}{}_{k j_M} T^{i_1 \dots i_N}{}_{j_1 \dots j_{M-1} l}.\end{aligned} \tag{1.3.8}$$

[15]By using eq. (1.3.1) we may infer how the Christoffel symbols themselves must transform – they are not tensors. Firstly,

$$\begin{aligned}\nabla_{\xi^i} T_j(\vec{\xi}) &= \partial_{\xi^i} T_j(\vec{\xi}) - \Gamma^l{}_{ij}(\vec{\xi}) T_l(\vec{\xi}) \\ &= \frac{\partial x^a}{\partial \xi^i} \frac{\partial x^b}{\partial \xi^j} \partial_{x^a} T_b\left(\vec{x}(\vec{\xi})\right) + \left(\frac{\partial^2 x^b}{\partial \xi^j \partial \xi^i} - \Gamma^l{}_{ij}(\vec{\xi}) \frac{\partial x^b(\vec{\xi})}{\partial \xi^l} \right) T_b\left(\vec{x}(\vec{\xi})\right)\end{aligned} \tag{1.3.9}$$

On the other hand,

$$\begin{aligned}\nabla_{\xi^i} T_j(\vec{\xi}) &= \frac{\partial x^a}{\partial \xi^i} \frac{\partial x^b}{\partial \xi^j} \nabla_{x^a} T_b\left(\vec{x}(\vec{\xi})\right) \\ &= \frac{\partial x^a}{\partial \xi^i} \frac{\partial x^b}{\partial \xi^j} \left\{ \partial_{x^a} T_b\left(\vec{x}(\vec{\xi})\right) - \Gamma^l{}_{ab}\left(\vec{x}(\vec{\xi})\right) T_l\left(\vec{x}(\vec{\xi})\right) \right\}\end{aligned} \tag{1.3.10}$$

Comparing equations (1.3.9) and (1.3.10) leads us to relate the Christoffel symbol written in $\vec{\xi}$ coordinates $\Gamma^l{}_{ij}(\vec{\xi})$ and that written in $\vec{x}$ coordinates $\Gamma^l{}_{ij}(\vec{x})$.

$$\Gamma^l{}_{ij}(\vec{\xi}) = \Gamma^k{}_{mn}\left(\vec{x}(\vec{\xi})\right) \frac{\partial \xi^l}{\partial x^k(\vec{\xi})} \frac{\partial x^m(\vec{\xi})}{\partial \xi^i} \frac{\partial x^n(\vec{\xi})}{\partial \xi^j} + \frac{\partial \xi^l}{\partial x^k(\vec{\xi})} \frac{\partial^2 x^k(\vec{\xi})}{\partial \xi^j \partial \xi^i}. \tag{1.3.11}$$

On the right hand side, all $\vec{x}$ have been replaced with $\vec{x}(\vec{\xi})$.[16]

The covariant derivative, like its partial derivative counterpart, obeys the product rule. Suppressing the indices, if T_1 and T_2 are both tensors, we have

$$\nabla(T_1 T_2) = (\nabla T_1) T_2 + T_1 (\nabla T_2). \tag{1.3.12}$$

As you will see below, the metric is parallel transported in all directions,

$$\nabla_i g_{jk} = \nabla_i g^{jk} = 0. \tag{1.3.13}$$

Combined with the product rule in eq. (1.3.12), this means when raising and lowering of indices of a covariant derivative of a tensor, the metric may be passed in and out of the ∇. For example,

$$\begin{aligned}g_{ia} \nabla_j T^{kal} &= \nabla_j g_{ia} \cdot T^{kal} + g_{ia} \nabla_j T^{kal} = \nabla_j (g_{ia} T^{kal}) \\ &= \nabla_j T^k{}_i{}^l.\end{aligned} \tag{1.3.14}$$

[15]The semi-colon is sometimes also used to denote the covariant derivative. For example, $\nabla_l \nabla_i T^{jk} \equiv T^{jk}{}_{;il}$.

[16]We note in passing that in gauge theory – which encompasses humanity's current description of the non-gravitational forces (electromagnetic-weak $(\mathrm{SU}_2)_{\text{left-handed fermions}} \times (U_1)_{\text{hypercharge}}$ and strong nuclear $(\mathrm{SU}_3)_{\text{color}}$) – the fundamental fields there $\{A^b{}_\mu\}$ transforms (in a group theory sense) in a very similar fashion as the Christoffel symbols do (under a coordinate transformation) in eq. (1.3.11).

Remark I have introduced the Christoffel symbol here by showing how it allows us to define a derivative operator on a tensor that returns a tensor. I should mention here that, alternatively, it is also possible to view $\Gamma^i{}_{jk}$ as "rotation matrices," describing the failure of parallel transporting the basis bras $\{\langle \mathrm{d}x^i|\}$ and kets $\{|\partial_i\rangle\}$ as they are moved from one point in space to a neighboring point infinitesimally far away. Specifically,

$$\nabla_i \left\langle \mathrm{d}x^j \right| = -\Gamma^j{}_{ik} \left\langle \mathrm{d}x^k \right| \qquad \text{and} \qquad \nabla_i \left|\partial_j\right\rangle = \Gamma^l{}_{ij} \left|\partial_l\right\rangle . \tag{1.3.15}$$

Within this perspective, the tensor components are scalars. The product rule then yields, for instance,

$$\begin{aligned}\nabla_i \left(V_a \left\langle \mathrm{d}x^a\right|\right) &= \left(\nabla_i V_a\right) \left\langle \mathrm{d}x^a\right| + V_a \nabla_i \left\langle \mathrm{d}x^a\right| \\ &= \left(\partial_i V_j - V_a \Gamma^a{}_{ij}\right) \left\langle \mathrm{d}x^j\right| .\end{aligned} \tag{1.3.16}$$

Riemann and Ricci tensors I will not use them very much in the rest of our discussion in this section (§(1)), but I should still highlight that the Riemann and Ricci tensors are fundamental to describing curvature. The Riemann tensor is built out of the Christoffel symbols via

$$R^i{}_{jkl} = \partial_k \Gamma^i{}_{lj} - \partial_l \Gamma^i{}_{kj} + \Gamma^i{}_{sk}\Gamma^s{}_{lj} - \Gamma^i{}_{sl}\Gamma^s{}_{kj}. \tag{1.3.17}$$

The failure of parallel transport of some vector V^i around an infinitesimally small loop, is characterized by

$$[\nabla_k, \nabla_l] V^i \equiv (\nabla_k \nabla_l - \nabla_l \nabla_k) V^i = R^i{}_{jkl} V^j, \tag{1.3.18}$$

$$[\nabla_k, \nabla_l] V_j \equiv (\nabla_k \nabla_l - \nabla_l \nabla_k) V_j = -R^i{}_{jkl} V_i. \tag{1.3.19}$$

The generalization to higher rank tensors is

$$\begin{aligned}[\nabla_i, \nabla_j] T^{k_1\dots k_N}{}_{l_1\dots l_M} &= R^{k_1}{}_{aij} T^{a k_2 \dots k_N}{}_{l_1\dots l_M} + R^{k_2}{}_{aij} T^{k_1 a k_3 \dots k_N}{}_{l_1\dots l_M} + \dots + R^{k_N}{}_{aij} T^{k_1\dots k_{N-1} a}{}_{l_1\dots l_M} \\ &\quad - R^a{}_{l_1 ij} T^{k_1\dots k_N}{}_{a l_2 \dots l_M} - R^a{}_{l_2 ij} T^{k_1\dots k_N}{}_{l_1 a l_3 \dots l_M} - \dots - R^a{}_{l_M ij} T^{k_1\dots k_N}{}_{l_1\dots l_{M-1} a}.\end{aligned} \tag{1.3.20}$$

The Riemann tensor obeys the following symmetries.

$$R_{ijab} = R_{abij}, \qquad R_{ijab} = -R_{jiab}, \qquad R_{abij} = -R_{abji}. \tag{1.3.21}$$

The Riemann tensor also obeys the Bianchi identities[17]

$$R^i{}_{[jkl]} = \nabla_{[i} R^{jk}{}_{lm]} = 0. \tag{1.3.22}$$

In D dimensions, the Riemann tensor has $D^2(D^2-1)/12$ algebraically independent components. In particular, in $D=1$ dimension, space is always flat because $R_{1111} = -R_{1111} = 0$.

[17]The symbol $[\dots]$ means the indices within it are fully anti-symmetrized; in particular, $T_{[ijk]} = T_{ijk} - T_{ikj} - T_{jik} + T_{jki} - T_{kji} + T_{kij}$. We will have more to say about this operation later on.

The Ricci tensor is defined as the non-trivial contraction of a pair of the Riemann tensor's indices.

$$R_{jl} \equiv R^{i}{}_{jil}. \tag{1.3.23}$$

It is symmetric

$$R_{ij} = R_{ji}. \tag{1.3.24}$$

Finally the Ricci scalar results from a contraction of the Ricci tensor's indices.

$$\mathcal{R} \equiv g^{jl} R_{jl}. \tag{1.3.25}$$

Contracting eq. (1.3.22) appropriately yields the Bianchi identities involving the Ricci tensor and scalar

$$\nabla^i \left(R_{ij} - \frac{g_{ij}}{2} \mathcal{R} \right) = 0. \tag{1.3.26}$$

This is a good place to pause and state, the Christoffel symbols in eq. (1.3.2), covariant derivatives, and the Riemann/Ricci tensors, etc., are in general very tedious to compute. If you ever have to do so on a regular basis, say for research, I highly recommend familiarizing yourself with one of the various software packages available that could do them for you.

Geodesics Recall the distance integral in eq. (1.1.22). If you wish to determine the shortest path (aka geodesic) between some given pair of points $\vec{x}_1$ and $\vec{x}_2$, you will need to minimize eq. (1.1.22). This is a "calculus of variation" problem. The argument runs as follows. Suppose you found the path $\vec{z}(\lambda)$ that yields the shortest ℓ. Then, if you consider a slight variation $\delta\vec{z}$ of the path, namely consider

$$\vec{x}(\lambda) = \vec{z}(\lambda) + \delta\vec{z}(\lambda), \tag{1.3.27}$$

we must find the contribution to ℓ at first order in $\delta\vec{z}$ to be zero. This is analogous to the vanishing of the first derivatives of a function at its minimum.[18] In other words, in the integrand of eq. (1.1.22) we must replace

$$g_{ij}\left(\vec{x}(\lambda)\right) \to g_{ij}\left(\vec{z}(\lambda) + \delta\vec{z}(\lambda)\right) = g_{ij}\left(\vec{z}(\lambda)\right) + \delta z^k(\lambda) \frac{\partial g_{ij}\left(\vec{z}(\lambda)\right)}{\partial z^k} + \mathcal{O}(\delta z^2) \tag{1.3.28}$$

$$\frac{\mathrm{d}x^i(\lambda)}{\mathrm{d}\lambda} \to \frac{\mathrm{d}z^i(\lambda)}{\mathrm{d}\lambda} + \frac{\mathrm{d}\delta z^i(\lambda)}{\mathrm{d}\lambda}. \tag{1.3.29}$$

Since $\delta\vec{z}$ was arbitrary, at first order, its coefficient within the integrand must vanish. If we further specialize to affine parameters λ such that

$$\sqrt{g_{ij}(\mathrm{d}z^i/\mathrm{d}\lambda)(\mathrm{d}z^j/\mathrm{d}\lambda)} = \text{constant along the entire path } \vec{z}(\lambda), \tag{1.3.30}$$

[18]There is some smoothness condition being assumed here. For instance, the tip of the pyramid (or a cone) is the maximum height achieved, but the derivative slightly away from the tip is negative in all directions.

then one would arrive at the following second order non-linear ODE. Minimizing the distance ℓ between $\vec{x}_1$ and $\vec{x}_2$ leads to the shortest path $\vec{z}(\lambda)$ ($\equiv$ geodesic) obeying:

$$0 = \frac{\mathrm{d}^2 z^i}{\mathrm{d}\lambda^2} + \Gamma^i{}_{jk}\left(g_{ab}(\vec{z})\right)\frac{\mathrm{d}z^j}{\mathrm{d}\lambda}\frac{\mathrm{d}z^k}{\mathrm{d}\lambda}, \tag{1.3.31}$$

with the boundary conditions

$$\vec{z}(\lambda_1) = \vec{x}_1, \qquad \vec{z}(\lambda_2) = \vec{x}_2. \tag{1.3.32}$$

The converse is also true, in that – if the geodesic equation in eq. (1.3.31) holds, then g_{ij} $(\mathrm{d}z^i/\mathrm{d}\lambda)(\mathrm{d}z^j/\mathrm{d}\lambda)$ is a constant along the entire geodesic. Denoting $\ddot{z}^i \equiv \mathrm{d}^2 z^i/\mathrm{d}\lambda^2$ and $\dot{z}^i \equiv \mathrm{d}z^i/\mathrm{d}\lambda$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{ij}\dot{z}^i\dot{z}^j\right) &= 2\ddot{z}^i\dot{z}^j g_{ij} + \dot{z}^k\partial_k g_{ij}\dot{z}^i\dot{z}^j \\ &= 2\ddot{z}^i\dot{z}^j g_{ij} + \dot{z}^k\dot{z}^i\dot{z}^j\left(\partial_k g_{ij} + \partial_i g_{kj} - \partial_j g_{ik}\right)\end{aligned} \tag{1.3.33}$$

Note that the last two terms inside the parenthesis of the second equality cancels. The reason for inserting them is because the expression contained within the parenthesis is related to the Christoffel symbol; keeping in mind eq. (1.3.2),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{ij}\dot{z}^i\dot{z}^j\right) &= 2\dot{z}^i\left\{\ddot{z}^j g_{ij} + \dot{z}^k\dot{z}^j g_{il}\frac{g^{lm}}{2}\left(\partial_k g_{jm} + \partial_j g_{km} - \partial_m g_{jk}\right)\right\} \\ &= 2g_{il}\dot{z}^i\left\{\ddot{z}^l + \dot{z}^k\dot{z}^j\Gamma^l{}_{kj}\right\} = 0.\end{aligned} \tag{1.3.34}$$

The last equality follows because the expression in the $\{\dots\}$ is the left hand side of eq. (1.3.31). This constancy of g_{ij} $(\mathrm{d}z^i/\mathrm{d}\lambda)(\mathrm{d}z^j/\mathrm{d}\lambda)$ is useful for solving the geodesic equation itself.

Problem 1.7. Noether's theorem for Lagrangian mechanics Show that the affine parameter form of the geodesic (1.3.31) follows from demanding the following integral be extremized:

$$\ell^2 = (\lambda_2 - \lambda_1)\int_{\lambda_1}^{\lambda_2}\mathrm{d}\lambda g_{ij}\left(\vec{z}(\lambda)\right)\frac{\mathrm{d}z^i}{\mathrm{d}\lambda}\frac{\mathrm{d}z^j}{\mathrm{d}\lambda}. \tag{1.3.35}$$

(In the General Relativity literature, $\ell^2/2$ (half of eq. (1.3.35)) is known as Synge's world function.) That is, show that eq. (1.3.31) follows from applying the Euler-Lagrange equations to the Lagrangian

$$L \equiv \frac{1}{2}g_{ij}\dot{z}^i\dot{z}^j, \qquad \dot{z}^i \equiv \frac{\mathrm{d}z^i}{\mathrm{d}\lambda}. \tag{1.3.36}$$

Now argue that the Hamiltonian H is equal to the Lagrangian L. Can you prove that H, and therefore L, is a constant of motion? Moreover, if the geodesic equation (1.3.31) is satisfied by $z^\mu(\lambda)$, argue that the integral in eq. (1.3.35) yields the square of the geodesic distance between $\vec{x}_1 \equiv \vec{z}(\lambda_1)$ and $\vec{x}_2 \equiv \vec{z}(\lambda_2)$?

Conserved quantities from symmetries Finally, suppose ∂_k is a Killing vector. Explain why

$$\frac{\partial L}{\partial \dot{z}^k} = \text{constant}. \tag{1.3.37}$$

This is an example of Noether's theorem. For example, in flat Euclidean space, since the metric in Cartesian coordinates is a constant δ_{ij}, all the $\{\partial_i | i = 1, 2, \dots, D\}$ are Killing vectors. Therefore, from $L = (1/2)\delta_{ij}\dot{z}^i\dot{z}^j$, and we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\mathrm{d}z^i}{\mathrm{d}\lambda} = 0 \qquad \Rightarrow \qquad \frac{\mathrm{d}z^i}{\mathrm{d}\lambda} = \text{constant}. \tag{1.3.38}$$

This is, in fact, the statement that the center of mass of an isolated system obeying Newtonian mechanics moves with a constant velocity. By re-writing the Euclidean metric in spherical coordinates, provide the proper definition of angular momentum (about the $D-$axis) and proceed to prove that it is conserved.

Geodesics on a 2−sphere How many geodesics are there joining any two points on the 2−sphere? How many geodesics are there joining the North Pole and South Pole? Solve the geodesic equation (cf. eq. (1.3.31)) on the unit 2−sphere described by

$$\mathrm{d}\ell^2 = \mathrm{d}\theta^2 + \sin(\theta)^2\mathrm{d}\phi^2. \tag{1.3.39}$$

Explain how your answer would change if the sphere were of radius R instead. Hint: To solve the geodesic equation it helps to exploit the spherical symmetry of the problem; for e.g., what are the geodesics emanating from the North Pole? Then transform the answer to the more general case.

Christoffel symbols from Lagrangian As an example of how the action principle in eq. (1.3.35) allows us to extract the Christoffel symbols, let us consider the following $D-$dimensional metric:

$$\mathrm{d}\ell^2 \equiv a(\vec{x})^2\mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}, \tag{1.3.40}$$

where $a(\vec{x})$ is an arbitrary function. The Lagrangian in eq. (1.3.36) is now

$$L = \frac{1}{2}a^2\delta_{ij}\dot{z}^i\dot{z}^j, \qquad \dot{z}^i \equiv \frac{\mathrm{d}z^i}{\mathrm{d}\lambda}. \tag{1.3.41}$$

Applying the Euler-Lagrange equations,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{z}^i} - \frac{\partial L}{\partial z^i} = 0 \tag{1.3.42}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(a^2\dot{z}^i\right) - a\partial_i a\dot{\vec{z}}^2 = 0 \tag{1.3.43}$$

$$2a\dot{z}^j\partial_j a\ \dot{z}^i + a^2\ddot{z}^i - a\partial_i a\dot{\vec{z}}^2 = 0 \tag{1.3.44}$$

$$\ddot{z}^i + \left(\frac{\partial_j a}{a}\delta^i_l + \frac{\partial_l a}{a}\delta^i_j - \frac{\partial_i a}{a}\delta_{lj}\right)\dot{z}^l\dot{z}^j = \ddot{z}^i + \Gamma^i{}_{lj}\dot{z}^l\dot{z}^j = 0. \tag{1.3.45}$$

Using $\{\dots\}$ to indicate symmetrization of the indices, we have derived

$$\begin{aligned}\Gamma^i{}_{lj} &= \frac{1}{a}\left(\partial_{\{j} a \delta^i_{l\}} - \partial_i a \delta_{lj}\right) \\ &= \left(\delta^k_{\{j}\delta^i_{l\}} - \delta^{ki}\delta_{lj}\right)\partial_k \ln a. \end{aligned} \tag{1.3.46}$$

Problem 1.8. It is always possible to find a coordinate system with coordinates $\vec{y}$ such that, as $\vec{y} \to \vec{y}_0$, the Christoffel symbols vanish

$$\Gamma^k{}_{ij}(\vec{y}_0) = 0. \tag{1.3.47}$$

Can you demonstrate why this is true from the equivalence principle encoded in eq. (1.2.1)? Hint: it is important that, locally, the first deviation from flat space is quadratic in the displacement vector $(y - y_0)^i$. □

Remark That there is always an orthonormal frame where the metric is flat – recall eq. (1.2.47) – as well as the existence of a locally flat coordinate system, is why the measure of curvature, in particular the Riemann tensor in eq. (1.3.17), depends on gradients (second derivatives) of the metric.

Problem 1.9. Christoffel $\Gamma^i{}_{jk}$ contains as much information as $\partial_i g_{ab}$ Why do the Christoffel symbols take on the form in eq. (1.3.2)? It comes from assuming that the Christoffel symbol obeys the symmetry $\Gamma^i{}_{jk} = \Gamma^i{}_{kj}$ – this is the torsion-free condition – and demanding that the covariant derivative of a metric is a zero tensor,

$$\nabla_i g_{jk} = 0. \tag{1.3.48}$$

This can be expanded as

$$\nabla_i g_{jk} = 0 = \partial_i g_{jk} - \Gamma^l{}_{ij} g_{lk} - \Gamma^l{}_{ik} g_{jl}. \tag{1.3.49}$$

Expand also $\nabla_j g_{ki}$ and $\nabla_k g_{ij}$, and show that

$$2\Gamma^l{}_{ij} g_{lk} = \partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}. \tag{1.3.50}$$

Divide both sides by 2 and contract both sides with g^{km} to obtain $\Gamma^m{}_{ij}$ in eq. (1.3.2).

Remark Incidentally, while eq. (1.3.2) tells us the Christoffel symbol can be written in terms of the first derivatives of the metric; eq. (1.3.49) indicates the first derivative of the metric can also always be expressed in terms of the Christoffel symbols. In other words, $\partial_i g_{ab}$ contains as much information as $\Gamma^i{}_{jk}$, provided of course that g_{ij} itself is known. □

Problem 1.10. Can you show that the $\delta_\xi g_{ij}$ in eq. (1.2.42) can be re-written in a more covariant looking expression

$$\delta_\xi g_{ij}(\vec{x}') = \nabla_i \xi_j + \nabla_j \xi_i? \tag{1.3.51}$$

$\delta_\xi g_{ij} = \nabla_i \xi_j + \nabla_j \xi_i = 0$ is known as Killing's equation,[19] and a vector that satisfies Killing's equation is called a Killing vector. Showing that $\delta_\xi g_{ij}$ is a tensor indicate such a characterization of symmetry is a generally covariant statement.

Hint: Convert all partial derivatives into covariant ones by adding/subtracting Christoffel symbols appropriately; for instance $\partial_a \xi^i = \nabla_a \xi^i - \Gamma^i{}_{ab}\xi^b$. □

[19] The maximum number of linearly independent Killing vectors in D dimensions is $D(D+1)/2$. See Chapter 13 of Weinberg's *Gravitation and Cosmology* for a discussion.

Problem 1.11. Argue that, if a tensor $T^{i_1 i_2 \dots i_N}$ is zero in some coordinate system, it must be zero in any other coordinate system. □

Problem 1.12. Prove that the tensor $T_{i_1}{}^{i_2 \dots i_N}$ is zero if and only if the corresponding tensor $T_{i_1 i_2 \dots i_N}$ is zero. Then, using the product rule, explain why $\nabla_i g_{jk} = 0$ implies $\nabla_i g^{jk} = 0$. Hint: start with $\nabla_i (g_{aj} g_{bk} g^{jk})$. □

Problem 1.13. Calculate the Christoffel symbols of the 3-dimensional Euclidean metric in Cartesian coordinates δ_{ij}. Then calculate the Christoffel symbols for the same space, but in spherical coordinates: $(\mathrm{d}\ell)^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + (\sin\theta)^2 \mathrm{d}\phi^2)$. To start you off, the non-zero components of the metric are

$$g_{rr} = 1, \qquad g_{\theta\theta} = r^2, \qquad g_{\phi\phi} = r^2 (\sin\theta)^2; \tag{1.3.52}$$

$$g^{rr} = 1, \qquad g^{\theta\theta} = r^{-2}, \qquad g^{\phi\phi} = \frac{1}{r^2 (\sin\theta)^2}. \tag{1.3.53}$$

Also derive the Christoffel symbols in spherical coordinates from their Cartesian counterparts using eq. (1.3.11). This lets you cross-check your results; you should also feel free to use software to help. Partial answer: the non-zero components in spherical coordinates are

$$\Gamma^r{}_{\theta\theta} = -r, \qquad \Gamma^r{}_{\phi\phi} = -r(\sin\theta)^2, \tag{1.3.54}$$

$$\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{1}{r}, \qquad \Gamma^\theta{}_{\phi\phi} = -\cos\theta \cdot \sin\theta, \tag{1.3.55}$$

$$\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{1}{r}, \qquad \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \cot\theta. \tag{1.3.56}$$

To provide an example, let us calculate the Christoffel symbols of 2D flat space written in cylindrical coordinates $\xi^i \equiv (r, \phi)$,

$$\mathrm{d}\ell^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\phi, \qquad r \geq 0, \ \phi \in [0, 2\pi). \tag{1.3.57}$$

This means the non-zero components of the metric are

$$g_{rr} = 1, \qquad g_{\phi\phi} = r^2, \qquad g^{rr} = 1, \qquad g^{\phi\phi} = r^{-2}. \tag{1.3.58}$$

Keeping the diagonal nature of the metric in mind, let us start with

$$\begin{aligned} \Gamma^r{}_{ij} &= \frac{1}{2} g^{rk} \left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \right) = \frac{1}{2} g^{rr} \left(\partial_i g_{jr} + \partial_j g_{ir} - \partial_r g_{ij} \right) \\ &= \frac{1}{2} \left(\delta^r_j \partial_i g_{rr} + \delta^r_i \partial_j g_{rr} - \delta^\phi_i \delta^\phi_j \partial_r r^2 \right) = -\delta^\phi_i \delta^\phi_j r. \end{aligned} \tag{1.3.59}$$

In the third equality we have used the fact that the only g_{ij} that depends on r (and therefore yield a non-zero r-derivative) is $g_{\phi\phi}$. Now for the

$$\begin{aligned} \Gamma^\phi{}_{ij} &= \frac{1}{2} g^{\phi\phi} \left(\partial_i g_{j\phi} + \partial_j g_{i\phi} - \partial_\phi g_{ij} \right) \\ &= \frac{1}{2r^2} \left(\delta^\phi_j \partial_i g_{\phi\phi} + \delta^\phi_i \partial_j g_{\phi\phi} \right) = \frac{1}{2r^2} \left(\delta^\phi_j \delta^r_i \partial_r r^2 + \delta^\phi_i \delta^r_j \partial_r r^2 \right) \\ &= \frac{1}{r} \left(\delta^\phi_j \delta^r_i + \delta^\phi_i \delta^r_j \right). \end{aligned} \tag{1.3.60}$$

If we had started from Cartesian coordinates x^i,

$$x^i = r(\cos\phi, \sin\phi), \tag{1.3.61}$$

we know the Christoffel symbols in Cartesian coordinates are all zero, since the metric components are constant. If we wish to use eq. (1.3.11) to calculate the Christoffel symbols in (r, ϕ), the first term on the right hand side is zero and what we need are the $\partial x/\partial\xi$ and $\partial^2 x/\partial\xi\partial\xi$ matrices. The first derivative matrices are

$$\frac{\partial x^i}{\partial \xi^j} = \begin{bmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{bmatrix}^i{}_j \tag{1.3.62}$$

$$\frac{\partial \xi^i}{\partial x^j} = \left(\left(\frac{\partial x}{\partial \xi}\right)^{-1}\right)^i{}_j = \begin{bmatrix} \cos\phi & \sin\phi \\ -r^{-1}\sin\phi & r^{-1}\cos\phi \end{bmatrix}^i{}_j, \tag{1.3.63}$$

whereas the second derivative matrices are

$$\frac{\partial^2 x^1}{\partial \xi^i \xi^j} = \begin{bmatrix} 0 & -\sin\phi \\ -\sin\phi & -r\cos\phi \end{bmatrix} \tag{1.3.64}$$

$$\frac{\partial^2 x^2}{\partial \xi^i \xi^j} = \begin{bmatrix} 0 & \cos\phi \\ \cos\phi & -r\sin\phi \end{bmatrix}. \tag{1.3.65}$$

Therefore, from eq. (1.3.11),

$$\begin{aligned} \Gamma^r{}_{ij}(r,\phi) &= \frac{\partial r}{\partial x^k}\frac{\partial x^k}{\partial \xi^i \partial \xi^j} \qquad (1.3.66)\\ &= \cos\phi \cdot \begin{bmatrix} 0 & -\sin\phi \\ -\sin\phi & -r\cos\phi \end{bmatrix} + \sin\phi \cdot \begin{bmatrix} 0 & \cos\phi \\ \cos\phi & -r\sin\phi \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -r \end{bmatrix}. \end{aligned}$$

Similarly,

$$\begin{aligned} \Gamma^\phi{}_{ij}(r,\phi) &= \frac{\partial \phi}{\partial x^k}\frac{\partial x^k}{\partial \xi^i \partial \xi^j} \qquad (1.3.67)\\ &= -r^{-1}\sin\phi \begin{bmatrix} 0 & -\sin\phi \\ -\sin\phi & -r\cos\phi \end{bmatrix} + r^{-1}\cos\phi \begin{bmatrix} 0 & \cos\phi \\ \cos\phi & -r\sin\phi \end{bmatrix} = \begin{bmatrix} 0 & r^{-1} \\ r^{-1} & 0 \end{bmatrix}. \end{aligned}$$

□

Parallel transport Let v^i be a (tangent) vector field and $T^{j_1 \dots j_N}$ be some tensor. (Here, the placement of indices on the T is not important, but we will assume for convenience, all of them are upper indices.) We say that the tensor T is invariant under parallel transport along the vector v when

$$v^i \nabla_i T^{j_1 \dots j_N} = 0. \tag{1.3.68}$$

Problem 1.14. As an example, let's calculate the Christoffel symbols of the metric on the 2-sphere with unit radius,

$$(\mathrm{d}\ell)^2 = \mathrm{d}\theta^2 + (\sin\theta)^2 \mathrm{d}\phi^2. \tag{1.3.69}$$

Do not calculate from scratch – remember you have already computed the Christoffel symbols in 3D Euclidean space. How do you extract the 2-sphere Christoffel symbols from that calculation?

In the coordinate system (θ, ϕ), define the vector $v^i = (v^\theta, v^\phi) = (1, 0)$, i.e., $v = \partial_\theta$. This is the vector tangent to the sphere, at a given location $(0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi)$ on the sphere, such that it points away from the North and towards the South pole, along a constant longitude line. Show that it is parallel transported along itself, as quantified by the statement

$$v^i \nabla_i v^j = \nabla_\theta v^j = 0. \tag{1.3.70}$$

Also calculate $\nabla_\phi v^j$; comment on the result at $\theta = \pi/2$. Hint: recall our earlier 2-sphere discussion, where we considered parallel transporting a tangent vector from the North pole to the equator, along the equator, then back up to the North pole. □

Variation of the metric & divergence of tensors If we perturb the metric slightly

$$g_{ij} \to g_{ij} + h_{ij}, \tag{1.3.71}$$

where the components of h_{ij} are to be viewed as "small", the inverse metric will become

$$g^{ij} \to g^{ij} - h^{ij} + h^{ik} h_k^{\ j} + \mathcal{O}\left(h^3\right), \tag{1.3.72}$$

then the square root of the determinant of the metric will change as

$$\sqrt{|g|} \to \sqrt{|g|}\left(1 + \frac{1}{2} g^{ab} h_{ab} + \mathcal{O}(h^2)\right). \tag{1.3.73}$$

Problem 1.15. Use the matrix identity, where for any square matrix X,

$$\det e^X = e^{\mathrm{Tr}[X]}, \tag{1.3.74}$$

[20]to prove eq. (1.3.73). (The Tr X means the trace of the matrix X – sum over its diagonal terms.) Hint: Start with $\det(g_{ij} + h_{ij}) = \det(g_{ij}) \cdot \det(\delta^i_j + h^i_{\ j})$, with $h^i_{\ j} \equiv g^{ik} h_{kj}$. Then massage $\delta^i_j + h^i_{\ j} = \exp(\ln(\delta^i_j + h^i_{\ j}))$. □

Problem 1.16. Use eq. (1.3.73) and the definition of the Christoffel symbol to show that

$$\partial_i \ln \sqrt{|g|} = \frac{1}{2} g^{ab} \partial_i g_{ab} = \Gamma^s_{\ is}. \tag{1.3.75}$$

□

Problem 1.17. Divergence of tensors. Verify the following formulas for the divergence of a vector V^i, a fully antisymmetric rank-($N \leq D$) tensor $F^{i_1 i_2 \dots i_N}$ and a symmetric tensor $S^{ij} = S^{ji}$,

$$\nabla_i V^i = \frac{\partial_i\left(\sqrt{|g|} V^i\right)}{\sqrt{|g|}}, \tag{1.3.76}$$

[20]See, for e.g., Theorem 3.10 of arXiv: math-ph/0005032.

$$\nabla_j F^{ji_2 \dots i_N} = \frac{\partial_j \left(\sqrt{|g|} F^{ji_2 \dots i_N} \right)}{\sqrt{|g|}}, \tag{1.3.77}$$

$$\nabla_i S^{ij} = \frac{\partial_i \left(\sqrt{|g|} S^{ij} \right)}{\sqrt{|g|}} + \Gamma^j{}_{ab} S^{ab}. \tag{1.3.78}$$

Note that, fully antisymmetric means, swapping any pair of indices costs a minus sign,

$$F^{i_1 \dots i_{a-1} i_a i_{a+1} \dots i_{b-1} i_b i_{b+1} \dots i_N} = -F^{i_1 \dots i_{a-1} i_b i_{a+1} \dots i_{b-1} i_a i_{b+1} \dots i_N}. \tag{1.3.79}$$

Comment on how these expressions, equations (1.3.76)-(1.3.78), transform under a coordinate transformation, i.e., $\vec{x} \to \vec{x}(\vec{\xi})$. $\square$

Gradient of a scalar It is worth highlighting that the gradient of a scalar, with upper indices, depends on the metric; whereas the covariant derivative on the same scalar, with lower indices, does not.

$$\nabla^i \varphi = g^{ij} \nabla_j \varphi = g^{ij} \partial_j \varphi. \tag{1.3.80}$$

This means, even in flat space, $\nabla^i \varphi$ is not always equal to $\nabla_i \varphi$. (They are equal in Cartesian coordinates.) For instance, in spherical coordinates (r, θ, ϕ), where

$$g^{ij} = \text{diag}(1, r^{-2}, r^{-2}(\sin\theta)^{-2}); \tag{1.3.81}$$

the gradient of a scalar is

$$\nabla^i \varphi = \left(\partial_r \varphi, r^{-2} \partial_\theta \varphi, r^{-2} (\sin\theta)^{-2} \partial_\phi \varphi \right). \tag{1.3.82}$$

while the same object with lower indices is simply

$$\nabla_i \varphi = \left(\partial_r \varphi, \partial_\theta \varphi, \partial_\phi \varphi \right). \tag{1.3.83}$$

Divergence of a vector The divergence of a vector V^i is

$$\nabla_i V^i = \nabla^i V_i. \tag{1.3.84}$$

Laplacian of a scalar The Laplacian of a scalar ψ can be thought of as the divergence of its gradient. In 3D vector calculus you would write is as $\vec{\nabla}^2$ but in curved spaces we may also write it as $\square$ or $\nabla_i \nabla^i$:

$$\square \psi \equiv \vec{\nabla}^2 \psi = \nabla_i \nabla^i \psi = g^{ij} \nabla_i \nabla_j \psi. \tag{1.3.85}$$

Problem 1.18. Show that the Laplacian of a scalar can be written more explicitly in terms of the determinant of the metric and the inverse metric as

$$\square \psi \equiv \nabla_i \nabla^i \psi = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j \psi \right). \tag{1.3.86}$$

Hint: Start with the expansion $\nabla_i \nabla^i \psi = \partial_i \nabla^i \psi + \Gamma^i{}_{ij} \nabla^j \psi$. $\square$

Levi-Civita Tensor We have just seen how to write the divergence in any curved or flat space. We will now see that the curl from vector calculus also has a differential geometric formulation as an antisymmetric tensor, which will allow us to generalize the former to not only curved spaces but also arbitrary dimensions greater than 2. But first, we have to introduce the Levi-Civita tensor, and with it, the Hodge dual.

In D spatial dimensions we first define a Levi-Civita *symbol*

$$\epsilon_{i_1 i_2 \dots i_{D-1} i_D}. \tag{1.3.87}$$

It is defined by the following properties.

- It is completely antisymmetric in its indices. This means swapping any of the indices $i_a \leftrightarrow i_b$ (for $a \neq b$) will return

$$\epsilon_{i_1 i_2 \dots i_{a-1} i_a i_{a+1} \dots i_{b-1} i_b i_{b+1} \dots i_{D-1} i_D} = -\epsilon_{i_1 i_2 \dots i_{a-1} i_b i_{a+1} \dots i_{b-1} i_a i_{b+1} \dots i_{D-1} i_D}. \tag{1.3.88}$$

- For a given ordering of the D distinct coordinates $\{x^i | i = 1, 2, 3, \dots, D\}$, $\epsilon_{123\dots D} \equiv 1$. Below, we will have more to say about this choice.

These are sufficient to define every component of the Levi-Civita symbol. From the first definition, if any of the D indices are the same, say $i_a = i_b$, then the Levi-Civita symbol returns zero. (Why?) From the second definition, when all the indices are distinct, $\epsilon_{i_1 i_2 \dots i_{D-1} i_D}$ is a $+1$ if it takes even number of swaps to go from $\{1, \dots, D\}$ to $\{i_1, \dots, i_D\}$; and is a -1 if it takes an odd number of swaps to do the same.

For example, in the (perhaps familiar) 3 dimensional case, in Cartesian coordinates (x^1, x^2, x^3),

$$1 = \epsilon_{123} = -\epsilon_{213} = -\epsilon_{321} = -\epsilon_{132} = \epsilon_{231} = \epsilon_{312}. \tag{1.3.89}$$

The Levi-Civita *tensor* $\widetilde{\epsilon}_{i_1 \dots i_D}$ is defined as

$$\widetilde{\epsilon}_{i_1 i_2 \dots i_D} \equiv \sqrt{|g|}\epsilon_{i_1 i_2 \dots i_D}. \tag{1.3.90}$$

Let us understand why it is a (pseudo-)tensor. Because the Levi-Civita *symbol* is just a multi-index array of ± 1 and 0, it does not change under coordinate transformations. Equation (1.2.24) then implies

$$\sqrt{|g(\vec{\xi})|}\epsilon_{a_1 a_2 \dots a_D} = \sqrt{\left|g\left(\vec{x}(\vec{\xi})\right)\right|}\left|\det \frac{\partial x^i(\vec{\xi})}{\partial \xi^j}\right| \epsilon_{a_1 a_2 \dots a_D}. \tag{1.3.91}$$

On the right hand side, $\left|g\left(\vec{x}(\vec{\xi})\right)\right|$ is the absolute value of the determinant of g_{ij} written in the coordinates $\vec{x}$ but with $\vec{x}$ replaced with $\vec{x}(\vec{\xi})$.

If $\widetilde{\epsilon}_{i_1 i_2 \dots i_D}$ were a tensor, on the other hand, it must obey eq. (1.2.15),

$$\begin{aligned}
\sqrt{|g(\vec{\xi})|}\epsilon_{a_1 a_2 \dots a_D} &\overset{?}{=} \sqrt{\left|g\left(\vec{x}(\vec{\xi})\right)\right|}\epsilon_{i_1 \dots i_D} \frac{\partial x^{i_1}}{\partial \xi^{a_1}} \dots \frac{\partial x^{i_D}}{\partial \xi^{a_D}}, \\
&= \sqrt{\left|g\left(\vec{x}(\vec{\xi})\right)\right|}\left(\det \frac{\partial x^i}{\partial \xi^j}\right) \epsilon_{a_1 \dots a_D},
\end{aligned} \tag{1.3.92}$$

where in the second line we have recalled the co-factor expansion determinant of any matrix M,

$$\epsilon_{a_1\dots a_D} \det M = \epsilon_{i_1\dots i_D} M^{i_1}{}_{a_1} \dots M^{i_D}{}_{a_D}. \tag{1.3.93}$$

Comparing equations (1.3.91) and (1.3.92) tells us the Levi-Civita $\widetilde{\epsilon}_{a_1\dots a_D}$ transforms as a tensor for *orientation-preserving* coordinate transformations, namely for all coordinate transformations obeying

$$\det \frac{\partial x^i}{\partial \xi^j} = \epsilon_{i_1 i_2 \dots i_D} \frac{\partial x^{i_1}}{\partial \xi^1} \frac{\partial x^{i_2}}{\partial \xi^2} \dots \frac{\partial x^{i_D}}{\partial \xi^D} > 0. \tag{1.3.94}$$

Parity flips This restriction on the sign of the determinant of the Jacobian means the Levi-Civita tensor is invariant under "parity", and is why I call it a pseudo-tensor. Parity flips are transformations that reverse the orientation of some coordinate axis, say $\xi^i \equiv -x^i$ (for some fixed i) and $\xi^j = x^j$ for $j \neq i$. For the Levi-Civita tensor,

$$\sqrt{g(\vec{x})}\epsilon_{i_1\dots i_D} = \sqrt{g(\vec{\xi})} \left| \det \text{diag}[1, \dots, 1, \underbrace{-1}_{i\text{th component}}, 1, \dots, 1] \right| \epsilon_{i_1\dots i_D} = \sqrt{g(\vec{\xi})}\epsilon_{i_1\dots i_D}; \tag{1.3.95}$$

whereas, under the usual rules of coordinate transformations (eq. (1.2.15)) we would have expected a 'true' tensor $T_{i_1\dots i_D}$ to behave, for instance, as

$$T_{(1)(2)\dots(i-1)(i)(i+1)\dots(D)}(\vec{x}) \frac{\partial x^i}{\partial \xi^i} = -T_{(1)(2)\dots(i-1)(i)(i+1)\dots(D)}(\vec{\xi}). \tag{1.3.96}$$

Orientation of coordinate system What is orientation? It is the choice of how one orders the coordinates in use, say $(x^1, x^2, \dots, x^D)$, together with the convention that $\epsilon_{12\dots D} \equiv 1$.

In 2D flat spacetime, for example, we may choose the 'right-handed' (x^1, x^2) as Cartesian coordinates, $\epsilon_{12} \equiv 1$, and obtain the infinitesimal volume $\mathrm{d}^2\vec{x} = \mathrm{d}x^1 \mathrm{d}x^2$. We can switch to cylindrical coordinates

$$\vec{x}(\vec{\xi}) = r(\cos\phi, \sin\phi). \tag{1.3.97}$$

so that

$$\frac{\partial x^i}{\partial r} = (\cos\phi, \sin\phi), \qquad \frac{\partial x^i}{\partial \phi} = r(-\sin\phi, \cos\phi), \qquad r \geq 0,\ \phi \in [0, 2\pi). \tag{1.3.98}$$

If we ordered $(\xi^1, \xi^2) = (r, \phi)$, we would have

$$\epsilon_{i_1 i_2} \frac{\partial x^{i_1}}{\partial r} \frac{\partial x^{i_2}}{\partial \phi} = \det \begin{bmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{bmatrix} = r(\cos\phi)^2 + r(\sin\phi)^2 = r. \tag{1.3.99}$$

If we instead ordered $(\xi^1, \xi^2) = (\phi, r)$, we would have

$$\epsilon_{i_1 i_2} \frac{\partial x^{i_1}}{\partial \phi} \frac{\partial x^{i_2}}{\partial r} = \det \begin{bmatrix} -r\sin\phi & \cos\phi \\ r\cos\phi & \sin\phi \end{bmatrix} = -r(\sin\phi)^2 - r(\cos\phi)^2 = -r. \tag{1.3.100}$$

We can see that going from (x^1, x^2) to $(\xi^1, \xi^2) \equiv (r, \phi)$ is orientation preserving; and we should also choose $\epsilon_{r\phi} \equiv 1$.[21]

[21] We have gone from a 'right-handed' coordinate system (x^1, x^2) to a 'right-handed' (r, ϕ); we could also have gone from a 'left-handed' one (x^2, x^1) to a 'left-handed' (ϕ, r) and this would still be orientation-preserving.

Problem 1.19. By going from Cartesian coordinates (x^1, x^2, x^3) to spherical ones,

$$\vec{x}(\vec{\xi}) = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{1.3.101}$$

determine what is the orientation preserving ordering of the coordinates of $\vec{\xi}$, and is $\epsilon_{r\theta\phi}$ equal $+1$ or -1? □

Infinitesimal volume re-visited The infinitesimal volume we encountered earlier can really be written as

$$\mathrm{d}(\text{vol.}) = \mathrm{d}^D\vec{x}\sqrt{|g(\vec{x})|}\epsilon_{12\dots D} = \mathrm{d}^D\vec{x}\sqrt{|g(\vec{x})|}, \tag{1.3.102}$$

so that under a coordinate transformation $\vec{x} \to \vec{x}(\vec{\xi})$, the necessarily positive infinitesimal volume written in $\vec{x}$ transforms into another positive infinitesimal volume, but written in $\vec{\xi}$:

$$\mathrm{d}^D\vec{x}\sqrt{|g(\vec{x})|}\epsilon_{12\dots D} = \mathrm{d}^D\vec{\xi}\sqrt{\left|g(\vec{\xi})\right|}\epsilon_{12\dots D}. \tag{1.3.103}$$

Below, we will see that $\mathrm{d}^D\vec{x}\sqrt{|g(\vec{x})|}$ in modern integration theory is viewed as a differential $D-$form.

Problem 1.20. We may consider the infinitesimal volume in 3D flat space in Cartesian coordinates

$$\mathrm{d}(\text{vol.}) = \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3. \tag{1.3.104}$$

Now, let us switch to spherical coordinates $\vec{\xi}$, with the ordering in the previous problem. Show that it is given by

$$\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 = \mathrm{d}^3\vec{\xi}\sqrt{|g(\vec{\xi})|}, \qquad \sqrt{|g(\vec{\xi})|} = \epsilon_{i_1 i_2 i_3}\frac{\partial x^{i_1}}{\partial\xi^1}\frac{\partial x^{i_2}}{\partial\xi^2}\frac{\partial x^{i_3}}{\partial\xi^3}. \tag{1.3.105}$$

Can you compare $\sqrt{|g(\vec{\xi})|}$ with the volume of the parallelepiped formed by $\partial_{\xi^1}x^i$, $\partial_{\xi^2}x^i$ and $\partial_{\xi^3}x^i$?[22] □

Cross-Product in Flat 3D, Right-hand rule Notice the notion of orientation in 3D is closely tied to the "right-hand rule" in vector calculus. Let $\vec{X}$ and $\vec{Y}$ be vectors in Euclidean 3-space. In Cartesian coordinates, where $g_{ij} = \delta_{ij}$, you may check that their cross product is

$$\left(\vec{X}\times\vec{Y}\right)^k = \epsilon^{ijk}X^iY^j. \tag{1.3.106}$$

For example, if $\vec{X}$ is parallel to the positive x^1 axis and $\vec{Y}$ parallel to the positive x^2-axis, so that $\vec{X} = |\vec{X}|(1,0,0)$ and $\vec{Y} = |\vec{Y}|(0,1,0)$, the cross product reads

$$\left(\vec{X}\times\vec{Y}\right)^k \to |\vec{X}||\vec{Y}|\epsilon^{12k} = |\vec{X}||\vec{Y}|\delta_3^k, \tag{1.3.107}$$

[22] Because of the existence of locally flat coordinates $\{y^i\}$, the interpretation of $\sqrt{|g(\xi)|}$ as the volume of parallelepiped formed by $\{\partial_{\xi^1}y^i, \dots, \partial_{\xi^D}y^i\}$ actually holds very generally.

i.e., it is parallel to the positive x^3 axis. (Remember k cannot be either 1 or 2 because ϵ^{ijk} is fully antisymmetric.) If we had chosen $\epsilon_{123} = \epsilon^{123} \equiv -1$, then the cross product would become the "left-hand rule". Below, I will continue to point out, where appropriate, how this issue of orientation arises in differential geometry.

Problem 1.21. Show that the Levi-Civita tensor with all upper indices is given by

$$\widetilde{\epsilon}^{i_1 i_2 \dots i_D} = \frac{\text{sgn}\ \det(g_{ab})}{\sqrt{|g|}} \epsilon_{i_1 i_2 \dots i_D}. \tag{1.3.108}$$

In curved spaces, the sign of the $\det g_{ab} = 1$; whereas in curved spacetimes it depends on the signature used for the flat metric.[23] Hint: Raise the indices by contracting with inverse metrics, then recall the cofactor expansion definition of the determinant. □

Problem 1.22. Show that the covariant derivative of the Levi-Civita tensor is zero.

$$\nabla_j \widetilde{\epsilon}_{i_1 i_2 \dots i_D} = 0. \tag{1.3.109}$$

(Hint: Start by expanding the covariant derivative in terms of Christoffel symbols; then go through some combinatoric reasoning or invoke the equivalence principle.) From this, explain why the following equalities are true; for some vector V,

$$\nabla_j \left(\widetilde{\epsilon}^{i_1 i_2 \dots i_{D-2} jk} V_k \right) = \widetilde{\epsilon}^{i_1 i_2 \dots i_{D-2} jk} \nabla_j V_k = \widetilde{\epsilon}^{i_1 i_2 \dots i_{D-2} jk} \partial_j V_k. \tag{1.3.110}$$

Why is $\nabla_i V_j - \nabla_j V_i = \partial_i V_j - \partial_j V_i$ for any V_i? Hint: expand the covariant derivatives in terms of the partial derivatives and the Christoffel symbols. □

Combinatorics This is an appropriate place to state how to actually construct a fully antisymmetric tensor from a given tensor $T_{i_1 \dots i_N}$. Denoting $\Pi(i_1 \dots i_N)$ to be a permutation of the indices $\{i_1 \dots i_N\}$, the antisymmetrization procedure is given by

$$\begin{aligned} T_{[i_1 \dots i_N]} &= \sum_{\text{permutations } \Pi \text{ of } \{i_1, i_2, \dots, i_N\}}^{N!} \sigma_\Pi \cdot T_{\Pi(i_1 \dots i_N)} \\ &= \sum_{\text{even permutations } \Pi \text{ of } \{i_1, i_2, \dots, i_N\}} T_{\Pi(i_1 \dots i_N)} - \sum_{\text{odd permutations } \Pi \text{ of } \{i_1, i_2, \dots, i_N\}} T_{\Pi(i_1 \dots i_N)}. \end{aligned} \tag{1.3.111}$$

In words: for a rank$-N$ tensor, $T_{[i_1 \dots i_N]}$ consists of a sum of $N!$ terms. The first is $T_{i_1 \dots i_N}$. Each and every other term consists of T with its indices permuted over all the $N! - 1$ distinct remaining possibilities, multiplied by $\sigma_\Pi = +1$ if it took even number of index swaps to get to the given permutation, and $\sigma_\Pi = -1$ if it took an odd number of swaps. (The σ_Π is often called the sign of the permutation Π.) For example,

$$T_{[ij]} = T_{ij} - T_{ji}, \qquad T_{[ijk]} = T_{ijk} - T_{ikj} - T_{jik} + T_{jki} + T_{kij} - T_{kji}. \tag{1.3.112}$$

Can you see why eq. (1.3.111) yields a fully antisymmetric object? Consider any pair of distinct indices, say i_a and i_b, for $1 \leq (a \neq b) \leq N$. Since the sum on its right hand side contains every

[23] See eq. (1.2.51) to understand why the sign of the determinant of the metric is always determined by the sign of the determinant of its flat counterpart.

permutation (multiplied by the sign) – we may group the terms in the sum of eq. (1.3.111) into pairs, say $\sigma_{\Pi_\ell} T_{j_1 \dots i_a \dots i_b \dots j_N} - \sigma_{\Pi_\ell} T_{j_1 \dots i_b \dots i_a \dots j_N}$. That is, for a given term $\sigma_{\Pi_\ell} T_{j_1 \dots i_a \dots i_b \dots j_N}$ there must be a counterpart with $i_a \leftrightarrow i_b$ swapped, multipled by a minus sign, because – if the first term involved even (odd) number of swaps to get to, then the second must have involved an odd (even) number. If we now considered swapping $i_a \leftrightarrow i_b$ in every term in the sum on the right hand side of eq. (1.3.111),

$$T_{[i_1 \dots i_a \dots i_b \dots i_N]} = \sigma_{\Pi_\ell} T_{j_1 \dots i_a \dots i_b \dots j_N} - \sigma_{\Pi_\ell} T_{j_1 \dots i_b \dots i_a \dots j_N} + \dots, \tag{1.3.113}$$

$$T_{[i_1 \dots i_b \dots i_a \dots i_N]} = -\left(\sigma_{\Pi_\ell} T_{j_1 \dots i_a \dots i_b \dots j_N} - \sigma_{\Pi_\ell} T_{j_1 \dots i_b \dots i_a \dots j_N} + \dots \right). \tag{1.3.114}$$

Problem 1.23. Given $T_{i_1 i_2 \dots i_N}$, how do we construct a fully symmetric object from it, i.e., such that swapping any two indices returns the same object? □

Problem 1.24. If the Levi-Civita symbol is subject to the convention $\epsilon_{12\dots D} \equiv 1$, explain why it is equivalent to the following expansion in Kronecker δs.

$$\epsilon_{i_1 i_2 \dots i_D} = \delta^1_{[i_1} \delta^2_{i_2} \dots \delta^{D-1}_{i_{D-1}} \delta^D_{i_D]} \tag{1.3.115}$$

Can you also explain why the following is true?

$$\epsilon_{a_1 a_2 \dots a_{D-1} a_D} \det A = \epsilon_{i_1 i_2 \dots i_{D-1} i_D} A^{i_1}{}_{a_1} A^{i_2}{}_{a_2} \dots A^{i_{D-1}}{}_{a_{D-1}} A^{i_D}{}_{a_D} \tag{1.3.116}$$

Problem 1.25. Argue that

$$\begin{aligned} T_{[i_1 \dots i_N]} = T_{[i_1 \dots i_{N-1}] i_N} - T_{[i_N i_2 \dots i_{N-1}] i_1} - T_{[i_1 i_N i_3 \dots i_{N-1}] i_2} \\ - T_{[i_1 i_2 i_N i_4 \dots i_{N-1}] i_3} - \dots - T_{[i_1 \dots i_{N-2} i_N] i_{N-1}}. \end{aligned} \tag{1.3.117}$$

□

Product of Levi-Civita tensors The product of two Levi-Civita tensors will be important for the discussions to come. We have

$$\widetilde{\epsilon}^{i_1 \dots i_N k_1 \dots k_{D-N}} \widetilde{\epsilon}_{j_1 \dots j_N k_1 \dots k_{D-N}} = \text{sgn} \det(g_{ab}) \cdot A_N \delta^{i_1}_{[j_1} \dots \delta^{i_N}_{j_N]}, \qquad 1 \leq N \leq D, \tag{1.3.118}$$

$$\widetilde{\epsilon}^{k_1 \dots k_D} \widetilde{\epsilon}_{k_1 \dots k_D} = \text{sgn} \det(g_{ab}) \cdot A_0, \qquad A_{N \geq 0} \equiv (D-N)!. \tag{1.3.119}$$

(Remember $0! = 1! = 1$; also, $\delta^{i_1}_{[j_1} \dots \delta^{i_N}_{j_N]} = \delta^{[i_1}_{j_1} \dots \delta^{i_N]}_{j_N}$.) Let us first understand why there are a bunch of Kronecker deltas on the right hand side, starting from the $N = D$ case – where no indices are contracted.

$$\text{sgn} \det(g_{ab}) \widetilde{\epsilon}^{i_1 \dots i_D} \widetilde{\epsilon}_{j_1 \dots j_D} = \epsilon_{i_1 \dots i_D} \epsilon_{j_1 \dots j_D} = \delta^{i_1}_{[j_1} \dots \delta^{i_D}_{j_D]} \tag{1.3.120}$$

(This means $A_D = 1$.) The first equality follows from eq. (1.3.108). The second may seem a bit surprising, because the indices $\{i_1, \dots, i_D\}$ are attached to a completely different $\widetilde{\epsilon}$ tensor from the $\{j_1, \dots, j_D\}$. However, if we manipulate

$$\delta^{i_1}_{[j_1} \dots \delta^{i_D}_{j_D]} = \delta^{i_1}_{[1} \dots \delta^{i_D}_{D]} \sigma_j = \delta^1_{[1} \dots \delta^D_{D]} \sigma_i \sigma_j = \sigma_i \sigma_j = \epsilon_{i_1 \dots i_D} \epsilon_{j_1 \dots j_D}, \tag{1.3.121}$$

where $\sigma_i = 1$ if it took even number of swaps to re-arrange $\{i_1, \dots, i_D\}$ to $\{1, \dots, D\}$ and $\sigma_i = -1$ if it took odd number of swaps; similarly, $\sigma_j = 1$ if it took even number of swaps to re-arrange $\{j_1, \dots, j_D\}$ to $\{1, \dots, D\}$ and $\sigma_j = -1$ if it took odd number of swaps. But σ_i is precisely the Levi-Civita *symbol* $\epsilon_{i_1 \dots i_D}$ and likewise $\sigma_j = \epsilon_{j_1 \dots j_D}$. The (≥ 1)-contractions between the $\widetilde{\epsilon}$s can, in principle, be obtained by contracting the right hand side of (1.3.120). Because one contraction of the $(N+1)$ Kronecker deltas have to return N Kronecker deltas, by induction, we now see why the right hand side of eq. (1.3.118) takes the form it does for any N.

What remains is to figure out the actual value of A_N. We will do so recursively, by finding a relationship between A_N and A_{N-1}. We will then calculate A_1 and use it to generate all the higher A_Ns. Starting from eq. (1.3.118), and employing eq. (1.3.117),

$$\begin{aligned}
&\widetilde{\epsilon}^{i_1 \dots i_{N-1} \sigma k_1 \dots k_{D-N}} \widetilde{\epsilon}_{j_1 \dots j_{N-1} \sigma k_1 \dots k_{D-N}} = A_N \delta^{i_1}_{[j_1} \dots \delta^{i_{N-1}}_{j_{N-1}} \delta^{\sigma}_{\sigma]} \qquad (1.3.122) \\
&= A_N \left(\delta^{i_1}_{[j_1} \dots \delta^{i_{N-1}}_{j_{N-1}]} \delta^{\sigma}_{\sigma} - \delta^{i_1}_{[\sigma} \delta^{i_2}_{j_2} \dots \delta^{i_{N-1}}_{j_{N-1}]} \delta^{\sigma}_{j_1} - \delta^{i_1}_{[j_1} \delta^{i_2}_{\sigma} \delta^{i_3}_{j_3} \dots \delta^{i_{N-1}}_{j_{N-1}]} \delta^{\sigma}_{j_2} - \dots - \delta^{i_1}_{[j_1} \dots \delta^{i_{N-2}}_{j_{N-2}} \delta^{i_{N-1}}_{\sigma]} \delta^{\sigma}_{j_{N-1}} \right) \\
&= A_N \cdot (D - (N-1)) \delta^{i_1}_{[j_1} \dots \delta^{i_{N-1}}_{j_{N-1}]} \equiv A_{N-1} \delta^{i_1}_{[j_1} \dots \delta^{i_{N-1}}_{j_{N-1}]}.
\end{aligned}$$

(The last equality is a definition, because A_{N-1} *is* the coefficient of $\delta^{i_1}_{[j_1} \dots \delta^{i_{N-1}}_{j_{N-1}]}$.) We have the relationship

$$A_N = \frac{A_{N-1}}{D - (N-1)}. \qquad (1.3.123)$$

If we contract every index, we have to sum over all the $D!$ (non-zero components of the Levi-Civita symbol)^2,

$$\widetilde{\epsilon}^{i_1 \dots i_D} \widetilde{\epsilon}_{i_1 \dots i_D} = \operatorname{sgn} \det(g_{ab}) \cdot \sum_{i_1, \dots, i_D} (\epsilon_{i_1 \dots i_D})^2 = \operatorname{sgn} \det(g_{ab}) \cdot D! \qquad (1.3.124)$$

That means $A_0 = D!$. If we contracted every index but one,

$$\widetilde{\epsilon}^{i k_1 \dots k_D} \widetilde{\epsilon}_{j k_1 \dots k_D} = \operatorname{sgn} \det(g_{ab}) A_1 \delta^i_j. \qquad (1.3.125)$$

Contracting the i and j indices, and invoking eq. (1.3.124),

$$\operatorname{sgn} \det(g_{ab}) \cdot D! = \operatorname{sgn} \det(g_{ab}) A_1 \cdot D \qquad \Rightarrow \qquad A_1 = (D-1)!. \qquad (1.3.126)$$

That means we may use A_1 (or, actually, A_0) to generate all other $A_{N \geq 0}$s,

$$\begin{aligned}
A_N &= \frac{A_{N-1}}{(D - (N-1))} = \frac{1}{D - (N-1)} \frac{A_{N-2}}{D - (N-2)} = \dots \\
&= \frac{A_1}{(D-1)(D-2)(D-3) \dots (D - (N-1))} = \frac{(D-1)!}{(D-1)(D-2)(D-3) \dots (D - (N-1))} \\
&= \frac{(D-1)(D-2)(D-3) \dots (D-(N-1))(D-N)(D-(N+1)) \dots 3 \cdot 2 \cdot 1}{(D-1)(D-2)(D-3) \dots (D-(N-1))} \\
&= (D-N)!. \qquad (1.3.127)
\end{aligned}$$

Note that $0! = 1$, so $A_D = 1$ as we have found earlier.

Problem 1.26. Matrix determinants revisited Explain why the cofactor expansion definition of a square matrix in eq. (**??**) can also be expressed as

$$\det A = \epsilon^{i_1 i_2 \dots i_{D-1} i_D} A^1{}_{i_1} A^2{}_{i_2} \dots A^{D-1}{}_{i_{D-1}} A^D{}_{i_D} \tag{1.3.128}$$

provided we define $\epsilon^{i_1 i_2 \dots i_{D-1} i_D}$ in the same way we defined its lower index counterpart, including $\epsilon^{123\dots D} \equiv 1$. That is, why can we cofactor expand about either the rows or the columns of a matrix, to obtain its determinant? What does that tell us about the relation $\det A^T = \det A$? Can you also prove, using our result for the product of two Levi-Civita symbols, that $\det(A \cdot B) = (\det A)(\det B)$? □

Problem 1.27. In 3D vector calculus, the curl of a gradient of a scalar is zero – how would you express that using the $\widetilde{\epsilon}$ tensor? What about the statement that the divergence of a curl of a vector field is zero? Can you also derive, using the $\widetilde{\epsilon}$ tensor in Cartesian coordinates and eq. (1.3.118), the 3D vector cross product identity

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}? \tag{1.3.129}$$

□

Hodge dual We are now ready to define the Hodge dual. Given a fully antisymmetric rank-N tensor $T_{i_1 \dots i_N}$, its Hodge dual – which I shall denote as $\widetilde{T}^{j_1 \dots j_{D-N}}$ – is a fully antisymmetric rank-$(D-N)$ tensor whose components are

$$\widetilde{T}^{j_1 \dots j_{D-N}} \equiv \frac{1}{N!} \widetilde{\epsilon}^{j_1 \dots j_{D-N} i_1 \dots i_N} T_{i_1 \dots i_N}. \tag{1.3.130}$$

Invertible Note that the Hodge dual is an invertible operation, as long as we are dealing with fully antisymmetric tensors, in that given $\widetilde{T}^{j_1 \dots j_{D-N}}$ we can recover $T_{i_1 \dots i_N}$ and vice versa.[24] All you have to do is contract both sides with the Levi-Civita tensor, namely

$$T_{i_1 \dots i_N} = \text{sgn}(\det g_{ab}) \frac{(-)^{N(D-N)}}{(D-N)!} \widetilde{\epsilon}_{i_1 \dots i_N j_1 \dots j_{D-N}} \widetilde{T}^{j_1 \dots j_{D-N}}. \tag{1.3.131}$$

In other words $\widetilde{T}^{j_1 \dots j_{D-N}}$ and $T_{i_1 \dots i_N}$ contain the same amount of information.

Problem 1.28. Using eq. (1.3.118), verify the proportionality constant $(-)^{N(D-N)}\text{sgn}g$ in the inverse Hodge dual of eq. (1.3.131), and thereby prove that the Hodge dual is indeed invertible for fully antisymmetric tensors. □

Curl The curl of a vector field A_i can now either be defined as the antisymmetric rank-2 tensor

$$F_{ij} \equiv \partial_{[i} A_{j]} \tag{1.3.132}$$

[24] The fully antisymmetric property is crucial here: any symmetric portion of a tensor contracted with the Levi-Civita tensor would be lost. For example, an arbitrary rank-2 tensor can always be decomposed as $T_{ij} = (1/2)T_{\{ij\}} + (1/2)T_{[ij]}$; then, $\widetilde{\epsilon}^{i_1 \dots i_{D-2} jk} T_{jk} = \widetilde{\epsilon}^{i_1 \dots i_{D-2} jk}((1/2)T_{\{jk\}} + (1/2)T_{[jk]}) = (1/2)\widetilde{\epsilon}^{i_1 \dots i_{D-2} jk} T_{[jk]}$. The symmetric part is lost because $\widetilde{\epsilon}^{i_1 \dots i_{D-2} jk} T_{\{jk\}} = -\widetilde{\epsilon}^{i_1 \dots i_{D-2} kj} T_{\{kj\}}$.

or its rank-$(D-2)$ Hodge dual

$$\widetilde{F}^{i_1 i_2 \dots i_{D-2}} \equiv \frac{1}{2}\widetilde{\epsilon}^{i_1 i_2 \dots i_{D-2} jk} \partial_{[j} A_{k]}. \tag{1.3.133}$$

$(D=3)$-dimensional space is a special case where both the original vector field A^i and the Hodge dual $\widetilde{F}^i$ are rank-1 tensors. This is usually how electromagnetism is taught: that in 3D the magnetic field is a vector arising from the curl of the vector potential A_i:

$$B^i = \frac{1}{2}\widetilde{\epsilon}^{ijk}\partial_{[j}A_{k]} = \widetilde{\epsilon}^{ijk}\partial_j A_k. \tag{1.3.134}$$

In particular, when we specialize to 3D flat space with Cartesian coordinates:

$$\left(\vec{\nabla}\times\vec{A}\right)^i = \epsilon^{ijk}\partial_j A_k, \qquad \text{(Flat 3D Cartesian)}. \tag{1.3.135}$$

$$\left(\vec{\nabla}\times\vec{A}\right)^1 = \epsilon^{123}\partial_2 A_3 + \epsilon^{132}\partial_3 A_2 = \partial_2 A_3 - \partial_3 A_2, \qquad \text{etc.} \tag{1.3.136}$$

By setting $i = 1, 2, 3$ we can recover the usual definition of the curl in 3D vector calculus. But you may have noticed from equations (1.3.132) and (1.3.133), in any other dimension, that the magnetic field is really not a (rank-1) vector but should be viewed either as a rank-2 curl or a rank$-(D-2)$ Hodge dual of this curl. □

Divergence versus Curl We can extend the definition of a curl of a vector field to that of a rank$-N$ fully antisymmetric $B_{i_1 \dots i_N}$ as

$$\nabla_{[\sigma} B_{i_1 \dots i_N]} = \partial_{[\sigma} B_{i_1 \dots i_N]}. \tag{1.3.137}$$

(Can you explain why the ∇ can be replaced with ∂?) With the Levi-Civita tensor, we can convert the curl of an antisymmetric tensor into the divergence of its dual,

$$\nabla_\sigma \widetilde{B}^{j_1 \dots j_{D-N-1}\sigma} = \frac{1}{N!}\widetilde{\epsilon}^{j_1 \dots j_{D-N-1}\sigma i_1 \dots i_N}\nabla_\sigma B_{i_1 \dots i_N} \tag{1.3.138}$$

$$= (N+1)\cdot\widetilde{\epsilon}^{j_1 \dots j_{D-N-1}\sigma i_1 \dots i_N}\partial_{[\sigma} B_{i_1 \dots i_N]}. \tag{1.3.139}$$

Problem 1.29. Show, by contracting both sides of eq. (1.3.134) with an appropriate $\widetilde{\epsilon}$-tensor, that

$$\widetilde{\epsilon}_{ijk} B^k = \frac{1}{2}\partial_{[i} A_{j]}. \tag{1.3.140}$$

Assume sgn $\det(g_{ab}) = 1$. □

Problem 1.30. In D-dimensional space, is the Hodge dual of a rank-D fully antisymmetric tensor $F_{i_1 \dots i_D}$ invertible? Hint: If $F_{i_1 \dots i_D}$ is fully antisymmetric, how many independent components does it have? Can you use that observation to relate $\widetilde{F}$ and $F_{i_1 \dots i_D}$ in

$$\widetilde{F} \equiv \frac{1}{D!}\widetilde{\epsilon}^{i_1 \dots i_D} F_{i_1 \dots i_D}? \tag{1.3.141}$$

□

Problem 1.31. All 2D Metrics Are Conformally Flat[25] A metric g_{ij} is said to be conformally flat if it is equal to the flat metric multiplied by a scalar function (which we shall denote as Ω^2 – not to be confused with the solid angle):

$$g_{ij} = \Omega^2 \bar{g}_{ij}. \tag{1.3.142}$$

Here, $\bar{g}_{ij} = \text{diag}[1, 1]$ if we are working with a curved space; whereas (in the following Chapter) $\bar{g}_{ij} = \text{diag}[1, -1]$ if we are dealing with a curved spacetime instead.

In this problem, we will prove that:

> In a 2D curved space(time), it is always possible to find a set of local coordinates such that the metric takes the conformally flat form in eq. (1.3.142).

Suppose we begin with the metric $g_{i'j'}(\vec{x}')\mathrm{d}x'^i \mathrm{d}x'^j$. To show that we can find a coordinate transformation $\vec{x}'(\vec{x})$ such that eq. (1.3.142) is achieved, explain why

$$\frac{\partial x^1}{\partial x'^m}\frac{\partial x^2}{\partial x'^n} g^{m'n'}(\vec{x}') = 0. \tag{1.3.143}$$

If we view x^1 and x^2 as scalar fields in the curved space(time), then $\partial x^{1,2}/\partial x'^m$ are 1-forms (for e.g., $\mathrm{d}x^1 = (\partial x^1/\partial x'^m)\mathrm{d}x'^m$), and eq. (1.3.143) tells us they are orthogonal. Show that eq. (1.3.143) may be solved by demanding one is the Hodge dual of the other – namely,

$$\frac{\partial x^1}{\partial x'^m} = \widetilde{\epsilon}_{m'}{}^{n'} \frac{\partial x^2}{\partial x'^n}. \tag{1.3.144}$$

Provided eq. (1.3.144) holds, next show that

$$\frac{\partial x^1}{\partial x'^m}\frac{\partial x^1}{\partial x'^n} g^{m'n'} = (\text{sgn}\det g)\, \frac{\partial x^2}{\partial x'^m}\frac{\partial x^2}{\partial x'^n} g^{m'n'}. \tag{1.3.145}$$

Now explain why eq. (1.3.143) implies eq. (1.3.142) for both curved space and spacetime. □

Problem 1.32. Curl, divergence, and all that The electromagnetism textbook by J.D.Jackson contains on its very last page explicit forms of the gradient and Laplacian of a scalar as well as divergence and curl of a vector – in Cartesian, cylindrical, and spherical coordinates in 3-dimensional flat space. Can you derive them with differential geometric techniques? Note that the vectors there are expressed in an orthonormal basis.

Cartesian coordinates In Cartesian coordinates $\{x^1, x^2, x^3\} \in \mathbb{R}^3$, we have the metric

$$\mathrm{d}\ell^2 = \delta_{ij}\mathrm{d}x^i \mathrm{d}x^j. \tag{1.3.146}$$

Show that the gradient of a scalar ψ is

$$\vec{\nabla}\psi = (\partial_1\psi, \partial_2\psi, \partial_3\psi) = (\partial^1\psi, \partial^2\psi, \partial^3\psi); \tag{1.3.147}$$

the Laplacian of a scalar ψ is

$$\nabla_i\nabla^i\psi = \delta^{ij}\partial_i\partial_j\psi = \left(\partial_1^2 + \partial_2^2 + \partial_3^2\right)\psi; \tag{1.3.148}$$

[25] This problem is based on appendix 11C of [**?**].

the divergence of a vector A is

$$\nabla_i A^i = \partial_i A^i; \tag{1.3.149}$$

and the curl of a vector A is

$$(\vec{\nabla} \times \vec{A})^i = \epsilon^{ijk} \partial_j A_k. \tag{1.3.150}$$

Cylindrical coordinates In cylindrical coordinates $\{\rho \geq 0, 0 \leq \phi < 2\pi, z \in \mathbb{R}\}$, employ the following parametrization for the Cartesian components of the 3D Euclidean coordinate vector

$$\vec{x} = (\rho \cos\phi, \rho \sin\phi, z) \tag{1.3.151}$$

to argue that the flat metric is translated from $g_{ij} = \delta_{ij}$ to

$$\mathrm{d}\ell^2 = \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\phi^2 + \mathrm{d}z^2. \tag{1.3.152}$$

Show that the gradient of a scalar ψ is

$$\nabla^{\widehat{\rho}} \psi = \partial_\rho \psi, \qquad \nabla^{\widehat{\phi}} \psi = \frac{1}{\rho} \partial_\phi \psi, \qquad \nabla^{\widehat{z}} \psi = \partial_z \psi; \tag{1.3.153}$$

the Laplacian of a scalar ψ is

$$\nabla_i \nabla^i \psi = \frac{1}{\rho} \partial_\rho \left(\rho \partial_\rho \psi\right) + \frac{1}{\rho^2} \partial_\phi^2 \psi + \partial_z^2 \psi; \tag{1.3.154}$$

the divergence of a vector A is

$$\nabla_i A^i = \frac{1}{\rho} \left(\partial_\rho \left(\rho A^{\widehat{\rho}}\right) + \partial_\phi A^{\widehat{\phi}} \right) + \partial_z A^{\widehat{z}}; \tag{1.3.155}$$

and the curl of a vector A is

$$\widetilde{\epsilon}^{\widehat{\rho} jk} \partial_j A_k = \frac{1}{\rho} \partial_\phi A^{\widehat{z}} - \partial_z A^{\widehat{\phi}}, \qquad \widetilde{\epsilon}^{\widehat{\phi} jk} \partial_j A_k = \partial_z A^{\widehat{\rho}} - \partial_\rho A^{\widehat{z}},$$
$$\widetilde{\epsilon}^{\widehat{z} jk} \partial_j A_k = \frac{1}{\rho} \left(\partial_\rho \left(\rho A^{\widehat{\phi}} \right) - \partial_\phi A^{\widehat{\rho}} \right). \tag{1.3.156}$$

Spherical coordinates In spherical coordinates $\{r \geq 0, 0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi\}$ the Cartesian components of the 3D Euclidean coordinate vector reads

$$\vec{x} = \left(r \sin(\theta) \cos(\phi), r \sin(\theta) \sin(\phi), r \cos(\theta)\right). \tag{1.3.157}$$

Show that the flat metric is now

$$\mathrm{d}\ell^2 = \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + (\sin\theta)^2 \mathrm{d}\phi^2 \right); \tag{1.3.158}$$

the gradient of a scalar ψ is

$$\nabla^{\widehat{r}} \psi = \partial_r \psi, \qquad \nabla^{\widehat{\theta}} \psi = \frac{1}{r} \partial_\theta \psi, \qquad \nabla^{\widehat{\phi}} \psi = \frac{1}{r \sin\theta} \partial_\phi \psi; \tag{1.3.159}$$

the Laplacian of a scalar ψ is

$$\nabla_i \nabla^i \psi = \frac{1}{r^2}\partial_r \left(r^2 \partial_r \psi\right) + \frac{1}{r^2 \sin\theta}\partial_\theta \left(\sin\theta \cdot \partial_\theta \psi\right) + \frac{1}{r^2 (\sin\theta)^2}\partial_\phi^2 \psi; \tag{1.3.160}$$

the divergence of a vector A reads

$$\nabla_i A^i = \frac{1}{r^2}\partial_r \left(r^2 A^{\widehat{r}}\right) + \frac{1}{r\sin\theta}\partial_\theta \left(\sin\theta \cdot A^{\widehat{\theta}}\right) + \frac{1}{r\sin\theta}\partial_\phi A^{\widehat{\phi}}; \tag{1.3.161}$$

and the curl of a vector A is given by

$$\widetilde{\epsilon}^{\widehat{r}jk}\partial_j A_k = \frac{1}{r\sin\theta}\left(\partial_\theta (\sin\theta \cdot A^{\widehat{\phi}}) - \partial_\phi A^{\widehat{\theta}}\right), \qquad \widetilde{\epsilon}^{\widehat{\theta}jk}\partial_j A_k = \frac{1}{r\sin\theta}\partial_\phi A^{\widehat{r}} - \frac{1}{r}\partial_r (r A^{\widehat{\phi}}),$$
$$\widetilde{\epsilon}^{\widehat{\phi}jk}\partial_j A_k = \frac{1}{r}\left(\partial_r \left(r A^{\widehat{\theta}}\right) - \partial_\theta A^{\widehat{r}}\right). \tag{1.3.162}$$

$\square$

1.4 Hypersurfaces

1.4.1 Induced Metrics

There are many physical and mathematical problems where we wish to study some $(N < D)$-dimensional (hyper)surface residing (aka embedded) in a D dimensional ambient space. One way to describe this surface is to first endow it with N coordinates $\{\xi^{\rm I} | {\rm I} = 1, 2, \dots, N\}$, whose indices we will denote with capital letters to distinguish from the D coordinates $\{x^i\}$ parametrizing the ambient space. Then the position of the point $\vec{\xi}$ on this hypersurface in the ambient perspective is given by $\vec{x}(\vec{\xi})$. Distances on this hypersurface can be measured using the ambient metric by restricting the latter on the former, i.e.,

$$g_{ij} \mathrm{d}x^i \mathrm{d}x^j \to g_{ij}\left(\vec{x}(\vec{\xi})\right) \frac{\partial x^i(\vec{\xi})}{\partial \xi^{\rm I}} \frac{\partial x^j(\vec{\xi})}{\partial \xi^{\rm J}} \mathrm{d}\xi^{\rm I} \mathrm{d}\xi^{\rm J} \equiv H_{\rm IJ}(\vec{\xi}) \mathrm{d}\xi^{\rm I} \mathrm{d}\xi^{\rm J}. \tag{1.4.1}$$

The $H_{\rm IJ}$ is the (induced) metric on the hypersurface.[26]

Observe that the N vectors

$$\left\{ \frac{\partial x^i}{\partial \xi^{\rm I}} \partial_i \middle| {\rm I} = 1, 2, \dots, N \right\}, \tag{1.4.2}$$

are tangent to this hypersurface. They form a basis set of tangent vectors at a given point $\vec{x}(\vec{\xi})$, but from the ambient D-dimensional perspective. On the other hand, the $\partial/\partial\xi^{\rm I}$ themselves form a basis set of tangent vectors, from the perspective of an observer confined to live on this hypersurface.

Example A simple example is provided by the 2-sphere of radius R embedded in 3D flat space. We already know that it can be parametrized by two angles $\xi^{\rm I} \equiv (0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi)$, such that from the ambient perspective, the sphere is described by

$$x^i(\vec{\xi}) = R(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \qquad \text{(Cartesian components)}. \tag{1.4.3}$$

(Remember R is a fixed quantity here.) The induced metric on the sphere itself, according to eq. (1.4.1), will lead us to the expected result

$$H_{\rm IJ}(\vec{\xi}) \mathrm{d}\xi^{\rm I} \mathrm{d}\xi^{\rm J} = R^2 \left(\mathrm{d}\theta^2 + (\sin\theta)^2 \mathrm{d}\phi^2\right). \tag{1.4.4}$$

Area of 2D surface in 3D flat space A common vector calculus problem is to give some function $f(x, y)$ of two variables, where x and y are to be interpreted as Cartesian coordinates on a flat plane; then proceed to ask what its area is for some specified domain on the (x, y)-plane. We see such a problem can be phrased as a differential geometric one. First, we view f as the z coordinate of some hypersurface embedded in 3-dimensional flat space, so that

$$X^i \equiv (x, y, z) = (x, y, f(x, y)). \tag{1.4.5}$$

[26] The Lorentzian signature of curved space*time*s, as opposed to the Euclidean one in curved spaces, complicates the study of hypersurfaces in the former. One has to distinguish between timelike, spacelike and null surfaces. For a pedagogical discussion see Eric Poisson's *A Relativist's Toolkit* – in fact, much of the material in this section is heavily based on its Chapter 3. Note, however, it is not necessary to know General Relativity to study hypersurfaces in curved spacetimes.

The tangent vectors $(\partial X^i/\partial \xi^{\rm I})$ are

$$\frac{\partial X^i}{\partial x} = (1, 0, \partial_x f), \qquad \frac{\partial X^i}{\partial y} = (0, 1, \partial_y f). \tag{1.4.6}$$

The induced metric, according to eq. (1.4.1), is given by

$$\begin{aligned} H_{\rm IJ}(\vec{\xi}){\rm d}\xi^{\rm I}{\rm d}\xi^{\rm J} &= \delta_{ij}\left(\frac{\partial X^i}{\partial x}\frac{\partial X^j}{\partial x}({\rm d}x)^2 + \frac{\partial X^i}{\partial y}\frac{\partial X^j}{\partial y}({\rm d}y)^2 + 2\frac{\partial X^i}{\partial x}\frac{\partial X^j}{\partial y}{\rm d}x{\rm d}y\right), \\ H_{\rm IJ}(\vec{\xi}) &\doteq \left[\begin{array}{cc} 1+(\partial_x f)^2 & \partial_x f \partial_y f \\ \partial_x f \partial_y f & 1+(\partial_y f)^2 \end{array}\right], \qquad \xi^{\rm I} \equiv (x, y), \end{aligned} \tag{1.4.7}$$

where on the second line the "$\doteq$" means it is "represented by" the matrix to its right – the first row corresponds, from left to right, to the xx, xy components; the second row yx and yy components. Recall that the infinitesimal volume (= 2D area) is given in any coordinate system $\vec{\xi}$ by ${\rm d}^2\xi\sqrt{\det H_{\rm IJ}(\vec{\xi})}$. That means from taking the det of eq. (1.4.7), if the domain on (x, y) is denoted as $\mathfrak{D}$, the corresponding area swept out by f is given by the 2D integral

$$\begin{aligned} \int_{\mathfrak{D}} {\rm d}x{\rm d}y\sqrt{\det H_{\rm IJ}(x,y)} &= \int_{\mathfrak{D}} {\rm d}x{\rm d}y\sqrt{(1+(\partial_x f)^2)(1+(\partial_y f)^2) - (\partial_x f \partial_y f)^2} \\ &= \int_{\mathfrak{D}} {\rm d}x{\rm d}y\sqrt{1+(\partial_x f(x,y))^2 + (\partial_y f(x,y))^2}. \end{aligned} \tag{1.4.8}$$

Differential Forms and Volume Although we have not (and shall not) employ differential forms very much, it is very much part of modern integration theory. One no longer writes $\int {\rm d}^3\vec{x} f(\vec{x})$, for instance, but rather

$$\int f(\vec{x}){\rm d}x^1 \wedge {\rm d}x^2 \wedge {\rm d}x^3. \tag{1.4.9}$$

More generally, whenever the following $N-$form occur under an integral sign, we have the definition

$$\underbrace{{\rm d}x^1 \wedge {\rm d}x^2 \wedge \dots \wedge {\rm d}x^{N-1} \wedge {\rm d}x^N}_{\text{(Differential form notation)}} \equiv \underbrace{{\rm d}^N\vec{x}}_{\text{Physicists' colloquial math-speak}}. \tag{1.4.10}$$

(Here $N \leq D$, where D is the dimension of space.) This needs to be supplemented with the constraint that it is a fully antisymmetric object:

$${\rm d}x^{i_1} \wedge {\rm d}x^{i_2} \wedge \dots \wedge {\rm d}x^{i_{N-1}} \wedge {\rm d}x^{i_N} = \epsilon_{i_1 \dots i_N}{\rm d}x^1 \wedge {\rm d}x^2 \wedge \dots \wedge {\rm d}x^{N-1} \wedge {\rm d}x^N. \tag{1.4.11}$$

The superposition of rank-($N \leq D$) differential forms spanned by $\{(1/N!)F_{i_1 \dots i_N}{\rm d}x^{i_1} \wedge \dots \wedge {\rm d}x^{i_N}\}$, for arbitrary but fully antisymmetric $\{F_{i_1 \dots i_N}\}$, forms a vector space.

Why differential forms are fundamental to integration theory is because, it is this antisymmetry that allows its proper definition as the volume spanned by an $N-$parallelepiped. For one, the antisymmetric nature of forms is responsible for the Jacobian upon a change-of-variables $\vec{x}(\vec{y})$ familiar from multi-variable calculus – using eq. (1.4.11):

$$\begin{aligned}
\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^{N-1} \wedge \mathrm{d}x^N &= \frac{\partial x^1}{\partial y^{i_1}} \frac{\partial x^2}{\partial y^{i_2}} \cdots \frac{\partial x^N}{\partial y^{i_N}} \mathrm{d}y^{i_1} \wedge \mathrm{d}y^{i_2} \wedge \cdots \wedge \mathrm{d}y^{i_{N-1}} \wedge \mathrm{d}y^{i_N} \\
&= \frac{\partial x^1}{\partial y^{i_1}} \frac{\partial x^2}{\partial y^{i_2}} \cdots \frac{\partial x^N}{\partial y^{i_N}} \epsilon^{i_1 \dots i_N} \mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \cdots \wedge \mathrm{d}y^{N-1} \wedge \mathrm{d}y^N \\
&= \left(\det \frac{\partial x^a}{\partial y^b}\right) \mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \cdots \wedge \mathrm{d}y^{N-1} \wedge \mathrm{d}y^N. \qquad (1.4.12)
\end{aligned}$$

In a $(D \geq 2)-$dimensional flat space, you might be familiar with the statement that D linearly independent vectors define a $D-$parallelepiped. Its volume, in turn, is computed through the determinant of the matrix whose columns (or rows) are these vectors. If we now consider the $(N \leq D)-$form built out of N scalar fields $\{\Phi^{\mathrm{I}} | \mathrm{I} = 1, 2, \dots, N\}$, i.e.,

$$\mathrm{d}\Phi^1 \wedge \cdots \wedge \mathrm{d}\Phi^N, \qquad (1.4.13)$$

let us see how it defines an infinitesimal $N-$volume by generalizing the notion of volume-as-determinants.[27] Focusing on the $N = 2$ case, if $\vec{v} \equiv (p_1 \mathrm{d}x^1, \dots, p_D \mathrm{d}x^D)$ and $\vec{w} \equiv (q_1 \mathrm{d}x^1, \dots, q_D \mathrm{d}x^D)$ are two linearly independent vectors formed from $p_i = \partial_i \Phi^1$ and $q_i = \partial_i \Phi^2$, then

$$\mathrm{d}\Phi^1 \wedge \mathrm{d}\Phi^2 = (p_i \mathrm{d}x^i) \wedge (q_j \mathrm{d}x^j) = p_i q_j \mathrm{d}x^i \wedge \mathrm{d}x^j \qquad (1.4.14)$$

is in fact the 2D area spanned by the parallelepiped defined by $\vec{v}$ and $\vec{w}$. For, since $\mathrm{d}\Phi^1 \wedge \mathrm{d}\Phi^2$ is a coordinate scalar, we may choose a locally flat coordinate system $\{y^i\}$ such that p_i and q_i lie on the $(1, 2)-$plane; i.e., $p_{i>2} = q_{i>2} = 0$ and

$$\begin{aligned}
\mathrm{d}\Phi^1 \wedge \mathrm{d}\Phi^2 = (p_i \mathrm{d}y^i) \wedge (q_j \mathrm{d}y^j) &= p_1 q_2 \mathrm{d}y^1 \wedge \mathrm{d}y^2 + p_2 q_1 \mathrm{d}y^2 \wedge \mathrm{d}y^1 \\
&= (p_1 q_2 - p_2 q_1) \mathrm{d}x^1 \mathrm{d}x^2 = \det \left[\begin{array}{cc} \vec{v} & \vec{w} \end{array} \right]; \qquad (1.4.15)
\end{aligned}$$

where now

$$\vec{v} = \left(\partial_1 \Phi^1 \mathrm{d}y^1, \partial_2 \Phi^1 \mathrm{d}y^2, \vec{0}\right)^T, \qquad (1.4.16)$$

$$\vec{w} = \left(\partial_1 \Phi^2 \mathrm{d}y^1, \partial_2 \Phi^2 \mathrm{d}y^2, \vec{0}\right)^T. \qquad (1.4.17)$$

This argument can be readily extended to higher $2 < N \leq D$.

1.4.2 Fluxes, Gauss-Stokes' theorems, Poincaré lemma

Normal to hypersurface Suppose the hypersurface is $(D-1)$ dimensional, sitting in a D dimensional ambient space. Then it could also be described by first identifying a scalar function of the ambient space $f(\vec{x})$ such that some constant-f surface coincides with the hypersurface,

$$f(\vec{x}) = C \equiv \text{constant}. \qquad (1.4.18)$$

[27] These scalar fields $\{\Phi^{\mathrm{I}}\}$ can also be thought of as coordinates parametrizing some $N-$dimensional sub-space of the ambient $D-$dimensional space.

For example, a 2-sphere of radius R can be defined in Cartesian coordinates $\vec{x}$ as

$$f(\vec{x}) = R^2, \qquad \text{where} \qquad f(\vec{x}) = \vec{x}^2. \tag{1.4.19}$$

Given the function f, we now show that $\mathrm{d}f = 0$ can be used to define a unit normal n^i through

$$n^i \equiv \frac{\nabla^i f}{\sqrt{\nabla^j f \nabla_j f}} = \frac{g^{ik}\partial_k f}{\sqrt{g^{lm}\nabla_l f \nabla_m f}}. \tag{1.4.20}$$

That n^i is of unit length can be checked by a direct calculation. For n^i to be normal to the hypersurface means, when dotted into the latter's tangent vectors from our previous discussion, it returns zero:

$$\frac{\partial x^i(\vec{\xi})}{\partial \xi^{\mathrm{I}}} \partial_i f(\vec{x}) \bigg|_{\text{on hypersurface}} = \frac{\partial}{\partial \xi^{\mathrm{I}}} f\left(\vec{x}(\vec{\xi})\right) = \partial_{\mathrm{I}} f(\vec{\xi}) = 0. \tag{1.4.21}$$

The second and third equalities constitute just a re-statement that f is constant on our hypersurface. Using n^i we can also write down the induced metric on the hypersurface as

$$H_{ij} = g_{ij} - n_i n_j. \tag{1.4.22}$$

By induced metric H_{ij} on the hypersurface of one lower dimension than that of the ambient D-space, we mean that the "dot product" of two vectors v^i and w^i, say, is

$$H_{ij} v^i w^j = g_{ij} v^i_{\parallel} w^j_{\parallel}; \tag{1.4.23}$$

where $v^i_{\parallel}$ and $w^i_{\parallel}$ are v^i and w^i projected along the hyper-surface at hand. In words: $H_{ij} v^i w^i$ is the dot product computed using the ambient metric but with the components of v and w orthogonal to the hypersurface removed. Now,

$$v^i_{\parallel} = H^i{}_j v^j \qquad \text{and} \qquad w^i_{\parallel} = H^i{}_j w^j. \tag{1.4.24}$$

That this construction of $v^i_{\parallel}$ and $w^i_{\parallel}$ yields vectors perpendicular to n^i is because

$$H_{ij} n^j = \left(g_{ij} - n_i n_j\right) n^j = n_i - n_i = 0. \tag{1.4.25}$$

Furthermore, because

$$H^i{}_l H^l{}_j = H^i{}_j, \tag{1.4.26}$$

we deduce

$$H_{ij} v^i w^j = g_{ij} H^i{}_a H^j{}_b v^a w^b. \tag{1.4.27}$$

Problem 1.33. For the 2-sphere in 3-dimensional flat space, defined by eq. (1.4.19), calculate the components of the induced metric H_{ij} in eq. (1.4.22) and compare it that in eq. (1.4.4). Hint: compute $\mathrm{d}\sqrt{\vec{x}^2}$ in terms of $\{\mathrm{d}x^i\}$ and exploit the constraint $\vec{x}^2 = R^2$; then consider what is the $-(n_i \mathrm{d}x^i)^2$ occurring in $H_{ij}\mathrm{d}x^i\mathrm{d}x^j$, when written in spherical coordinates? □

Problem 1.34. Consider some 2-dimensional surface parametrized by $\xi^{\rm I} = (\sigma, \rho)$, whose trajectory in D-dimensional flat space is provided by the Cartesian coordinates $\vec{x}(\sigma, \rho)$. What is the formula analogous to eq. (1.4.8), which yields the area of this 2D surface over some domain $\mathfrak{D}$ on the (σ, ρ) plane? Hint: First ask, "what is the 2D induced metric?" Answer:

$$\text{Area} = \int_{\mathfrak{D}} \mathrm{d}\sigma \mathrm{d}\rho \sqrt{(\partial_\sigma \vec{x})^2 (\partial_\rho \vec{x})^2 - (\partial_\sigma \vec{x} \cdot \partial_\rho \vec{x})^2}, \qquad (\partial_{\rm I} \vec{x})^2 \equiv \partial_{\rm I} x^i \partial_{\rm I} x^j \delta_{ij}. \tag{1.4.28}$$

(This is not too far from the Nambu-Goto action of string theory.) □

Directed surface elements What is the analog of $\mathrm{d}(\vec{\text{Area}})$ from vector calculus? This question is important for the discussion of the curved version of Gauss' theorem, as well as the description of fluxes – rate of flow of, say, a fluid – across surface areas. If we have a $(D-1)$ dimensional hypersurface with induced metric $H_{\rm IJ}(\xi^{\rm K})$, determinant $H \equiv \det H_{\rm IJ}$, and a unit normal n^i to it, then the answer is

$$\mathrm{d}^{D-1}\Sigma_i \equiv \mathrm{d}^{D-1}\vec{\xi} \sqrt{|H(\vec{\xi})|} n_i \left(\vec{x}(\vec{\xi}) \right) \tag{1.4.29}$$

$$= \mathrm{d}^{D-1}\vec{\xi}\, \widetilde{\epsilon}_{i j_1 j_2 \dots j_{D-1}} \left(\vec{x}(\vec{\xi}) \right) \frac{\partial x^{j_1}(\vec{\xi})}{\partial \xi^1} \frac{\partial x^{j_2}(\vec{\xi})}{\partial \xi^2} \dots \frac{\partial x^{j_{D-1}}(\vec{\xi})}{\partial \xi^{D-1}}. \tag{1.4.30}$$

The difference between equations (1.4.29) and (1.4.30) is that the first requires knowing the normal vector beforehand, while the second description is purely intrinsic to the hypersurface and can be computed once its parametrization $\vec{x}(\vec{\xi})$ is provided. Also be aware that the choice of orientation of the $\{\xi^{\rm I}\}$ should be consistent with that of the ambient $\{\vec{x}\}$ and the infinitesimal volume $\mathrm{d}^D \vec{x} \sqrt{|g|} \epsilon_{12\dots D}$.

The $\mathrm{d}^{D-1}\xi \sqrt{|H|}$ is the (scalar) infinitesimal area ($= (D-1)$-volume) and n_i provides the direction. The second equality requires justification. Let's define $\{\mathcal{E}_{\rm I}{}^i | {\rm I} = 1, 2, 3, \dots, D-1\}$ to be the $(D-1)$ vector fields

$$\mathcal{E}_{\rm I}{}^i(\vec{\xi}) \equiv \frac{\partial x^i(\vec{\xi})}{\partial \xi^{\rm I}}. \tag{1.4.31}$$

Problem 1.35. Show that the tensor in eq. (1.4.30),

$$\widetilde{n}_i \equiv \widetilde{\epsilon}_{i j_1 j_2 \dots j_{D-1}} \mathcal{E}_1{}^{j_1} \dots \mathcal{E}_{D-1}{}^{j_{D-1}} \tag{1.4.32}$$

is orthogonal to all the $(D-1)$ vectors $\{\mathcal{E}_{\rm I}{}^i\}$. Since n_i is the sole remaining direction in the D space, $\widetilde{n}_i$ must be proportional to n_i

$$\widetilde{n}_i = \varphi \cdot n_i. \tag{1.4.33}$$

To find φ we merely have to dot both sides with n^i,

$$\varphi(\vec{\xi}) = \sqrt{|g(\vec{x}(\vec{\xi}))|} \epsilon_{i j_1 j_2 \dots j_{D-1}} n^i \frac{\partial x^{j_1}(\vec{\xi})}{\partial \xi^1} \dots \frac{\partial x^{j_{D-1}}(\vec{\xi})}{\partial \xi^{D-1}}. \tag{1.4.34}$$

Given a point of the surface $\vec{x}(\vec{\xi})$ we can always choose the coordinates $\vec{x}$ of the ambient space such that, at least in a neighborhood of this point, x^1 refers to the direction orthogonal to the surface and the $\{x^2, x^3, \dots, x^D\}$ lie on the surface itself. Argue that, in this coordinate system, eq. (1.4.20) becomes

$$n^i = \frac{g^{(i)(1)}}{\sqrt{g^{(1)(1)}}}, \tag{1.4.35}$$

and therefore eq. (1.4.34) reads

$$\varphi(\vec{\xi}) = \sqrt{|g(\vec{x}(\vec{\xi}))|}\sqrt{g^{(1)(1)}}. \tag{1.4.36}$$

Cramer's rule (cf. (**??**)) from matrix algebra reads: the ij component (the ith row and jth column) of the inverse of a matrix $(A^{-1})_{ij}$ is $((-)^{i+j}/\det A)$ times the determinant of A with the jth row and ith column removed. Use this and the definition of the induced metric to conclude that

$$\varphi(\vec{\xi}) = \sqrt{|H(\vec{\xi})|}, \tag{1.4.37}$$

thereby proving the equality of equations (1.4.29) and (1.4.30). □

Gauss' theorem We are now ready to state (without proof) *Gauss' theorem.* In 3D vector calculus, Gauss tells us the volume integral, over some domain $\mathfrak{D}$, of the divergence of a vector field is equal to the flux of the same vector field across the boundary $\partial\mathfrak{D}$ of the domain. Exactly the same statement applies in a D dimensional ambient curved space with some closed $(D-1)$ dimensional hypersurface that defines $\partial\mathfrak{D}$.

> Let V^i be an arbitrary vector field, and let $\vec{x}(\vec{\xi})$ describe this closed boundary surface so that it has an (outward) directed surface element $\mathrm{d}^{D-1}\Sigma_i$ given by equations (1.4.29) and (1.4.30). Then
>
> $$\int_{\mathfrak{D}} \mathrm{d}^D x \sqrt{|g(\vec{x})|}\nabla_i V^i(\vec{x}) = \int_{\partial\mathfrak{D}} \mathrm{d}^{D-1}\Sigma_i V^i\left(\vec{x}(\vec{\xi})\right). \tag{1.4.38}$$

Flux Just as in 3D vector calculus, the $\mathrm{d}^{D-1}\Sigma_i V^i$ can be viewed as the flux of some fluid described by V^i across an infinitesimal element of the hypersurface $\partial\mathfrak{D}$.
Remark Gauss' theorem is not terribly surprising if you recognize the integrand as a total derivative,

$$\sqrt{|g|}\nabla_i V^i = \partial_i(\sqrt{|g|}V^i) \tag{1.4.39}$$

(recall eq. (1.3.76)) and therefore it should integrate to become a surface term ($\equiv (D-1)$-dimensional integral). The right hand side of eq. (1.4.38) merely makes this surface integral explicit, in terms of the coordinates $\vec{\xi}$ describing the boundary $\partial\mathfrak{D}$.
Closed surface Note that if you apply Gauss' theorem eq. (1.4.38), on a closed surface such as the sphere, the result is immediately zero. A closed surface is one where there are no boundaries. (For the 2-sphere, imagine starting with the Northern Hemisphere; the boundary is then the equator. By moving this boundary south-wards, i.e., from one latitude line to the next, until it vanishes at the South Pole – our boundary-less surface becomes the 2-sphere.) Since there are no boundaries, the right hand side of eq. (1.4.38) is automatically zero.

Problem 1.36. We may see this directly for the 2-sphere case. The metric on the 2-sphere of radius R is

$$\mathrm{d}\ell^2 = R^2(\mathrm{d}\theta^2 + (\sin\theta)^2\mathrm{d}\phi^2), \qquad \theta \in [0,\pi],\ \phi \in [0,2\pi). \tag{1.4.40}$$

Let V^i be an arbitrary smooth vector field on the 2-sphere. Show explicitly – namely, do the integral – that

$$\int_{\mathbb{S}^2} \mathrm{d}^2 x \sqrt{|g(\vec{x})|}\nabla_i V^i = 0. \tag{1.4.41}$$

Hint: For the ϕ-integral, remember that $\phi = 0$ and $\phi = 2\pi$ refer to the same point, for a fixed θ. □

Problem 1.37. Hudge dual formulation of Gauss' theorem in D-space. Let us consider the Hodge dual of the vector field in eq. (1.4.38),

$$\widetilde{V}_{i_1 \dots i_{D-1}} \equiv \widetilde{\epsilon}_{i_1 \dots i_{D-1} j} V^j. \tag{1.4.42}$$

First show that

$$\widetilde{\epsilon}^{j i_1 \dots i_{D-1}} \nabla_j \widetilde{V}_{i_1 \dots i_{D-1}} \propto \partial_{[1} \widetilde{V}_{23 \dots D]} \propto \nabla_i V^i. \tag{1.4.43}$$

(Find the proportionality factors.) Then deduce the dual formulation of Gauss' theorem, namely, the relationship between

$$\int_{\mathfrak{D}} \mathrm{d}^D x \partial_{[1} \widetilde{V}_{23 \dots D]} \qquad \text{and} \qquad \int_{\partial\mathfrak{D}} \mathrm{d}^{D-1}\xi \widetilde{V}_{i_1 \dots i_{D-1}} \left(\vec{x}(\vec{\xi})\right) \frac{\partial x^{i_1}(\vec{\xi})}{\partial \xi^1} \cdots \frac{\partial x^{i_{D-1}}(\vec{\xi})}{\partial \xi^{D-1}}. \tag{1.4.44}$$

The $\widetilde{V}_{i_1 \dots i_{D-1}} \partial_{\xi^1} x^{i_1} \dots \partial_{\xi^{D-1}} x^{i_{D-1}}$ can be viewed as the original tensor $\widetilde{V}_{i_1 \dots i_{D-1}}$, but projected onto the boundary $\partial\mathfrak{D}$.

In passing, I should point out, what you have shown in eq. (1.4.44) can be written in a compact manner using differential forms notation:

$$\int_{\mathfrak{D}} \mathrm{d}\widetilde{V} = \int_{\partial\mathfrak{D}} \widetilde{V}, \tag{1.4.45}$$

by viewing the fully antisymmetric object $\widetilde{V}$ as a differential $(D-1)$-form. □

Example: Coulomb potential in flat space A basic application of Gauss' theorem is the derivation of the (spherically symmetric) Coulomb potential of a unit point charge in $D \geq 3$ spatial dimensions, satisfying

$$\nabla_i \nabla^i \psi = -\delta^{(D)}(\vec{x} - \vec{x}') \tag{1.4.46}$$

in flat space. Let us consider as domain $\mathfrak{D}$ the sphere of radius r centered at the point charge at $\vec{x}'$. Using spherical coordinates, $\vec{x} = r\widehat{n}(\vec{\xi})$, where $\widehat{n}$ is the unit radial vector emanating from $\vec{x}'$, the induced metric on the boundary $\partial\mathfrak{D}$ is simply the metric of the $(D-1)$-sphere. We now identify in eq. (1.4.38) $V^i = \nabla^i \psi$. The normal vector is simply $n^i \partial_i = \partial_r$, and so Gauss' law using eq. (1.4.29) reads

$$-1 = \int_{\mathbb{S}^{D-1}} \mathrm{d}^{D-1}\vec{\xi}\sqrt{|H|} r^{D-1} \partial_r \psi(r). \tag{1.4.47}$$

The $\int_{\mathbb{S}^{D-1}} \mathrm{d}^{D-1}\vec{\xi}\sqrt{|H|} = 2\pi^{D/2}/\Gamma(D/2)$ is simply the solid angle subtended by the $(D-1)$-sphere ($\equiv$ volume of the $(D-1)$-sphere of unit radius). So at this point we have

$$\partial_r \psi(r) = -\frac{\Gamma(D/2)}{2\pi^{D/2} r^{D-1}} \qquad \Rightarrow \qquad \psi(r) = \frac{\Gamma(D/2)}{4((D-2)/2)\pi^{D/2} r^{D-2}} = \frac{\Gamma(\frac{D}{2}-1)}{4\pi^{D/2} r^{D-2}}. \tag{1.4.48}$$

I have used the Gamma-function identity $\Gamma(z)z = \Gamma(z+1)$. Replacing $r \to |\vec{x} - \vec{x}'|$, we conclude that the Coulomb potential due to a unit strength electric charge is

$$\psi(\vec{x}) = \frac{\Gamma(\frac{D}{2}-1)}{4\pi^{D/2}|\vec{x}-\vec{x}'|^{D-2}}, \qquad D \geq 3. \tag{1.4.49}$$

It is instructive to also use Gauss' law using eq. (1.4.30).

$$-1 = \int_{\mathbb{S}^{D-1}} \mathrm{d}^{D-1}\vec{\xi} \epsilon_{i_1 \dots i_{D-1} j} \frac{\partial x^{i_1}}{\partial \xi^1} \cdots \frac{\partial x^{i_{D-1}}}{\partial \xi^{D-1}} g^{jk}(\vec{x}(\vec{\xi})) \partial_k \psi(r \equiv \sqrt{\vec{x}^2}). \tag{1.4.50}$$

On the surface of the sphere, we have the completeness relation (cf. (**??**)):

$$g^{jk}(\vec{x}(\vec{\xi})) = \delta^{\mathrm{IJ}} \frac{\partial x^j}{\partial \xi^{\mathrm{I}}} \frac{\partial x^k}{\partial \xi^{\mathrm{J}}} + \frac{\partial x^j}{\partial r} \frac{\partial x^k}{\partial r}. \tag{1.4.51}$$

(This is also the coordinate transformation for the inverse metric from Cartesian to Spherical coordinates.) At this point,

$$\begin{aligned}
-1 &= \int_{\mathbb{S}^{D-1}} \mathrm{d}^{D-1}\vec{\xi} \epsilon_{i_1 \dots i_{D-1} j} \frac{\partial x^{i_1}}{\partial \xi^1} \cdots \frac{\partial x^{i_{D-1}}}{\partial \xi^{D-1}} \left(\delta^{\mathrm{IJ}} \frac{\partial x^j}{\partial \xi^{\mathrm{I}}} \frac{\partial x^k}{\partial \xi^{\mathrm{J}}} + \frac{\partial x^j}{\partial r} \frac{\partial x^k}{\partial r} \right) \partial_k \psi(r \equiv \sqrt{\vec{x}^2}) \\
&= \int_{\mathbb{S}^{D-1}} \mathrm{d}^{D-1}\vec{\xi} \epsilon_{i_1 \dots i_{D-1} j} \frac{\partial x^{i_1}}{\partial \xi^1} \cdots \frac{\partial x^{i_{D-1}}}{\partial \xi^{D-1}} \frac{\partial x^j}{\partial r} \left(\frac{\partial x^k}{\partial r} \partial_k \psi(r \equiv \sqrt{\vec{x}^2}) \right).
\end{aligned} \tag{1.4.52}$$

The Levi-Civita symbol contracted with the Jacobians can now be recognized as simply the determinant of the D-dimensional metric written in spherical coordinates $\sqrt{|g(r, \vec{\xi})|}$. (Note the determinant is positive because of the way we ordered our coordinates.) That is in fact equal to $\sqrt{|H(r, \vec{\xi})|}$ because $g_{rr} = 1$. Whereas $(\partial x^k / \partial r) \partial_k \psi = \partial_r \psi$. We have therefore recovered the previous result using eq. (1.4.29).

Problem 1.38. Coulomb Potential in 2D Use the above arguments to show, the solution to

$$\nabla_i \nabla^i \psi = -\delta^{(2)}(\vec{x} - \vec{x}') \tag{1.4.53}$$

is

$$\psi(\vec{x}) = -\frac{\ln(L^{-1}|\vec{x} - \vec{x}'|)}{2\pi}. \tag{1.4.54}$$

Here, L is an arbitrary length scale. Why is there is a restriction $D \geq 3$ in eq. (1.4.49)? □

Tensor elements Suppose we have a $(N < D)$-dimensional domain $\mathfrak{D}$ parametrized by $\{\vec{x}(\xi^{\mathrm{I}})|\mathrm{I} = 1, 2, \dots, N\}$ whose boundary $\partial\mathfrak{D}$ is parametrized by $\{\vec{x}(\theta^{\mathfrak{A}})|\mathfrak{A} = 1, 2, \dots, N-1\}$. We may define a $(D-N)$-tensor element that generalizes the one in eq. (1.4.30)

$$\mathrm{d}^N\Sigma_{i_1\dots i_{D-N}} \equiv \mathrm{d}^N\xi\, \widetilde{\epsilon}_{i_1\dots i_{D-N}j_1 j_2\dots j_N}\left(\vec{x}(\vec{\xi})\right) \frac{\partial x^{j_1}(\vec{\xi})}{\partial \xi^1}\frac{\partial x^{j_2}(\vec{\xi})}{\partial \xi^2}\dots\frac{\partial x^{j_N}(\vec{\xi})}{\partial \xi^N}. \tag{1.4.55}$$

We may further define the boundary surface element

$$\mathrm{d}^{N-1}\Sigma_{i_1\dots i_{D-N}k} \equiv \mathrm{d}^{N-1}\theta\, \widetilde{\epsilon}_{i_1\dots i_{D-N}kj_1\dots j_{N-1}}\left(\vec{x}(\vec{\theta})\right) \frac{\partial x^{j_1}(\vec{\theta})}{\partial \theta^1}\frac{\partial x^{j_2}(\vec{\theta})}{\partial \theta^2}\dots\frac{\partial x^{j_{N-1}}(\vec{\theta})}{\partial \theta^{N-1}}. \tag{1.4.56}$$

Stokes' theorem[28] Stokes' theorem is the assertion that, in a $(N < D)$-dimensional simply connected subregion $\mathfrak{D}$ of some D-dimensional ambient space, the divergence of a fully antisymmetric rank $(D-N+1)$ tensor field $B^{i_1\dots i_{D-N}k}$ integrated over the domain $\mathfrak{D}$ can also be expressed as the integral of $B^{i_1\dots i_{D-N}k}$ over its boundary $\partial\mathfrak{D}$. Namely,

$$\int_{\mathfrak{D}} \mathrm{d}^N\Sigma_{i_1\dots i_{D-N}}\nabla_k B^{i_1\dots i_{D-N}k} = \frac{1}{D-N+1}\int_{\partial\mathfrak{D}} \mathrm{d}^{N-1}\Sigma_{i_1\dots i_{D-N}k}B^{i_1\dots i_{D-N}k}, \tag{1.4.57}$$
$$N < D,\ B^{[i_1\dots i_{D-N}k]} = (D-N+1)!B^{i_1\dots i_{D-N}k}.$$

Problem 1.39. Hodge dual formulation of Stokes' theorem. Define

$$\widetilde{B}_{j_1\dots j_{N-1}} \equiv \frac{1}{(D-N+1)!}\widetilde{\epsilon}_{j_1\dots j_{N-1}i_1\dots i_{D-N}k}B^{i_1\dots i_{D-N}k}. \tag{1.4.58}$$

Can you convert eq. (1.4.57) into a relationship between

$$\int_{\mathfrak{D}} \mathrm{d}^N\vec{\xi}\partial_{[i_1}\widetilde{B}_{i_2\dots i_N]}\frac{\partial x^{i_1}}{\partial \xi^1}\dots\frac{\partial x^{i_N}}{\partial \xi^N} \qquad \text{and} \qquad \int_{\partial\mathfrak{D}} \mathrm{d}^{N-1}\vec{\theta}\widetilde{B}_{i_1\dots i_{N-1}}\frac{\partial x^{i_1}}{\partial \theta^1}\dots\frac{\partial x^{i_{N-1}}}{\partial \theta^{N-1}}? \tag{1.4.59}$$

Furthermore, explain why the Jacobians can be "brought inside the derivative".

$$\partial_{[i_1}\widetilde{B}_{i_2\dots i_N]}\frac{\partial x^{i_1}}{\partial \xi^1}\dots\frac{\partial x^{i_N}}{\partial \xi^N} = \frac{\partial x^{i_1}}{\partial \xi^{[1}}\partial_{|i_1|}\left(\frac{\partial x^{i_2}}{\partial \xi^2}\dots\frac{\partial x^{i_N}}{\partial \xi^{N]}}\widetilde{B}_{i_2\dots i_N}\right). \tag{1.4.60}$$

The $|\cdot|$ around i_1 indicate it is *not* to be part of the anti-symmetrization; only do so for the ξ-indices.

Like for Gauss' theorem, we point out that – by viewing $\widetilde{B}_{j_1\dots j_{N-1}}$ as components of a $(N-1)$-form, Stokes' theorem in eq. (1.4.57) reduces to the simple expression

$$\int_{\mathfrak{D}} \mathrm{d}\widetilde{B} = \int_{\partial\mathfrak{D}} \widetilde{B}. \tag{1.4.61}$$

□

[28] Just like for the Gauss' theorem case, in equations (1.4.55) and (1.4.56), the $\vec{\xi}$ and $\vec{\theta}$ coordinate systems need to be defined with orientations consistent with the ambient $\mathrm{d}^D\vec{x}\sqrt{|g(\vec{x})|}\epsilon_{12\dots D}$ one.

Relation to 3D vector calculus Stokes' theorem in vector calculus states that the flux of the curl of a vector field over some 2D domain $\mathfrak{D}$ sitting in the ambient 3D space, is equal to the line integral of the same vector field along the boundary $\partial\mathfrak{D}$ of the domain. Because eq. (1.4.57) may not appear, at first sight, to be related to the Stokes' theorem from 3D vector calculus, we shall work it out in some detail.

Problem 1.40. Consider some 2D hypersurface $\mathfrak{D}$ residing in a 3D curved space. For simplicity, let us foliate $\mathfrak{D}$ with constant ρ surfaces; let the other coordinate be ϕ, so $\vec{x}(0 \leq \rho \leq \rho_>, 0 \leq \phi \leq 2\pi)$ describes a given point on $\mathfrak{D}$ and the boundary $\partial\mathfrak{D}$ is given by the closed loop $\vec{x}(\rho = \rho_>, 0 \leq \phi \leq 2\pi)$. Let

$$B^{ik} \equiv \widetilde{\epsilon}^{ikj} A_j \tag{1.4.62}$$

for some vector field A^j. This implies in Cartesian coordinates,

$$\nabla_k B^{ik} = \left(\vec{\nabla} \times \vec{A}\right)^i. \tag{1.4.63}$$

Denote $\vec{\xi} = (\rho, \phi)$. Show that Stokes' theorem in eq. (1.4.57) reduces to the $N = 2$ vector calculus case:

$$\int_0^{\rho_>} \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\phi \sqrt{|H(\vec{\xi})|}\vec{n} \cdot \left(\vec{\nabla} \times \vec{A}\right) = \int_0^{2\pi} \mathrm{d}\phi \frac{\partial \vec{x}(\rho_>, \phi)}{\partial \phi} \cdot \vec{A}(\vec{x}(\rho_>, \phi)). \tag{1.4.64}$$

where the unit normal vector is given by

$$\vec{n} = \frac{(\partial \vec{x}(\vec{\xi})/\partial\rho) \times (\partial \vec{x}(\vec{\xi})/\partial\phi)}{\left|(\partial \vec{x}(\vec{\xi})/\partial\rho) \times (\partial \vec{x}(\vec{\xi})/\partial\phi)\right|}. \tag{1.4.65}$$

Of course, once you've verified Stokes' theorem for a particular coordinate system, you know by general covariance it holds in any coordinate system, i.e.,

$$\int_{\mathfrak{D}} \mathrm{d}^2\xi \sqrt{|H(\vec{\xi})|} n_i \widetilde{\epsilon}^{ijk} \partial_j A_k = \int_{\partial\mathfrak{D}} A_i \mathrm{d}x^i. \tag{1.4.66}$$

Step-by-step guide: Start with eq. (1.4.30), and show that in a Cartesian basis,

$$\mathrm{d}^2\Sigma_i = \mathrm{d}^2\xi \left(\frac{\partial \vec{x}}{\partial \rho} \times \frac{\partial \vec{x}}{\partial \phi}\right)^i. \tag{1.4.67}$$

The induced metric on the 2D domain $\mathfrak{D}$ is

$$H_{\mathrm{IJ}} = \delta_{ij} \partial_{\mathrm{I}} x^i \partial_{\mathrm{J}} x^j. \tag{1.4.68}$$

Work out its determinant. Then work out

$$\left|(\partial\vec{x}/\partial\rho) \times (\partial\vec{x}/\partial\phi)\right|^2 \tag{1.4.69}$$

using the identity

$$\widetilde{\epsilon}^{ijk}\widetilde{\epsilon}_{lmk} = \delta^i_l \delta^j_m - \delta^i_m \delta^j_l. \tag{1.4.70}$$

Can you thus relate $\sqrt{|H(\vec{\xi})|}$ to $|(\partial\vec{x}/\partial\rho) \times (\partial\vec{x}/\partial\phi)|$, and thereby verify the left hand side of eq. (1.4.57) yields the left hand side of (1.4.64)?

For the right hand side of eq. (1.4.64), begin by arguing that the boundary (line) element in eq. (1.4.56) becomes

$$\mathrm{d}\Sigma_{ki} = \mathrm{d}\phi\, \widetilde{\epsilon}_{kij} \frac{\partial x^j}{\partial \phi}. \tag{1.4.71}$$

Then use $\widetilde{\epsilon}^{ij_1j_2}\widetilde{\epsilon}_{kj_1j_2} = 2\delta^i_k$ to then show that the right hand side of eq. (1.4.57) is now that of eq. (1.4.64). □

Problem 1.41. Discuss how the tensor element in eq. (1.4.55) transforms under a change of hypersurface coordinates $\vec{\xi} \to \vec{\xi}(\vec{\xi}')$. Do the same for the tensor element in eq. (1.4.56): how does it transforms under a change of hypersurface coordinates $\vec{\theta} \to \vec{\theta}(\vec{\theta}')$? □

Poincaré Lemma In 3D vector calculus you have learned that a vector $\vec{B}$ is divergence-less everywhere in space iff it is the curl of another vector $\vec{A}$.

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \Leftrightarrow \qquad \vec{B} = \vec{\nabla} \times \vec{A}. \tag{1.4.72}$$

And, the curl of a vector $\vec{B}$ is zero everywhere in space iff it is the gradient of scalar ψ.

$$\vec{\nabla} \times \vec{B} = 0 \qquad \Leftrightarrow \qquad \vec{B} = \vec{\nabla}\psi. \tag{1.4.73}$$

Here, we shall see that these statements are special cases of the following.

> *Poincaré lemma* In an arbitrary D dimensional curved space, let $B_{i_1 \dots i_N}(\vec{x})$ be a fully antisymmetric rank-N tensor field, with $N \leq D$. Then, everywhere within a simply connected region of space,
>
> $$B_{i_1 \dots i_N} = \partial_{[i_1} C_{i_2 \dots i_N]}, \tag{1.4.74}$$
>
> – i.e., B is the "curl" of a fully antisymmetric rank-$(N-1)$ tensor C – if and only if
>
> $$\partial_{[j} B_{i_1 \dots i_N]} = 0. \tag{1.4.75}$$
>
> In differential form notation, by treating C as a $(N-1)$-form and B as a N-form, Poincaré would read: throughout a simply connected region of space,
>
> $$\mathrm{d}B = 0 \text{ iff } B = \mathrm{d}C. \tag{1.4.76}$$

Example I: Electromagnetism Let us recover the 3D vector calculus statement above, that the divergence-less nature of the magnetic field is equivalent to it being the curl of some vector field. Consider the dual of the magnetic field B^i:

$$\widetilde{B}^{ij} \equiv \widetilde{\epsilon}^{ijk} B_k. \tag{1.4.77}$$

The Poincaré Lemma says $\widetilde{B}_{ij} = \partial_{[i}A_{j]}$ if and only if $\partial_{[k}\widetilde{B}_{ij]} = 0$ everywhere in space. We shall proceed to take the dual of these two conditions. Via eq. (1.3.118), the first is equivalent to

$$\begin{aligned} \widetilde{\epsilon}^{kij}\widetilde{B}_{ij} &= \widetilde{\epsilon}^{kij}\partial_{[i}A_{j]}, \\ &= 2\widetilde{\epsilon}^{kij}\partial_i A_j. \end{aligned} \tag{1.4.78}$$

On the other hand, employing eq. (1.3.118),

$$\widetilde{\epsilon}^{kij}\widetilde{B}_{ij} = \widetilde{\epsilon}^{kij}\widetilde{\epsilon}_{ijl}B^l = 2B^k; \tag{1.4.79}$$

and therefore $\vec{B}$ is the curl of A_i:

$$B^k = \widetilde{\epsilon}^{kij}\partial_i A_j \tag{1.4.80}$$

While the latter condition $\mathrm{d}\widetilde{B} = 0$ is, again utilizing eq. (1.3.118), equivalent to

$$\begin{aligned} 0 &= \widetilde{\epsilon}^{kij}\partial_k\widetilde{B}_{ij} \\ &= \widetilde{\epsilon}_{kij}\widetilde{\epsilon}^{ijl}\nabla_k B_l = 2\nabla_l B^l. \end{aligned} \tag{1.4.81}$$

That is, the divergence of $\vec{B}$ is zero.

Example II A simple application is that of the line integral

$$I(\vec{x}, \vec{x}'; \mathfrak{P}) \equiv \int_{\mathfrak{P}} A_i \mathrm{d}x^i, \tag{1.4.82}$$

where $\mathfrak{P}$ is some path in D-space joining $\vec{x}'$ to $\vec{x}$. Poincaré tells us, if $\partial_{[i}A_{j]} = 0$ everywhere in space, then $A_i = \partial_i\varphi$, the A_i is a gradient of a scalar φ. Then $A_i\mathrm{d}x^i = \partial_i\varphi\mathrm{d}x^i = \mathrm{d}\varphi$, and the integral itself is actually path independent – it depends only on the end points:

$$\int_{\vec{x}'}^{\vec{x}} A_i \mathrm{d}x^i = \int_{\mathfrak{P}} \mathrm{d}\varphi = \varphi(\vec{x}) - \varphi(\vec{x}'), \qquad \text{whenever } \partial_{[i}A_{j]} = 0. \tag{1.4.83}$$

Problem 1.42. Make a similar translation, from the Poincaré Lemma, to the 3D vector calculus statement that a vector B is curl-less if and only if it is a pure gradient everywhere. □

Problem 1.43. Consider the vector potential, written in 3D Cartesian coordinates,

$$A_i \mathrm{d}x^i = \frac{x^1\mathrm{d}x^2 - x^2\mathrm{d}x^1}{(x^1)^2 + (x^2)^2}. \tag{1.4.84}$$

Can you calculate

$$F_{ij} = \partial_{[i}A_{j]}? \tag{1.4.85}$$

Consider a 2D surface whose boundary $\partial\mathfrak{D}$ circle around the $(0, 0, -\infty < x^3 < +\infty)$ line once. Can you use Stokes' theorem to show that

$$F_{ij} = 2\pi\epsilon_{ij3}\delta(x^1)\delta(x^2)? \tag{1.4.86}$$

Hint: Convert from Cartesian to polar coordinates $(x, y, z) = (r\cos\phi, r\sin\phi, z)$; the line integral on the right hand side of eq. (1.4.66) should simplify considerably. This problem illustrates the subtlety regarding the "simply connected" requirement of the Poincaré lemma. The magnetic field F_{ij} here describes that of a highly localized solenoid lying along the z-axis; its corresponding vector potential is a pure gradient in any simply connected 3−volume not containing the z-axis, but it is no longer a pure gradient in say a solid torus region encircling (but still not containing) it. □

2 Differential Geometry In Curved Spacetimes

We now move on to differential geometry in curved spacetimes. I assume the reader is familiar with basic elements of Special Relativity and with the discussion in §(1) – in many instances, I will simply bring over the results from there to the curved spacetime context. In §(2.1) I discuss Lorentz/Poincaré symmetry in flat spacetime, since it is fundamental to both Special and General Relativity. I then cover curved spacetime differential geometry proper from §(2.3) through §(2.5), focusing on issues not well developed in §(1). These three sections, together with §(1), are intended to form the first portion – the *kinematics* of curved space(time)s[29] – of a course on gravitation. Following that, §(2.6) contains somewhat specialized content regarding the expansion of geometric quantities off some fixed 'background' geometry; and finally, in §(2.7) we compile conformal transformation properties of geometric objects.

2.1 Constancy of c, Poincaré and Lorentz symmetry

We begin in flat (aka *Minkowski*) spacetime written in Cartesian coordinates $\{x^\mu \equiv (t, \vec{x})\}$. The 'square' of the distance between x^μ and $x^\mu + \mathrm{d}x^\mu$, is given by a 'modified Pythagoras' theorem' of sorts:

$$\begin{aligned} \mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu &= (\mathrm{d}x^0)^2 - \mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x} \\ &= (\mathrm{d}t)^2 - \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j; \end{aligned} \tag{2.1.1}$$

where the Minkowski metric tensor reads

$$\eta_{\mu\nu} \doteq \mathrm{diag}[1, -1, \dots, -1]. \tag{2.1.2}$$

The inverse metric $\eta^{\mu\nu}$ is simply the matrix inverse, $\eta^{\alpha\sigma}\eta_{\sigma\beta} = \delta^\alpha_\beta$; it is numerically equal to the flat metric itself:

$$\eta^{\mu\nu} \doteq \mathrm{diag}\left[1, -1, \dots, -1\right]. \tag{2.1.3}$$

Strictly speaking we should be writing eq. (2.1.1) in the 'dimensionally-correct' form

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}; \tag{2.1.4}$$

where c is the speed of light and $[\mathrm{d}s^2] = [\text{Length}^2]$. However, as explained in §(**??**), since the speed of light shows up frequently in relativity and gravitational physics, it is often advantageous to set $c = 1$, which in turn means all speeds are measured using c as the base unit. ($v = 0.23$ would mean $v = 0.23c$, for instance.) We shall do so throughout this section.

Notice too, we have switched from Latin/English alphabets in §(1), say $i, j, k, \dots \in \{1, 2, 3, \dots, D\}$ to Greek ones $\mu, \nu, \dots \in \{0, 1, 2, \dots, D \equiv d - 1\}$; the former run over the spatial coordinates while the latter over time (0th) and space $(1, \dots, D)$. Also note that the opposite 'mostly plus' sign convention $\eta_{\mu\nu} = \mathrm{diag}[-1, +1, \dots, +1]$ is equally valid and, in fact, more popular in the contemporary physics literature.

[29] As opposed to the dynamics of spacetime, which involves studying General Relativity, Einstein's field equations for the metric, and its applications.

Constancy of c One of the primary motivations that led Einstein to recognize eq. (2.1.1) as the proper geometric setting to describe physics, is the realization that the speed of light c is constant in all inertial frames. In modern physics, the latter is viewed as a consequence of spacetime translation and Lorentz symmetry, as well as the null character of the trajectories swept out by photons. That is, for transformation matrices $\{\Lambda\}$ satisfying

$$\Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}\eta_{\alpha\beta} = \eta_{\mu\nu}, \tag{2.1.5}$$

and constant vectors $\{a^\mu\}$ we have

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{\mu\nu}\mathrm{d}x'^\mu \mathrm{d}x'^\nu \tag{2.1.6}$$

whenever

$$x^\alpha = \Lambda^\alpha{}_\mu x'^\mu + a^\alpha. \tag{2.1.7}$$

The physical interpretation is that the frames parametrized by $\{x^\mu = (t, \vec{x})\}$ and $\{x'^\mu = (t', \vec{x}')\}$ are *inertial* frames: compact bodies with no external forces acting on them will sweep out geodesics $\mathrm{d}^2x^\mu/\mathrm{d}\tau^2 = 0 = \mathrm{d}^2x'^\mu/\mathrm{d}\tau'^2$, where the proper times τ and τ' are defined through the relations $\mathrm{d}\tau = \mathrm{d}t\sqrt{1-(\mathrm{d}\vec{x}/\mathrm{d}t)^2}$ and $\mathrm{d}\tau' = \mathrm{d}t'\sqrt{1-(\mathrm{d}\vec{x}'/\mathrm{d}t')^2}$. To interpret physical phenomenon taking place in one frame from the other frame's perspective, one would first have to figure out how to translate between x and x'.

Let x^μ be the spacetime Cartesian coordinates of a single photon; in a different Lorentz frame it has Cartesian coordinates x'^μ. Invoking its null character, namely $\mathrm{d}s^2 = 0$ – which holds in any inertial frame – we have $(\mathrm{d}x^0)^2 = \mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}$ and $(\mathrm{d}x'^0)^2 = \mathrm{d}\vec{x}'\cdot\mathrm{d}\vec{x}'$. This in turn tells us the speeds in both frames is unity:

$$\frac{|\mathrm{d}\vec{x}|}{\mathrm{d}x^0} = \frac{|\mathrm{d}\vec{x}'|}{\mathrm{d}x'^0} = 1. \tag{2.1.8}$$

A more thorough and hence deeper justification would be to recognize, it is the sign difference between the 'time' part and the 'space' part of the metric in eq. (2.1.1) – together with its Lorentz invariance – that gives rise to the wave equations obeyed by the photon. Equation (2.1.8) then follows as a consequence.

Problem 2.1. Explain why eq. (2.1.5) is equivalent to the matrix equation

$$\Lambda^T \eta \Lambda = \eta. \tag{2.1.9}$$

Hint: What are $\eta_{\mu\nu}\Lambda^\nu{}_\beta$ and $A^\nu{}_\beta B_{\nu\gamma}$ in matrix notation? □

Moving indices Just like in curved/flat time, tensor indices in flat spacetime are moved with the metric $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$. For example,

$$v^\mu = \eta^{\mu\nu}v_\nu, \qquad v_\mu = \eta_{\mu\nu}v^\nu; \tag{2.1.10}$$

$$T_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}T^{\alpha\beta}, \qquad T^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}T_{\alpha\beta}. \tag{2.1.11}$$

Symmetries We shall define Poincaré transformations[30] $x(x')$ to be the set of all coordinate transformations that leave the flat spacetime metric invariant (cf. eq. (2.1.6)). Poincaré and

[30] Poincaré transformations are also sometimes known as inhomogeneous Lorentz transformations.

Lorentz symmetries play fundamental roles in our understanding of both classical relativistic physics and quantum theories of elementary particle interactions; hence, this motivates us to study it in some detail. As we will now proceed to demonstrate, the most general invertible Poincaré transformation is in fact the one in eq. (2.1.7).

Derivation of eq. (2.1.6)[31] Now, under a coordinate transformation, eq. (2.1.6) reads

$$\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{\mu\nu}\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}\mathrm{d}x'^\alpha \mathrm{d}x'^\beta = \eta_{\alpha'\beta'}\mathrm{d}x'^\alpha \mathrm{d}x'^\beta. \tag{2.1.12}$$

Let us differentiate both sides of eq. (2.1.12) with respect to x'^σ.

$$\eta_{\mu\nu}\frac{\partial^2 x^\mu}{\partial x'^\sigma \partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} + \eta_{\mu\nu}\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial^2 x^\nu}{\partial x'^\sigma \partial x'^\beta} = 0. \tag{2.1.13}$$

Next, consider symmetrizing $\sigma\alpha$ and anti-symmetrizing $\sigma\beta$.

$$2\eta_{\mu\nu}\frac{\partial^2 x^\mu}{\partial x'^\sigma \partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} + \eta_{\mu\nu}\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial^2 x^\nu}{\partial x'^\sigma \partial x'^\beta} + \eta_{\mu\nu}\frac{\partial x^\mu}{\partial x'^\sigma}\frac{\partial^2 x^\nu}{\partial x'^\alpha \partial x'^\beta} = 0 \tag{2.1.14}$$

$$\eta_{\mu\nu}\frac{\partial^2 x^\mu}{\partial x'^\sigma \partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} - \eta_{\mu\nu}\frac{\partial^2 x^\mu}{\partial x'^\beta \partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\sigma} = 0 \tag{2.1.15}$$

Since partial derivatives commute, the second term from the left of eq. (2.1.13) vanishes upon anti-symmetrization of $\sigma\beta$. Adding equations (2.1.14) and (2.1.15) hands us

$$3\eta_{\mu\nu}\frac{\partial^2 x^\mu}{\partial x'^\sigma \partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} + \eta_{\mu\nu}\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial^2 x^\nu}{\partial x'^\sigma \partial x'^\beta} = 0. \tag{2.1.16}$$

Finally, subtracting eq. (2.1.13) from eq. (2.1.16) produces

$$2\eta_{\mu\nu}\frac{\partial^2 x^\mu}{\partial x'^\sigma \partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} = 0. \tag{2.1.17}$$

Because we have assumed Poincaré transformations are invertible, we may contract both sides with $\partial x'^\beta/\partial x^\kappa$.

$$\eta_{\mu\nu}\frac{\partial^2 x^\mu}{\partial x'^\sigma \partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}\frac{\partial x'^\beta}{\partial x^\kappa} = \eta_{\mu\nu}\frac{\partial^2 x^\mu}{\partial x'^\sigma \partial x'^\alpha}\delta^\nu_\kappa = 0. \tag{2.1.18}$$

Finally, we contract both sides with $\eta^{\kappa\rho}$:

$$\eta_{\mu'\kappa'}\eta^{\kappa'\rho}\frac{\partial^2 x^\mu}{\partial x'^\sigma \partial x'^\alpha} = \frac{\partial^2 x^\rho}{\partial x'^\sigma \partial x'^\alpha} = 0. \tag{2.1.19}$$

In words: since the second x'-derivative of x has to vanish, the transformation from x to x' can at most go linearly as x'; it cannot involve higher powers of x'. This implies the form in eq. (2.1.7). Plugging eq. (2.1.7) the latter into eq. (2.1.12), we recover the necessary definition of the Lorentz transformation in eq. (2.1.5).

[31] This argument can be found in Weinberg [1].

Poincaré Transformations The most general invertible coordinate transformations that leave the Cartesian Minkowski metric invariant involve the (spacetime-constant) Lorentz transformations $\{\Lambda^\mu{}_\alpha\}$ of eq (2.1.5) plus constant spacetime translations.

(Homogeneous) Lorentz Transformations form a Group[32] If $\Lambda^\mu{}_\alpha$ and $\Lambda'^\mu{}_\alpha$ denotes different Lorentz transformations, then notice the composition

$$\Lambda''^\mu{}_\alpha \equiv \Lambda^\mu{}_\sigma \Lambda'^\sigma{}_\alpha \tag{2.1.20}$$

is also a Lorentz transformation. For, keeping in mind the fundamental definition in eq. (2.1.5), we may directly compute

$$\begin{aligned} \Lambda''^\mu{}_\alpha \Lambda''^\nu{}_\beta \eta_{\mu\nu} &= \Lambda^\mu{}_\sigma \Lambda'^\sigma{}_\alpha \Lambda^\nu{}_\rho \Lambda'^\rho{}_\beta \eta_{\mu\nu} \\ &= \Lambda'^\sigma{}_\alpha \Lambda'^\rho{}_\beta \eta_{\sigma\rho} = \eta_{\alpha\beta}. \end{aligned} \tag{2.1.21}$$

To summarize:

The set of all Lorentz transformations $\{\Lambda^\mu{}_\alpha\}$ satisfying eq. (2.1.5), together with the composition law in eq. (2.1.20) for defining successive Lorentz transformations, form a *Group*.

Proof Let $\Lambda^\mu{}_\alpha$, $\Lambda'^\mu{}_\alpha$ and $\Lambda''^\mu{}_\alpha$ denote distinct Lorentz transformations.

- *Closure* Above, we have just verified that applying successive Lorentz transformations yields another Lorentz transformation; for e.g., $\Lambda^\mu{}_\sigma \Lambda'^\sigma{}_\nu$ and $\Lambda^\mu{}_\sigma \Lambda'^\sigma{}_\rho \Lambda''^\rho{}_\nu$ are Lorentz transformations.

- *Associativity* Because applying successive Lorentz transformations amount to matrix multiplication, and since the latter is associative, that means Lorentz transformations are associative:

$$\Lambda \cdot \Lambda' \cdot \Lambda'' = \Lambda \cdot (\Lambda' \cdot \Lambda'') = (\Lambda \cdot \Lambda') \cdot \Lambda''. \tag{2.1.22}$$

- *Identity* $\delta^\mu{}_\alpha$ is the identity Lorentz transformation:

$$\delta^\mu{}_\sigma \Lambda^\sigma{}_\nu = \Lambda^\mu{}_\sigma \delta^\sigma{}_\nu = \Lambda^\mu{}_\nu, \tag{2.1.23}$$

  and

$$\delta^\mu{}_\alpha \delta^\nu{}_\beta \eta_{\mu\nu} = \eta_{\alpha\beta}. \tag{2.1.24}$$

- *Inverse* Let us take the determinant of both sides of eq. (2.1.5) – by viewing the latter as matrix multiplication, we have $\Lambda^T \cdot \eta \cdot \Lambda = \eta$, which in turn means

$$(\det \Lambda)^2 = 1 \qquad \Rightarrow \qquad \det \Lambda = \pm 1. \tag{2.1.25}$$

[32] Refer to §(**??**) for the defining axioms of a Group.

Here, we have recalled $\det A^T = \det A$ for any square matrix A. Since the determinant of Λ is strictly non-zero, what eq. (2.1.25) teaches us is that Λ is always invertible: Λ^{-1} is guaranteed to exist. What remains is to check that, if Λ is a Lorentz transformation, so is Λ^{-1}. Starting with the matrix form of eq. (2.1.9), and utilizing $(\Lambda^{-1})^T = (\Lambda^T)^{-1}$,

$$\Lambda^T \eta \Lambda = \eta \tag{2.1.26}$$

$$(\Lambda^T)^{-1} \Lambda^T \eta \Lambda \Lambda^{-1} = (\Lambda^T)^{-1} \cdot \eta \cdot \Lambda^{-1} \tag{2.1.27}$$

$$\eta = (\Lambda^{-1})^T \cdot \eta \cdot \Lambda^{-1}. \tag{2.1.28}$$

Problem 2.2. Remember that indices are moved with the metric, so for example,

$$\Lambda^\mu{}_\alpha \eta_{\mu\nu} = \Lambda_{\nu\alpha}. \tag{2.1.29}$$

First explain how to go from eq. (2.1.5) to

$$\Lambda_\sigma{}^\alpha \Lambda^\sigma{}_\beta = \delta^\alpha_\beta \tag{2.1.30}$$

and deduce the inverse Lorentz transformation

$$\left(\Lambda^{-1}\right)^\alpha{}_\beta = \Lambda_\beta{}^\alpha = \eta_{\beta\nu} \eta^{\alpha\mu} \Lambda^\nu{}_\mu. \tag{2.1.31}$$

(Recall the inverse always exists because $\det \Lambda = \pm 1$.) $\square$

Jacobians Note that under the Poincaré transformation in eq. (2.1.7),

$$\frac{\partial x^\alpha}{\partial x'^\beta} = \Lambda^\alpha{}_\beta, \tag{2.1.32}$$

$$\frac{\partial x'^\alpha}{\partial x^\beta} = \Lambda_\beta{}^\alpha. \tag{2.1.33}$$

This implies

$$\mathrm{d}x^\alpha = \Lambda^\alpha{}_\beta \mathrm{d}x'^\beta, \tag{2.1.34}$$

$$\frac{\partial}{\partial x^\alpha} \equiv \partial_\alpha = \Lambda_\alpha{}^\beta \partial_{\beta'} \equiv \Lambda_\alpha{}^\beta \frac{\partial}{\partial x'^\beta}. \tag{2.1.35}$$

Problem 2.3. Explain why

$$\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta^{\alpha\beta} = \eta^{\mu\nu}. \tag{2.1.36}$$

Hint: Start from eq. (2.1.28). $\square$

Problem 2.4. Under the Poincaré transformation in eq. (2.1.7), show that

$$\eta^{\mu\nu} \partial_\mu \partial_\nu = \eta^{\mu\nu} \partial_{\mu'} \partial_{\nu'}; \tag{2.1.37}$$

where $\partial_\mu \equiv \partial/\partial x^\mu$ and $\partial_{\mu'} \equiv \partial/\partial x'^\mu$. How does

$$\partial^\mu \equiv \eta^{\mu\nu} \partial_\nu \tag{2.1.38}$$

transform under eq. (2.1.7)? $\square$

Problem 2.5. Prove that the Poincaré transformation in eq. (2.1.7) also defines a group. To systemize the discussion, first promote the spacetime coordinates to $d+1$ dimensional objects: $x^{\mathfrak{A}} = (x^\mu, 1)$ and $x'^{\mathfrak{A}} = (x'^\mu, 1)$, with $\mathfrak{A} = 0, 1, 2, \dots, d-1, d$. Then define the matrix

$$\Pi^{\mathfrak{A}}{}_{\mathfrak{B}}[\Lambda, a] = \left[\begin{array}{cc} \Lambda^\mu{}_\nu & a^\mu \\ 0 \dots 0 & 1 \end{array} \right]; \tag{2.1.39}$$

namely, its upper left $d \times d$ block is simply the Lorentz transformation $\Lambda^\mu{}_\nu$; while its rightmost column is $(a^\mu, 1)^T$ and its bottom row is $(0 \dots 0 \ 1)$. First check that $x^{\mathfrak{A}} = \Pi^{\mathfrak{A}}{}_{\mathfrak{B}}[\Lambda, a] x'^{\mathfrak{B}}$ is equivalent to eq. (2.1.7). Then proceed to verify that these set of matrices $\{\Pi^{\mathfrak{A}}{}_{\mathfrak{B}}[\Lambda, a]\}$ for different Lorentz transformations Λ and translation vectors a, with the usual rules of matrix multiplication, together define a group. □

Lorentzian 'inner product' is preserved That Λ is a Lorentz transformation means it is a linear operator that preserves the Lorentzian inner product. For suppose v and w are arbitrary vectors, the inner product of $v' \equiv \Lambda v$ and $w' \equiv \Lambda w$ is that between v and w.

$$v' \cdot w' \equiv \eta_{\alpha\beta} v'^\alpha w'^\beta = \eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu v^\mu w^\nu \tag{2.1.40}$$

$$= \eta_{\mu\nu} v^\mu w^\nu = v \cdot w. \tag{2.1.41}$$

This is very much analogous to rotations in $\mathbb{R}^D$ being the linear transformations that preserve the Euclidean inner product between spatial vectors: $\vec{v} \cdot \vec{w} = \vec{v}' \cdot \vec{w}'$ for all $\widehat{R}^T \widehat{R} = \mathbb{I}_{D\times D}$, where $\vec{v}' \equiv \widehat{R}\vec{v}$ and $\vec{w}' \equiv \widehat{R}\vec{w}$.

We wish to study in some detail what the most general form $\Lambda^\mu{}_\alpha$ may take. To this end, we shall do so by examining how it acts on some arbitrary vector field v^μ. Even though this section deals with Minkowski spacetime, this v^μ may also be viewed as a vector in a curved spacetime written in an orthonormal basis.

Rotations Let us recall that any spatial vector v^i may be rotated to point along the $1-$axis while preserving its Euclidean length. That is, there is always a $\widehat{R}$, obeying $\widehat{R}^T \widehat{R} = \mathbb{I}$ such that

$$\widehat{R}^i{}_j v^j \doteq \pm |\vec{v}| (1, 0, \dots, 0)^T, \qquad |\vec{v}| \equiv \sqrt{\delta_{ij} v^i v^j}. \tag{2.1.42}$$

[33]Conversely, since $\widehat{R}$ is necessarily invertible, any spatial vector v^i can be obtained by rotating it from $|\vec{v}|(1, \vec{0}^T)$. Moreover, in $D+1$ notation, these rotation matrices can be written as

$$\widehat{R}^\mu{}_\nu \doteq \left[\begin{array}{cc} 1 & \vec{0}^T \\ \vec{0} & \widehat{R}^i{}_j \end{array} \right] \tag{2.1.43}$$

$$\widehat{R}^0{}_\nu v^\nu = v^0, \tag{2.1.44}$$

$$\widehat{R}^i{}_\nu v^\nu = \widehat{R}^i{}_j v^j = (\pm|\vec{v}|, 0, \dots, 0)^T. \tag{2.1.45}$$

[33]This $\widehat{R}$ is not unique: for example, by choosing another rotation matrix $\widehat{R}''$ that only rotates the space orthogonal to v^i, $\widehat{R}\widehat{R}''\vec{v}$ and $\widehat{R}\vec{v}$ both yield the same result.

These considerations tell us, if we wish to study Lorentz transformations that are *not* rotations, we may reduce their study to the $(1+1)$D case. To see this, we first observe that

$$\Lambda \begin{bmatrix} v^0 \\ v^1 \\ \vdots \\ v^D \end{bmatrix} = \Lambda \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & \widehat{R} \end{bmatrix} \begin{bmatrix} v^0 \\ \pm|\vec{v}| \\ \vec{0} \end{bmatrix}. \tag{2.1.46}$$

And if the result of this matrix multiplication yields non-zero spatial components, namely $(v'^0, v'^1, \dots, v'^D)^T$, we may again find a rotation matrix $\widehat{R}'$ such that

$$\Lambda \begin{bmatrix} v^0 \\ v^1 \\ \vdots \\ v^D \end{bmatrix} = \begin{bmatrix} v'^0 \\ v'^1 \\ \vdots \\ v'^D \end{bmatrix} = \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & \widehat{R}' \end{bmatrix} \begin{bmatrix} v'^0 \\ \pm|\vec{v}'| \\ \vec{0} \end{bmatrix}. \tag{2.1.47}$$

At this point, we have reduced our study of Lorentz transformations to

$$\begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & \widehat{R}'^T \end{bmatrix} \Lambda \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & \widehat{R} \end{bmatrix} \begin{bmatrix} v^0 \\ v^1 \\ \vec{0} \end{bmatrix} \equiv \Lambda' \begin{bmatrix} v^0 \\ v^1 \\ \vec{0} \end{bmatrix} = \begin{bmatrix} v'^0 \\ v'^1 \\ \vec{0} \end{bmatrix}. \tag{2.1.48}$$

Because Λ was arbitrary so is Λ', since one can be gotten from another via rotations.

Time Reversal & Parity Flips Suppose the time component of the vector v^μ were negative $(v^0 < 0)$, we may write it as

$$\begin{bmatrix} -|v^0| \\ \vec{v} \end{bmatrix} = \widehat{T} \begin{bmatrix} |v^0| \\ \vec{v} \end{bmatrix}, \qquad \widehat{T} \equiv \begin{bmatrix} -1 & \vec{0}^T \\ \vec{0} & \mathbb{I}_{D\times D} \end{bmatrix}; \tag{2.1.49}$$

where $\widehat{T}$ is the time reversal matrix since it reverses the sign of the time component of the vector. You may readily check that $\widehat{T}$ itself is a Lorentz transformation in that it satisfies $\widehat{T}^T \eta \widehat{T} = \eta$.

Problem 2.6. Parity flip of the ith axis Suppose we wish to flip the sign of the ith spatial component of the vector, namely $v^i \to -v^i$. You can probably guess, this may be implemented via the diagonal matrix with all entries set to unity, except the ith component – which is set instead to -1.

$$_i\widehat{P}^\mu{}_\nu v^\nu = v^\mu, \qquad \mu \neq i, \tag{2.1.50}$$

$$_i\widehat{P}^i{}_\nu v^\nu = -v^i, \tag{2.1.51}$$

$$_i\widehat{P} \equiv \text{diag}[1, 1, \dots, 1, \underbrace{-1}_{(i+1)\text{th component}}, 1, \dots, 1]. \tag{2.1.52}$$

Define the rotation matrix $\widehat{R}^\mu{}_\nu$ such that it leaves all the axes orthogonal to the 1st and ith invariant, namely

$$\widehat{R}^\mu{}_\nu \widehat{e}^\nu_\ell = \widehat{e}^\nu_\ell, \tag{2.1.53}$$

$$\widehat{e}^\mu_\ell \equiv \delta^\mu_\ell, \qquad \ell \neq 1, i; \tag{2.1.54}$$

while rotating the $(1, i)$-plane clockwise by $\pi/2$:

$$\widehat{R} \cdot \widehat{e}_1 = -\widehat{e}_i, \qquad \widehat{R} \cdot \widehat{e}_i = +\widehat{e}_1. \tag{2.1.55}$$

Now argue that

$${}_i\widehat{P} = \widehat{R}^T \cdot {}_1\widehat{P} \cdot \widehat{R}. \tag{2.1.56}$$

Is ${}_i\widehat{P}$ a Lorentz transformation? □

Lorentz Boosts As already discussed, we may focus on the 2D case to elucidate the form of the most general Lorentz boost. This is the transformations that would mix time and space components, and yet leave the metric of spacetime $\eta_{\mu\nu} = \text{diag}[1, -1]$ invariant. (Neither time reversal, parity flips, nor spatial rotations mix time and space.) This is what revolutionized humanity's understanding of spacetime at the beginning of the 1900's: inspired by the fact that the speed of light is the same in all inertial frames, Einstein discovered *Special Relativity*, that the space and time coordinates of one frame have to become intertwined when being translated to those in another frame. We will turn this around later when discussing Maxwell's equations: the constancy of the speed of light in all inertial frames is in fact a consequence of the Lorentz covariance of the former.

Problem 2.7. We wish to find a 2×2 matrix Λ that obeys $\Lambda^T \cdot \eta \cdot \Lambda = \eta$, where $\eta_{\mu\nu} = \text{diag}[1, -1]$. By examining the diagonal terms of $\Lambda^T \cdot \eta \cdot \Lambda = \eta$, show that

$$\Lambda \doteq \left[\begin{array}{cc} \sigma_1 \cosh(\xi_1) & \sigma_2 \sinh(\xi_2) \\ \sigma_3 \sinh(\xi_1) & \sigma_4 \cosh(\xi_2) \end{array} \right], \tag{2.1.57}$$

where the $\sigma_{1,2,3,4}$ are either $+1$ or -1; altogether, there are 16 choices of signs. (Hint: $x^2 - y^2 = c^2$, for constant c, describes a hyperbola on the (x, y) plane.) From the off diagonal terms of $\Lambda^T \cdot \eta \cdot \Lambda = \eta$, argue that either $\xi_1 = \xi_2 \equiv \xi$ or $\xi_1 = -\xi_2 \equiv \xi$. Then explain why, if $\Lambda^0{}_0$ were not positive, we can always multiply it by a time reversal matrix to render it so; and likewise $\Lambda^1{}_1$ can always be rendered positive by multiplying it by a parity flip. By requiring $\Lambda^0{}_0$ and $\Lambda^1{}_1$ be both positive, therefore, prove that the resulting 2D Lorentz boost is

$$\Lambda^\mu{}_\nu(\xi) = \left[\begin{array}{cc} \cosh(\xi) & \sinh(\xi) \\ \sinh(\xi) & \cosh(\xi) \end{array} \right]. \tag{2.1.58}$$

This ξ is known as *rapidity.* In 2D, the rotation matrix is

$$\widehat{R}^i{}_j(\theta) = \left[\begin{array}{cc} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{array} \right]; \tag{2.1.59}$$

and therefore rapidity ξ is to the Lorentz boost in eq. (2.1.58) what the angle θ is to rotation $\widehat{R}^i{}_j(\theta)$ in eq. (2.1.59).

> **2D Lorentz Group:** In (1+1)D, the continuous boost in $\Lambda^\mu{}_\nu(\xi)$ in eq. (2.1.58) and the discrete time reversal and spatial reflection operators
>
> $$\widehat{T} = \left[\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right] \qquad \text{and} \qquad \widehat{P} = \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right]; \tag{2.1.60}$$
>
> altogether form the full set of Lorentz transformations – i.e., all solutions to eq. (2.1.5) consist of products of these three matrices.

$\square$

To understand the meaning of the rapidity ξ, let us consider applying it to an arbitrary 2D vector U^μ.

$$U' \equiv \Lambda \cdot U = \left[\begin{array}{c} U^0 \cosh(\xi) + U^1 \sinh(\xi) \\ U^1 \cosh(\xi) + U^0 \sinh(\xi) \end{array} \right]. \tag{2.1.61}$$

Lorentz Boost: Timelike case A vector U^μ is timelike if $U^2 \equiv \eta_{\mu\nu} U^\mu U^\nu > 0$; this often corresponds to vector tangent to the worldline of some material object. We will now show that it is always possible to Lorentz boost to its 'rest frame', namely $U'^\mu = \Lambda^\mu{}_\nu U^\nu = (U'^0, \vec{0})$.

In 2D, $U^2 > 0 \Rightarrow (U^0)^2 > (U^1)^2 \Rightarrow |U^0/U^1| > 1$. Then it is not possible to find a finite ξ such that $U'^0 = 0$, because that would amount to solving $\tanh(\xi) = -U^0/U^1$ but tanh lies between -1 and $+1$ while $-U^0/U^1$ is either less than -1 or greater than $+1$. On the other hand, it does mean we may solve for ξ that would set the spatial component to zero: $\tanh(\xi) = -U^1/U^0$. Recall that tangent vectors may be interpreted as the derivative of the spacetime coordinates with respect to some parameter λ, namely $U^\mu \equiv \mathrm{d}x^\mu/\mathrm{d}\lambda$. Therefore

$$\frac{U^1}{U^0} = \frac{\mathrm{d}x^1}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}x^0} = \frac{\mathrm{d}x^1}{\mathrm{d}x^0} \equiv v \tag{2.1.62}$$

is the velocity associated with U^μ in the frame $\{x^\mu\}$. Starting from $\tanh(\xi) = -v$, some algebra would then hand us (cf. eq. (2.1.58))

$$\cosh(\xi) = \gamma \equiv \frac{1}{\sqrt{1-v^2}}, \tag{2.1.63}$$

$$\sinh(\xi) = -\gamma \cdot v = -\frac{v}{\sqrt{1-v^2}}, \tag{2.1.64}$$

$$\Lambda^\mu{}_\nu = \left[\begin{array}{cc} \gamma & -\gamma \cdot v \\ -\gamma \cdot v & \gamma \end{array} \right]. \tag{2.1.65}$$

This in turn yields

$$U' = \left(\mathrm{sgn}(U^0) \sqrt{\eta_{\mu\nu} U^\mu U^\nu}, 0 \right)^T; \tag{2.1.66}$$

leading us to interpret the $\Lambda^\mu{}_\nu$ we have found in eq. (2.1.65) as the boost that bring observers to the frame where the flow associated with U^μ is 'at rest'. (Note that, if $U^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$, where τ is proper time, then $\eta_{\mu\nu} U^\mu U^\nu = 1$.)

As an important aside, we may generalize the two-dimensional Lorentz boost in eq. (2.1.65) to $D-$dimensions. One way to do it, is to simply append to the 2D Lorentz-boost matrix a $(D-2) \times (D-2)$ identity matrix (that leaves the $2-$ through $D-$spatial components unaltered) in a block diagonal form:

$$\Lambda^\mu{}_\nu \stackrel{?}{=} \left[\begin{array}{ccc} \gamma & -\gamma \cdot v & 0 \\ -\gamma \cdot v & \gamma & 0 \\ 0 & 0 & \mathbb{I}_{(D-2)\times(D-2)} \end{array} \right]. \tag{2.1.67}$$

But this is not doing much: we are still only boosting in the 1−direction. What if we wish to boost in v^i direction, where v^i is now some arbitrary spatial vector? To this end, we may promote the $(0,1)$ and $(1,0)$ components of eq. (2.1.65) to the spatial vectors $\Lambda^0{}_i$ and $\Lambda^i{}_0$ parallel to v^i. Whereas the $(1,1)$ component of eq. (2.1.65) is to be viewed as acting on the 1D space parallel to v^i, namely the operator $v^i v^j/\vec{v}^2$. (As a check: When $v^i = v(1,\vec{0})$, $v^i v^j/\vec{v}^2 = \delta^i_1 \delta^j_1$.) The identity operator acting on the orthogonal $(D-2)\times(D-2)$ space, i.e., the analog of $\mathbb{I}_{(D-2)\times(D-2)}$ in eq. (2.1.67), is $\Pi^{ij} = \delta^{ij} - v^i v^j/\vec{v}^2$. (Notice: $\Pi^{ij}v^j = (\delta^{ij} - v^i v^j/\vec{v}^2)v^j = 0$.) Altogether, the Lorentz boost in the v^i direction is given by

$$\Lambda^\mu{}_\nu(\vec{v}) \doteq \begin{bmatrix} \gamma & -\gamma v^i \\ -\gamma v^i & \gamma\frac{v^i v^j}{\vec{v}^2} + \left(\delta^{ij} - \frac{v^i v^j}{\vec{v}^2}\right) \end{bmatrix}, \qquad \vec{v}^2 \equiv \delta_{ab} v^a b^b. \tag{2.1.68}$$

It may be worthwhile to phrase this discussion in terms of the Cartesian coordinates $\{x^\mu\}$ and $\{x'^\mu\}$ parametrizing the two inertial frames. What we have shown is that the Lorentz boost in eq. (2.1.68) describes

$$U'^\mu = \Lambda^\mu{}_\nu(\vec{v}) U^\nu, \tag{2.1.69}$$

$$U^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}, \qquad U'^\mu = \frac{\mathrm{d}x'^\mu}{\mathrm{d}\lambda} = \left(\mathrm{sgn}(U^0)\sqrt{\eta_{\mu\nu}U^\mu U^\nu}, 0\right)^T. \tag{2.1.70}$$

λ is the intrinsic 1D coordinate parametrizing the worldlines, and by definition does not alter under Lorentz boost. The above statement is therefore equivalent to

$$\mathrm{d}x'^\mu = \Lambda^\mu{}_\nu(\vec{v})\mathrm{d}x^\nu, \tag{2.1.71}$$

$$x'^\mu = \Lambda^\mu{}_\nu(\vec{v})x^\nu + a^\mu, \tag{2.1.72}$$

where the spacetime translation a^μ shows up here as integration constants.

Problem 2.8. Lorentz boost in $(D+1)$−dimensions If $v^\mu \equiv (1, v^i)$, check via a direction calculation that the $\Lambda^\mu{}_\nu$ in eq. (2.1.68) produces a $\Lambda^\mu{}_\nu v^\nu$ that has no non-trivial spatial components. Also check that eq. (2.1.68) is, in fact, a Lorentz transformation. What is $\Lambda^\mu{}_\sigma(\vec{v})\Lambda^\sigma{}_\nu(-\vec{v})$?

Lorentz Boost: Spacelike case A vector U^μ is spacelike if $U^2 \equiv \eta_{\mu\nu}U^\mu U^\nu < 0$. As we will now show, it is always possible to find a Lorentz boost so that $U'^\mu = \Lambda^\mu{}_\nu U^\nu = (0, \vec{U}')$ has no time components – hence the term 'spacelike'. This can correspond, for instance, to the vector joining two spatial locations within a macroscopic body at a given time.

Suppose U were spacelike in 2D, $U^2 < 0 \Rightarrow (U^0)^2 < (U^1)^2 \Rightarrow |U^1/U^0| = |\mathrm{d}x^1/\mathrm{d}x^0| \equiv |v| > 1$. Then, recalling eq. (2.1.61), it is not possible to find a finite ξ such that $U'^1 = 0$, because that would amount to solving $\tanh(\xi) = -U^1/U^0$, but tanh lies between −1 and +1 whereas $-U^1/U^0 = -v$ is either less than −1 or greater than +1. On the other hand, it is certainly possible to have $U'^0 = 0$. Simply do $\tanh(\xi) = -U^0/U^1 = -1/v$. Similar algebra to the timelike case then hands us

$$\cosh(\xi) = \left(1 - v^{-2}\right)^{-1/2} = \frac{|v|}{\sqrt{v^2-1}}, \tag{2.1.73}$$

$$\sinh(\xi) = -(1/v)\left(1 - v^{-2}\right)^{-1/2} = -\frac{\mathrm{sgn}(v)}{\sqrt{v^2-1}}, \tag{2.1.74}$$

$$U' = \left(0, \mathrm{sgn}(v)\sqrt{-\eta_{\mu\nu}U^\mu U^\nu}\right)^T, \qquad v \equiv \frac{U^1}{U^0}. \tag{2.1.75}$$

We may interpret U'^μ and U^μ as infinitesimal vectors joining the same pair of spacetime points but in their respective frames. Specifically, U'^μ are the components in the frame where the pair lies on the same constant-time surface ($U'^0 = 0$). While U^μ are the components in a boosted frame.

Lorentz Boost: Null (aka lightlike) case The vector U^μ is null if $U^2 = \eta_{\mu\nu} U^\mu U^\nu = 0$. If U were null in 2D, that means $(U^0)^2 = (U^1)^2$, which in turn implies

$$U^\mu = \omega(1, \pm 1) \tag{2.1.76}$$

for some real number ω. Upon a Lorentz boost, eq. (2.1.61) tells us

$$U' \equiv \Lambda \cdot U = \omega \begin{bmatrix} \cosh(\xi) \pm \sinh(\xi) \\ \sinh(\xi) \pm \cosh(\xi) \end{bmatrix}. \tag{2.1.77}$$

As we shall see below, if U^μ describes the $d-$momentum of a photon, so that $|\omega|$ is its frequency in the un-boosted frame, the $U'^0/U^0 = \cosh(\xi) \pm \sinh(\xi)$ describes the photon's red- or blue-shift in the boosted frame. Notice it is not possible to set either the time nor the space component to zero, unless $\xi \to \pm\infty$.

> **Summary** Our analysis of the group of matrices $\{\Lambda\}$ obeying $\Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \eta_{\alpha\beta} = \eta_{\mu\nu}$ reveals that these Lorentz transformations consists of: time reversals, parity flips, spatial rotations and Lorentz boosts. A timelike vector can always be Lorentz-boosted so that all its spatial components are zero; while a spacelike vector can always be Lorentz-boosted so that its time component is zero.

Problem 2.9. Null, spacelike vs. timelike Do null vectors form a vector space? Simiarly, do spacelike or timelike vectors form a vector space? □

Exponential Form Lorentz transformations have continuous parameters that tell us how large/small a rotation and/or boost is being performed. Whenever these parameters may be tuned to set the said Lorentz transformation Λ to the identity, we may write it in an exponential form:

$$\Lambda^\mu{}_\nu = \left(e^X\right)^\mu{}_\nu, \tag{2.1.78}$$

where the exponential of the matrix X is defined through its power series, $\exp X = \sum_{\ell=0}^\infty X^\ell/\ell!$. Because we are moving indices with the metric $\eta_{\alpha\beta}$ – for e.g., $\eta_{\mu\nu} X^\mu{}_\alpha = X_{\nu\alpha}$ – the position of the indices on any object (upper or lower) is important. In particular, the Lorentz transformation itself $\Lambda^\mu{}_\nu$ has one upper and one lower index; and this means the X in $\Lambda = e^X$ must, too, have one upper and one lower index – for instance, the n-th term in the Taylor series reads:

$$\frac{1}{n!} X^\mu{}_{\sigma_1} X^{\sigma_1}{}_{\sigma_2} X^{\sigma_2}{}_{\sigma_3} \dots X^{\sigma_{n-2}}{}_{\sigma_{n-1}} X^{\sigma_{n-1}}{}_\nu. \tag{2.1.79}$$

If we use the defining relation in eq. (2.1.5), but consider it for small X only,

$$\left(\delta^\mu{}_\alpha + X^\mu{}_\alpha + \mathcal{O}\left(X^2\right)\right) \eta_{\mu\nu} \left(\delta^\nu{}_\beta + X^\nu{}_\beta + \mathcal{O}\left(X^2\right)\right) \tag{2.1.80}$$

$$= \eta_{\alpha\beta} + \delta^\mu{}_\alpha \eta_{\mu\nu} X^\nu{}_\beta + X^\mu{}_\alpha \eta_{\mu\nu} \delta^\nu{}_\beta + \mathcal{O}\left(X^2\right)$$

$$= \eta_{\alpha\beta} + X_{\alpha\beta} + X_{\beta\alpha} + \mathcal{O}\left(X^2\right) = \eta_{\alpha\beta}. \tag{2.1.81}$$

The order$-X$ terms will vanish iff $X_{\alpha\beta}$ itself (with both lower indices) or $X^{\alpha\beta}$ (with both upper indices) is anti-symmetric:

$$X_{\alpha\beta} = -X_{\beta\alpha}. \tag{2.1.82}$$

The general Lorentz transformation continuously connected to the identity must therefore be the exponential of the superposition of the basis of anti-symmetric matrices:

$$\Lambda^\alpha{}_\beta = \left(\exp\left(-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right)\right)^\alpha{}_\beta, \qquad \text{(Boosts \& Rotations)}, \tag{2.1.83}$$

$$-i\left(J^{\mu\nu}\right)^\alpha{}_\beta = \eta^{\mu\alpha}\delta^\nu{}_\beta - \eta^{\nu\alpha}\delta^\mu{}_\beta = +i\left(J^{\nu\mu}\right)^\alpha{}_\beta, \qquad \omega_{\mu\nu} = -\omega_{\nu\mu} \in \mathbb{R}. \tag{2.1.84}$$

Some words on the indices: $(J^{\mu\nu})^\alpha{}_\beta$ is the α-th row and β-th column of the (μ,ν)-th basis anti-symmetric matrix; with $\mu \neq \nu$. $\omega_{\mu\nu} = -\omega_{\nu\mu}$ are the parameters controlling the size of the rotations and boosts; they need to be real because $\Lambda^\alpha{}_\beta$ is real.

Problem 2.10. From eq. (2.1.84), write down $(J^{\mu\nu})^{\alpha\beta}$ and explain why these form a complete set of basis matrices for the generators of the Lorentz group. $\square$

Generators To understand the geometric meaning of eq. (2.1.84), let us figure out the form of X in eq. (2.1.78) that would generate individual Lorentz boosts and rotations in $(2+1)$D. The boost along the $1-$axis, according to eq. (2.1.58) is

$$\Lambda^\mu{}_\nu(\xi) = \begin{bmatrix} \cosh(\xi) & \sinh(\xi) & 0 \\ \sinh(\xi) & \cosh(\xi) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbb{I}_{3\times 3} - i\xi \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mathcal{O}\left(\xi^2\right). \tag{2.1.85}$$

The boost along the $2-$axis is

$$\Lambda^\mu{}_\nu(\xi) = \begin{bmatrix} \cosh(\xi) & 0 & \sinh(\xi) \\ 0 & 1 & 0 \\ \sinh(\xi) & 0 & \cosh(\xi) \end{bmatrix} = \mathbb{I}_{3\times 3} - i\xi \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} + \mathcal{O}\left(\xi^2\right). \tag{2.1.86}$$

Equations (2.1.85) and (2.1.86) tell us the generators of Lorentz boost, assuming $\Lambda^\mu{}_\nu(\xi)$ take the form $\exp(-i\xi K)$, is then

$$K^1 \equiv J^{01} \equiv -J^{10} \equiv \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \doteq i\left(\eta^{\mu 0}\delta^1_\nu - \eta^{\mu 1}\delta^0_\nu\right), \tag{2.1.87}$$

$$K^2 \equiv J^{02} \equiv -J^{20} \equiv \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \doteq i\left(\eta^{\mu 0}\delta^2_\nu - \eta^{\mu 2}\delta^0_\nu\right). \tag{2.1.88}$$

The counter-clockwise rotation on the $(1,2)$ plane, according to eq. (2.1.59), is

$$\Lambda^\mu{}_\nu(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = \mathbb{I}_{3\times 3} - i\theta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} + \mathcal{O}\left(\theta^2\right). \tag{2.1.89}$$

Assuming this rotation is $\Lambda^\mu{}_\nu(\theta) = \exp(-i\theta J^{12})$, the generator is.

$$J^{12} \equiv -J^{21} \equiv i\left(\eta^{\mu 1}\delta^2_\nu - \eta^{\mu 2}\delta^1_\nu\right). \tag{2.1.90}$$

We may gather, from equations (2.1.87), (2.1.88), and (2.1.90), the generators of boosts and rotations are in fact the ones in eq. (2.1.84).

Problem 2.11. Show, by a direct calculation, that $\exp(-i\xi K^1)$ and $\exp(-i\xi K^2)$ do indeed yield the boosts in equations (2.1.85) and (2.1.86) respectively. Show that $\exp(-i\theta J^{12})$ does indeed yield the rotation in eq. (2.1.89). Hint: You may write $K^j = i\,|0\rangle\langle j| + i\,|j\rangle\langle 0|$ and use a fictitious Hilbert space where $\langle\mu|\,\nu\rangle = \delta^{\mu\nu}$ and $(K^j)^\mu{}_\nu = \langle\mu\,|K^j|\,\nu\rangle$; then compute the Taylor series of $\exp(-i\xi K^j)$. □

Problem 2.12. We have only seen that eq. (2.1.84) generates individual boosts and rotations in $(2+1)$D. Explain why eq. (2.1.84) does in fact generalize to the generators of boosts and rotations in all dimensions $d \geq 3$. Hint: See previous problem. □

Determinants, Discontinuities By taking the determinant of eq. (2.1.5), and utilizing $\det(AB) = \det A \det B$ and $\det A^T = \det A$,

$$\det\Lambda^T \cdot \det\eta \cdot \det\Lambda = \det\eta \tag{2.1.91}$$

$$(\det\Lambda)^2 = 1 \tag{2.1.92}$$

$$\det\Lambda = \pm 1 \tag{2.1.93}$$

Notice the time reversal $\widehat{T}$ and parity flips $\{{}_{(i)}\widehat{P}\}$ matrices each has determinant -1. On the other hand, Lorentz boosts and rotations that may be tuned to the identity transformation must have determinant $+1$. This is because the identity itself has det $+1$ and since boosts and rotations depend continuously on their parameters, their determinant cannot jump abruptly from $+1$ and -1.

Problem 2.13. The determinant is a tool that can tell us there are certain Lorentz transformations that are disconnected from the identity – examples are

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} -\cosh\xi & \sinh\xi \\ -\sinh\xi & \cosh\xi \end{bmatrix}. \tag{2.1.94}$$

You can explain why these are disconnected from $\mathbb{I}$? □

Group multiplication Because matrices do not commute, it is not true in general that $e^X e^Y = e^{X+Y}$. Instead, the the Baker-Campbell-Hausdorff formula tells us

$$e^X e^Y = \exp\left(X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\left[X,[X,Y]\right] - \frac{1}{12}[Y,[X,Y]] + \dots\right), \tag{2.1.95}$$

$$[A,B] \equiv AB - BA; \tag{2.1.96}$$

where the exponent on the right hand involves sums of commutators $[\cdot,\cdot]$, commutators of commutators, commutators of commutators of commutators, etc.

If the generic form of the Lorentz transformation in eq. (2.1.78) holds, we would expect that the product of two Lorentz transformations to yield the same exponential form:

$$\exp\left(-\frac{i}{2}a_{\mu\nu}J^{\mu\nu}\right)\exp\left(-\frac{i}{2}b_{\alpha\beta}J^{\alpha\beta}\right)=\exp\left(-\frac{i}{2}c_{\delta\gamma}J^{\delta\gamma}\right). \tag{2.1.97}$$

Comparison with eq. (**??**) tells us, in order for the product of two Lorentz transformations to return the exponential form on the right hand side, the commutators of the generators $\{J^{\mu\nu}\}$ ought to return linear combinations of the generators. This way, higher commutators will continue to return further linear combinations of the generators, which then guarantees the form on the right hand side of eq. (2.1.97). More specifically, according to eq. (**??**), the first commutator would yield

$$\begin{aligned} e^{-\frac{i}{2}a_{\mu\nu}J^{\mu\nu}}e^{-\frac{i}{2}b_{\mu\nu}J^{\mu\nu}} &= \exp\Bigg[-\frac{i}{2}(a_{\mu\nu}+b_{\mu\nu})J^{\mu\nu}+\frac{1}{2}\left(-\frac{i}{2}\right)^2 a_{\mu\nu}b_{\alpha\beta}\left[J^{\mu\nu},J^{\alpha\beta}\right] \\ &\qquad +\frac{1}{12}\left(-\frac{i}{2}\right)^3 a_{\sigma\rho}a_{\mu\nu}b_{\alpha\beta}\left[J^{\sigma\rho},\left[J^{\mu\nu},J^{\alpha\beta}\right]\right]+\dots\Bigg] \qquad (2.1.98) \\ &= \exp\Bigg[-\frac{i}{2}(a_{\mu\nu}+b_{\mu\nu})J^{\mu\nu}+\frac{1}{2}\left(-\frac{i}{2}\right)^2 a_{\mu\nu}b_{\alpha\beta}Q^{\mu\nu\alpha\beta}{}_{\kappa\xi}J^{\kappa\xi} \\ &\qquad +\frac{1}{12}\left(-\frac{i}{2}\right)^3 a_{\sigma\rho}a_{\mu\nu}b_{\alpha\beta}Q^{\mu\nu\alpha\beta}{}_{\kappa\xi}Q^{\sigma\rho\kappa\xi}{}_{\omega\lambda}J^{\omega\lambda}+\dots\Bigg], \qquad (2.1.99) \end{aligned}$$

for appropriate complex numbers $\{Q^{\mu\nu\alpha\beta}{}_{\lambda\tau}\}$.

This is precisely what occurs. The commutation relations between generators of a general Lie group is known as its Lie algebra. For the Lorentz generators, a direct computation using eq. (2.1.84) would return:

Lie Algebra for SO$_{D,1}$

$$\left[J^{\mu\nu},J^{\rho\sigma}\right]=i\left(\eta^{\nu\rho}J^{\mu\sigma}-\eta^{\mu\rho}J^{\nu\sigma}+\eta^{\mu\sigma}J^{\nu\rho}-\eta^{\nu\sigma}J^{\mu\rho}\right). \tag{2.1.100}$$

Problem 2.14. Remember that linear operators acting on a Hilbert space themselves form a vector space. Consider a collection of linearly independent linear operators $\{L_1, L_2, \dots, L_N\}$. Suppose they are closed under commutation, namely

$$[L_i, L_j]=\sum_{k=1}^{N}c_{ijk}L_k; \tag{2.1.101}$$

for any i and j; and the c_{ijk} here are (complex) numbers. Prove that these N operators form a vector space. □

Problem 2.15. Non-singular Coordinate transformations form a group Let us verify explicitly that the Jacobians associated with general non-singular coordinate transformations form a group. Specifically, let us consider transforming from the coordinate system x^α to y^μ, and assume x^α in terms of y^μ has been provided (i.e., $x^\alpha(y^\mu)$ is known). We may also proceed to consider transforming to a third coordinate system, from y^μ to z^κ.

- *Closure* Denote the Jacobian as, for e.g., $\mathcal{J}^{\alpha}{}_{\mu}[x \to y] \equiv \partial x^{\alpha}/\partial y^{\mu}$. If we define the group operation as simply that of matrix multiplication, verify that

$$\mathcal{J}^{\alpha}{}_{\sigma}[x \to y]\mathcal{J}^{\sigma}{}_{\nu}[y \to z] = \mathcal{J}^{\alpha}{}_{\nu}[x \to z]. \tag{2.1.102}$$

  In words: multiplying the transformation matrix bringing us from x to y followed by that from y to z, yields the Jacobian that brings us from x directly to z. This composition law is what we would need, if the group operation is to implement coordinate transformations.

- *Associativity* Explain why the composition law for Jacobians is associative.

- *Identity* What is the identity Jacobian? What is the most general coordinate transformation it corresponds to?

- *Inverse* By non-singular, we mean $\det \mathcal{J}^{\alpha}{}_{\mu} \neq 0$. What does this imply about the existence of the inverse $(\mathcal{J}^{-1})^{\alpha}{}_{\mu}$?

□

2.2 Lorentz Transformations in 4 Dimensions

Lie Algebra for SO$(3,1)$ As far as we can tell, the world we live in has 3 space and 1 time dimensions. Let us now work out the Lie Algebra in eq. (2.1.100) more explicitly. Denoting the boost generator as

$$K^i \equiv J^{0i} \tag{2.2.1}$$

and the rotation generators as

$$J^i \equiv \frac{1}{2}\epsilon^{imn}J^{mn} \qquad \Leftrightarrow \qquad \epsilon^{imn}J^i \equiv J^{mn}; \tag{2.2.2}$$

with $\epsilon^{123} = \epsilon_{123} \equiv 1$. The generic Lorentz transformation continuously connected to the identity is

$$\Lambda(\vec{\xi},\vec{\theta}) = \exp\left(-i\xi_j K^j - i\theta_j J^j\right). \tag{2.2.3}$$

These $\{\Lambda(\vec{\xi},\vec{\theta})\}$ are not necessarily the 4×4 matrices obeying $\Lambda^{\mathrm{T}}\eta\Lambda = \eta$. Rather, their generators simply need to obey the same commutation relations in eq. (2.1.100).

We may compute from eq. (2.1.100) that

$$[J^m, J^n] = i\epsilon^{mnl}J^l, \tag{2.2.4}$$

$$[K^m, J^n] = i\epsilon^{mnl}K^l, \tag{2.2.5}$$

$$[K^m, K^n] = -i\epsilon^{mnl}J^l. \tag{2.2.6}$$

Problem 2.16. SU$(2)_{\mathbf{L}}\times$**SU**$(2)_{\mathbf{R}}$ Let us next define

$$J^i_+ \equiv \frac{1}{2}\left(J^i + iK^i\right), \tag{2.2.7}$$

$$J^i_- \equiv \frac{1}{2}\left(J^i - iK^i\right). \tag{2.2.8}$$

Use equations (2.2.4) through (2.2.6) to show that

$$\left[J_+^i, J_+^j\right] = i\epsilon^{ijk} J_+^k, \tag{2.2.9}$$

$$\left[J_-^i, J_-^j\right] = i\epsilon^{ijk} J_-^k, \tag{2.2.10}$$

$$\left[J_+^i, J_-^j\right] = 0. \tag{2.2.11}$$

Equations (2.2.9) and (2.2.10) tell us the $J_\pm^i$ obey the same algebra as the angular momentum ones in eq. (2.2.4); and eq. (2.2.11) says the two sets $\{\vec{J}_+, \vec{J}_-\}$ commute. □

2.2.1 $\mathrm{SL}_{2,\mathbb{C}}$ Spinors and Spin-Half

To describe spin$-1/2$ fermions in Nature – leptons (electrons, muons and taus) and quarks – one has to employ *spinors*. We will now build spinors in 4D Minkowski spacetime by viewing them as representations of the $\mathrm{SL}_{2,\mathbb{C}}$ group.

Basic Properties of $\{\sigma^\mu\}$ We begin by collecting the results in Problems (**??**) and (**??**) as well as the 'Pauli matrices from their algebra' discussion in §(**??**). A basis set of orthonormal 2×2 complex matrices is provided by $\{\sigma^\mu | \mu = 0, 1, 2, 3\}$, the 2×2 identity matrix

$$\sigma^0 \equiv \mathbb{I}_{2\times 2} \tag{2.2.12}$$

together with the Hermitian Pauli matrices $\{\sigma^i\}$. The $\{\sigma^i | i = 1, 2, 3\}$ may be viewed as arising from the algebra

$$\sigma^i \sigma^j = \delta^{ij} \mathbb{I}_{2\times 2} + i\epsilon^{ijk}\sigma^k, \tag{2.2.13}$$

which immediately implies the respective anti-commutator and commutator results:

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij} \qquad \text{and} \qquad [\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k. \tag{2.2.14}$$

As a result of eq. (2.2.13), the Pauli matrices have eigenvalues ± 1, namely

$$\sigma^i \left|\pm; i\right\rangle = \pm \left|\pm; i\right\rangle; \tag{2.2.15}$$

and thus -1 determinant (i.e., product of eigenvalues) and zero trace (i.e., sum of eigenvalues):

$$\det \sigma^i = -1, \qquad \mathrm{Tr}\, \sigma^i = 0. \tag{2.2.16}$$

An equivalent way of writing eq. (2.2.13) is to employ arbitrary complex vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. Denoting $\vec{a} \cdot \vec{\sigma} \equiv a_i \sigma^i$,

$$(\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma}) = \vec{a} \cdot \vec{b} + i(\vec{a} \times \vec{b}) \cdot \vec{\sigma}, \qquad (\vec{a} \times \vec{b})^k = \epsilon^{ijk} a_i b_j. \tag{2.2.17}$$

We may multiply by $(\vec{c} \cdot \vec{\sigma})$ from the right on both sides:

$$(\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma})(\vec{c} \cdot \vec{\sigma}) = i(\vec{a} \times \vec{b}) \cdot \vec{c} + \left\{(\vec{c} \cdot \vec{b})\vec{a} - (\vec{c} \cdot \vec{a})\vec{b} + (\vec{a} \cdot \vec{b})\vec{c}\right\} \cdot \vec{\sigma}. \tag{2.2.18}$$

Problem 2.17. Verify eq. (2.2.18). □

In the representation where σ^3 is diagonal,

$$\sigma^0 \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \sigma^1 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma^2 \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma^3 \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.2.19}$$

The inner product of $\{\sigma^\mu\}$ is provided by the matrix trace,

$$\langle \sigma^\mu | \, \sigma^\nu \rangle \equiv \frac{1}{2} \text{Tr} \left[\sigma^\mu \sigma^\nu \right] = \delta^{\mu\nu}. \tag{2.2.20}$$

Since the $\{\sigma^\mu\}$ form a basis, any 2×2 complex matrix A may be obtained as a superposition $A = q_\mu \sigma^\mu$ by choosing the appropriate complex parameters $\{q_\mu\}$. In addition, we will utilize

$$\bar{\sigma}^\mu \equiv \left(\mathbb{I}_{2\times 2}, -\sigma^i \right) = \sigma_\mu. \tag{2.2.21}$$

[34]We also need the 2D Levi-Civita symbol ϵ. Since ϵ is real and antisymmetric,

$$\epsilon^\dagger = \epsilon^{\mathrm{T}} = -\epsilon, \tag{2.2.22}$$

a direct calculation would reveal

$$\epsilon \cdot \epsilon^\dagger = -\epsilon^2 = \mathbb{I}. \tag{2.2.23}$$

According to eq. (2.2.13), because $\sigma^i \sigma^i = \mathbb{I}$ (for fixed i) that implies σ^i is its own inverse. We may then invoke eq. (**??**) to state

$$(\sigma^i)^{-1} = \sigma^i = -\frac{\epsilon (\sigma^i)^{\mathrm{T}} \epsilon}{\det \sigma^i} = \frac{\epsilon (\sigma^i)^{\mathrm{T}} \epsilon^\dagger}{\det \sigma^i} = \frac{\epsilon^\dagger (\sigma^i)^{\mathrm{T}} \epsilon}{\det \sigma^i}. \tag{2.2.24}$$

Since ϵ is real, $\det \sigma^i = -1$ (cf. eq. (2.2.16)), and σ^i is Hermitian, we may take the complex conjugate on both sides and deduce

$$(\sigma^i)^* = \epsilon \cdot \sigma^i \cdot \epsilon = \epsilon^\dagger \left(-\sigma^i \right) \epsilon = \epsilon \left(-\sigma^i \right) \epsilon^\dagger. \tag{2.2.25}$$

Since $\epsilon^2 = -\mathbb{I}$, we may multiply both sides by ϵ on the left and right,

$$\epsilon \cdot (\sigma^i)^* \cdot \epsilon = \epsilon^\dagger \cdot (-\sigma^i)^* \cdot \epsilon = \epsilon \cdot (-\sigma^i)^* \cdot \epsilon^\dagger = \sigma^i. \tag{2.2.26}$$

Problem 2.18. Using the notation in eq. (2.2.21), explain why

$$\epsilon \cdot (\bar{\sigma}^\mu)^* \cdot \epsilon^\dagger = \epsilon^\dagger \cdot (\bar{\sigma}^\mu)^* \cdot \epsilon = \sigma^\mu, \tag{2.2.27}$$

$$\epsilon \cdot (\sigma^\mu)^* \cdot \epsilon^\dagger = \epsilon^\dagger \cdot (\sigma^\mu)^* \cdot \epsilon = \bar{\sigma}^\mu; \tag{2.2.28}$$

and therefore

$$\epsilon \cdot \bar{\sigma}^\mu \cdot \epsilon^\dagger = \epsilon^\dagger \cdot \bar{\sigma}^\mu \cdot \epsilon = (\sigma^\mu)^*, \tag{2.2.29}$$

$$\epsilon \cdot \sigma^\mu \cdot \epsilon^\dagger = \epsilon^\dagger \cdot \sigma^\mu \cdot \epsilon = (\bar{\sigma}^\mu)^*. \tag{2.2.30}$$

Hint: Remember the properties of ϵ and σ^0. □

[34]Caution: The over-bar on $\bar{\sigma}$ is not complex conjugation.

Because $(\sigma^\mu)^2 = \mathbb{I}$ and $\sigma^\mu / \det \sigma^\mu = \bar{\sigma}^\mu = (\mathbb{I}, -\sigma^i) = \sigma_\mu$, eq. (**??**) informs us

$$\sigma^\mu = -\epsilon \cdot (\bar{\sigma}^\mu)^{\mathrm{T}} \cdot \epsilon = -\epsilon \cdot (\sigma_\mu)^{\mathrm{T}} \cdot \epsilon \tag{2.2.31}$$

$$= \epsilon^\dagger \cdot (\bar{\sigma}^\mu)^{\mathrm{T}} \cdot \epsilon = \epsilon^\dagger \cdot (\sigma_\mu)^{\mathrm{T}} \cdot \epsilon. \tag{2.2.32}$$

That $\bar{\sigma}^\mu = \sigma_\mu$ is because lowering the spatial components costs a minus sign.

Problem 2.19. Explain why

$$\bar{\sigma}^\mu = -\epsilon \cdot (\sigma^\mu)^{\mathrm{T}} \cdot \epsilon = -\epsilon \cdot (\bar{\sigma}_\mu)^{\mathrm{T}} \cdot \epsilon \tag{2.2.33}$$

$$= \epsilon^\dagger \cdot (\sigma^\mu)^{\mathrm{T}} \cdot \epsilon = \epsilon^\dagger \cdot (\bar{\sigma}_\mu)^{\mathrm{T}} \cdot \epsilon. \tag{2.2.34}$$

□

Exponential form Finally, for any complex $\{\psi_i\}$, we have from eq. (**??**),

$$\exp\left(-\frac{i}{2}\psi_i \sigma^i\right) = \cos\left(\frac{|\vec{\psi}|}{2}\right) - i\frac{\vec{\psi}\cdot\vec{\sigma}}{|\vec{\psi}|}\sin\left(\frac{|\vec{\psi}|}{2}\right), \tag{2.2.35}$$

$$\vec{\psi}\cdot\vec{\sigma} \equiv \psi_j \sigma^j, \qquad |\vec{\psi}| \equiv \sqrt{\psi_i \psi_i}. \tag{2.2.36}$$

One may readily check that its inverse is

$$\left(\exp\left(-\frac{i}{2}\psi_i\sigma^i\right)\right)^{-1} = \exp\left(+\frac{i}{2}\psi_i\sigma^i\right) = \cos\left(\frac{|\vec{\psi}|}{2}\right) + i\frac{\vec{\psi}\cdot\vec{\sigma}}{|\vec{\psi}|}\sin\left(\frac{|\vec{\psi}|}{2}\right). \tag{2.2.37}$$

(We will take the $\sqrt{\cdot}$ in the definition of $|\vec{\psi}|$ to be the positive square root.) Observe that the relation in eq. (**??**) is basis independent; namely, if we found a different representation of the Pauli matrices

$$\sigma'^i = U\sigma^i U^{-1} \qquad \Leftrightarrow \qquad U^{-1}\sigma'^i U = \sigma^i \tag{2.2.38}$$

then the algebra in eq. (2.2.13) and the exponential result in eq. (2.2.35) would respectively become

$$U^{-1}\sigma'^i U U^{-1}\sigma'^j U = U^{-1}\left(\delta^{ij} + i\epsilon^{ijk}\sigma'^k\right)U, \tag{2.2.39}$$

$$\sigma'^i\sigma'^j = \delta^{ij} + i\epsilon^{ijk}\sigma'^k \tag{2.2.40}$$

and

$$\exp\left(-\frac{i}{2}\psi_i U^{-1}\sigma'^i U\right) = U^{-1}\exp\left(-\frac{i}{2}\psi_i\sigma'^i\right)U = U^{-1}\left(\cos\left(\frac{|\vec{\psi}|}{2}\right) - i\frac{\psi_j\sigma'^j}{|\vec{\psi}|}\sin\left(\frac{|\vec{\psi}|}{2}\right)\right),$$

$$\exp\left(-\frac{i}{2}\psi_i\sigma'^i\right) = \cos\left(\frac{|\vec{\psi}|}{2}\right) - i\frac{\psi_j\sigma'^j}{|\vec{\psi}|}\sin\left(\frac{|\vec{\psi}|}{2}\right). \tag{2.2.41}$$

Lorentz Invariant p^2 & Helicity Eigenstates We now move on to the discussion of $\text{SL}_{2,\mathbb{C}}$ proper. If $p_\mu \equiv (p_0, p_1, p_2, p_3)$ is a real 4-component momentum vector, one would find that the determinant of $p_\mu \sigma^\mu$ yields the Lorentz invariant p^2:

$$\det p_\mu \sigma^\mu = \eta^{\mu\nu} p_\mu p_\nu \equiv p^2. \tag{2.2.42}$$

[35]If we exploited the representation in eq. (2.2.19),

$$p_\mu \sigma^\mu = \begin{bmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{bmatrix}, \tag{2.2.43}$$

from which eq. (2.2.42) may be readily verified. Furthermore, if we now multiply a 2×2 complex matrix L to the left and $L^\dagger$ to the right of the matrix $p_\mu \sigma^\mu$, namely

$$p_\mu \sigma^\mu \to L \cdot p_\mu \sigma^\mu \cdot L^\dagger; \tag{2.2.44}$$

– this transformation preserves the Hermitian nature of $p_\mu \sigma^\mu$ for real p_μ – then its determinant will transform as

$$p^2 = \det[p_\mu \sigma^\mu] \to \det\left[L \cdot p_\mu \sigma^\mu \cdot L^\dagger\right] = |\det L|^2 \, p^2. \tag{2.2.45}$$

If we choose

$$\det L = 1 \tag{2.2.46}$$

– this is the “S” $\equiv$ “special” $\equiv$ “unit determinant” in the $\text{SL}_{2,\mathbb{C}}$ – then we see from eq. (2.2.45) that such a transformation would preserve the inner product $p^2 \to p^2$. Therefore, we expect the group of $\text{SL}_{2,\mathbb{C}}$ matrices $\{L\}$ to implement Lorentz transformations via eq. (2.2.44).

We first note that the Hermitian object $(p_i/|\vec{p}|)\sigma^i$, for real p_i and $|\vec{p}| \equiv \sqrt{\delta^{ij} p_i p_j}$, may be diagonalized as

$$\frac{p_i}{|\vec{p}|} \left(\sigma^i\right)_{\text{A}\dot{\text{B}}} = \xi^+_{\text{A}} \overline{\xi^+}_{\dot{\text{B}}} - \xi^-_{\text{A}} \overline{\xi^-}_{\dot{\text{B}}}; \tag{2.2.47}$$

$$\frac{p_i}{|\vec{p}|} \sigma^i \xi^\pm = \pm \xi^\pm. \tag{2.2.48}$$

[36]In the representation of the Pauli matrices in eq. (2.2.19), the unit norm *helicity eigenstates* are, up to overall phases,

$$\xi^+{}_{\text{A}} = \left(e^{-i\phi_p} \cos\left[\frac{\theta_p}{2}\right], \sin\left[\frac{\theta_p}{2}\right]\right)^{\text{T}} \tag{2.2.49}$$

$$= \frac{1}{\sqrt{2}} \sqrt{1 - \frac{p_3}{|\vec{p}|}} \left(\frac{|\vec{p}| + p_3}{p_1 + ip_2}, 1\right)^{\text{T}} \tag{2.2.50}$$

---

[35]Although we are concerned with the full Lorentz group here, note that $\det p_i \sigma^i = -\vec{p}^2$; so one may also use Pauli matrices to analyze representations of the rotation group alone, i.e., all transformations that leave $\vec{p}^2$ invariant.

[36]This dotted/un-dotted notation will be explained shortly.

and

$$\xi^{-}{}_{\mathrm{A}} = \left(-e^{-i\phi_p}\sin\left[\frac{\theta_p}{2}\right], \cos\left[\frac{\theta_p}{2}\right]\right)^{\mathrm{T}} \tag{2.2.51}$$

$$= \frac{1}{\sqrt{2}}\sqrt{1+\frac{p_3}{|\vec{p}|}}\left(-\frac{|\vec{p}|-p_3}{p_1+ip_2}, 1\right)^{\mathrm{T}}. \tag{2.2.52}$$

Note that we have switched to spherical coordinates in momentum space, namely

$$p_i \equiv p\left(\sin\theta_p\cos\phi_p, \sin\theta_p\sin\phi_p, \cos\theta_p\right). \tag{2.2.53}$$

Also notice, under parity

$$\phi_p \to \phi_p + \pi \qquad \text{and} \qquad \theta_p \to \pi - \theta_p, \tag{2.2.54}$$

the helicity eigenstates in equations (2.2.49) and (2.2.51) transform into each other:

$$\xi^+ \to \xi^- \qquad \text{and} \qquad \xi^- \to \xi^+. \tag{2.2.55}$$

These eigenstates $\xi^{\pm}$ of the Hermitian $p_i\sigma^i$, in equations (2.2.49) and (2.2.51), span the 2D complex vector space, so their completeness relation is

$$\mathbb{I}_{\mathrm{A}\dot{\mathrm{B}}} = \xi^+_{\mathrm{A}}\overline{\xi^+}_{\dot{\mathrm{B}}} + \xi^-_{\mathrm{A}}\overline{\xi^-}_{\dot{\mathrm{B}}}; \tag{2.2.56}$$

Therefore, $p_\mu\sigma^\mu = p_0\mathbb{I} + p_i\sigma^i$ itself must be p_0 times of eq. (2.2.56) plus $|\vec{p}|$ times of eq. (2.2.47).

$$p_\mu\left(\sigma^\mu\right)_{\mathrm{A}\dot{\mathrm{B}}} \equiv p_{\mathrm{A}\dot{\mathrm{B}}} = \lambda_+\xi^+_{\mathrm{A}}\overline{\xi^+}_{\dot{\mathrm{B}}} + \lambda_-\xi^-_{\mathrm{A}}\overline{\xi^-}_{\dot{\mathrm{B}}}, \qquad \lambda_\pm \equiv p_0 \pm |\vec{p}|. \tag{2.2.57}$$

Massive particles If we define $\sqrt{p_\mu\sigma^\mu}$ to be the solution to $\sqrt{p_\mu\sigma^\mu}\sqrt{p_\mu\sigma^\mu} = p_\mu\sigma^\mu$, then

$$\sqrt{p\cdot\sigma} = \sqrt{p_\mu\sigma^\mu} = \sqrt{\lambda_+}\xi^+_{\mathrm{A}}\overline{\xi^+}_{\dot{\mathrm{B}}} + \sqrt{\lambda_-}\xi^-_{\mathrm{A}}\overline{\xi^-}_{\dot{\mathrm{B}}}. \tag{2.2.58}$$

In physical applications where p_μ is the momentum of a particle with mass m, $p_0 \geq |\vec{p}|$ and $p^2 = m^2$, the $\sqrt{\cdot}$ will often be chosen to the positive one – in the following sense. Firstly, the $\lambda_\pm$ in eq. (2.2.57), could have either positive or negative energy:

$$p^2 = m^2 \qquad \Rightarrow \qquad p_0 = \pm E_{\vec{p}} \equiv \pm\sqrt{\vec{p}^2 + m^2}. \tag{2.2.59}$$

We shall choose, for positive energy,

$$\sqrt{\lambda_\pm} = \sqrt{E_{\vec{p}} \pm |\vec{p}|} > 0; \tag{2.2.60}$$

and, for negative energy,

$$\sqrt{\lambda_\pm} = i\sqrt{E_{\vec{p}} \mp |\vec{p}|}, \tag{2.2.61}$$

where the $\sqrt{\cdot}$ is the positive one.

With such a choice, positive energy solutions obey

$$\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} \equiv \sqrt{p_\mu\sigma^\mu}\sqrt{p_\nu\bar{\sigma}^\nu} = \sqrt{\lambda_+\lambda_-}\left(\xi^+_{\rm A}\overline{\xi^+}_{\dot{\rm B}} + \xi^-_{\rm A}\overline{\xi^-}_{\dot{\rm B}}\right) \tag{2.2.62}$$

$$= \sqrt{p^2}\mathbb{I}_{2\times 2} = m\cdot\mathbb{I}_{2\times 2}, \tag{2.2.63}$$

where the orthonormality and completeness of the helicity eigenstates $\xi^\pm$ were used.

Whereas, negative energy solutions obey

$$\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} \equiv \sqrt{p_\mu\sigma^\mu}\sqrt{p_\nu\bar{\sigma}^\nu} = \sqrt{\lambda_+}\sqrt{\lambda_-}\left(\xi^+_{\rm A}\overline{\xi^+}_{\dot{\rm B}} + \xi^-_{\rm A}\overline{\xi^-}_{\dot{\rm B}}\right) \tag{2.2.64}$$

$$= i^2\sqrt{E_{\vec{p}}^2 - \vec{p}^2}\mathbb{I}_{2\times 2} = -m\cdot\mathbb{I}_{2\times 2}. \tag{2.2.65}$$

Additionally, since $(\sqrt{\lambda_\pm})^* = -i\sqrt{E_{\vec{p}} \mp |\vec{p}|}$, we have

$$\sqrt{p\cdot\sigma}^\dagger\sqrt{p\cdot\bar{\sigma}} = (\sqrt{\lambda_+})^*\sqrt{\lambda_-}\xi^+_{\rm A}\overline{\xi^+}_{\dot{\rm B}} + (\sqrt{\lambda_-})^*\sqrt{\lambda_+}\xi^-_{\rm A}\overline{\xi^-}_{\dot{\rm B}} \tag{2.2.66}$$

$$= \sqrt{E_{\vec{p}}^2 - \vec{p}^2}\left(\xi^+_{\rm A}\overline{\xi^+}_{\dot{\rm B}} + \xi^-_{\rm A}\overline{\xi^-}_{\dot{\rm B}}\right) = m\mathbb{I}_{2\times 2}, \tag{2.2.67}$$

$$\sqrt{p\cdot\bar{\sigma}}^\dagger\sqrt{p\cdot\sigma} = m\mathbb{I}_{2\times 2}. \tag{2.2.68}$$

Massless particles For massless particles, $m = 0$ and $p_0 = \pm|\vec{p}|$.

For positive energy $p_0 = |\vec{p}|$, the ξ^- mode becomes a null eigenvector because $\lambda_- = 0$. Whereas, eq. (2.2.57) now reads

$$p_{\rm A\dot{B}} = \xi_{\rm A}\overline{\xi_{\dot{\rm B}}}, \qquad \xi_{\rm A} \equiv \sqrt{2|\vec{p}|}\xi^+_{\rm A}. \tag{2.2.69}$$

For negative energy $p_0 = -|\vec{p}|$, the ξ^+ mode becomes a null eigenvector because $\lambda_+ = 0$. Whereas, eq. (2.2.57) now reads

$$p_{\rm A\dot{B}} = -\xi_{\rm A}\overline{\xi_{\dot{\rm B}}}, \qquad \xi_{\rm A} \equiv \sqrt{2|\vec{p}|}\xi^-_{\rm A}. \tag{2.2.70}$$

Construction of L We have discussed in §(**??**), any operator that is continuously connected to the identity can be written in the form $\exp X$. Since L has unit determinant (cf. (2.2.46)), let us focus on the case where it is continuously connected to the identity whenever it does depend on a set of complex parameters $\{q_\mu\}$, say:

$$L = e^{X(q)}. \tag{2.2.71}$$

Now, if we use eq. (1.3.74), $\det e^X = e^{\mathrm{Tr}[X]}$, we find that

$$\det L = e^{\mathrm{Tr}\ X(q)} = 1. \tag{2.2.72}$$

This implies

$$\mathrm{Tr}X(q) = 2\pi i n, \qquad n = 0, \pm 1, \pm 2, \dots. \tag{2.2.73}$$

Recalling that the $\{\sigma^\mu\}$ form a complete set, we may express

$$X(q) = q_\mu\sigma^\mu \tag{2.2.74}$$

and using the trace properties in eq. (2.2.16), we see that Tr $X(q) = 2q_0 = 2\pi in$. Since this $q_0\sigma^0 = i\pi n\mathbb{I}_{2\times 2}$, which commutes with all the other Pauli matrices, we have at this point

$$L = e^{i\pi n} e^{q_j\sigma^j} = (-)^n e^{q_j\sigma^j} \tag{2.2.75}$$

$$= (-)^n \left(\cos(i|\vec{q}|) - i\frac{q_j\sigma^j}{|\vec{q}|}\sin(i|\vec{q}|)\right) \tag{2.2.76}$$

$$= (-)^n \left(\cosh(|\vec{q}|) + \frac{q_j\sigma^j}{|\vec{q}|}\sinh(|\vec{q}|)\right). \tag{2.2.77}$$

Here, we have replaced $\theta_j \to 2iq_j$ in eq. (2.2.35); and note that $\sqrt{\theta_i\theta_i} = 2i\sqrt{q_iq_i}$ because we have defined the square root to be the positive one. To connect L to the identity, we need to set the $q_j\sigma^j$ terms to zero, since the Pauli matrices $\{\sigma^i\}$ are linearly independent and perpendicular to the identity $\mathbb{I}_{2\times 2}$. This is accomplished by putting $\vec{q} = \vec{0}$; which in turn means n must be even since the cosine/cosh would be unity. To summarize, at this stage:

> We have deduced that the most general unit determinant 2×2 complex matrix that is continuously connected to the identity is, in fact, given by eq. (2.2.35) for arbitrary complex $\vec{\psi}$, which we shall in turn parametrize as
>
> $$L = \exp\left(\frac{1}{2}(\xi_j - i\theta_j)\sigma^j\right), \tag{2.2.78}$$
>
> where the $\{\xi_j\}$ and $\{\theta_j\}$ are real (i.e., $q_j \equiv (1/2)(\xi_j - i\theta_j)$). Its inverse is
>
> $$L^{-1} = \exp\left(-\frac{1}{2}(\xi_j - i\theta_j)\sigma^j\right). \tag{2.2.79}$$
>
> By returning to the transformation in eq. (2.2.44), we will now demonstrate the $\{\xi_j\}$ correspond to Lorentz boosts and $\{\theta_j\}$ spatial rotations.

Problem 2.20. Use eq. (**??**) to argue that, for L belonging to the $\text{SL}_{2,\mathbb{C}}$ group, it obeys

$$L^{-1} = -\epsilon \cdot L^{\mathrm{T}} \cdot \epsilon. \tag{2.2.80}$$

Therefore

$$(L^{-1})^\dagger = -\epsilon \cdot L^* \cdot \epsilon = \epsilon^\dagger \cdot L^* \cdot \epsilon = \epsilon \cdot L^* \cdot \epsilon^\dagger. \tag{2.2.81}$$

$\square$

Rotations Set $\vec{\xi} = 0$ and focus on the case

$$\theta_j\sigma^j \to \theta\sigma^k \tag{2.2.82}$$

for a fixed $1 \le k \le 3$; so that eq. (2.2.78) is now

$$L = \exp\left(-\frac{i}{2}\theta\sigma^k\right) = \cos(\theta/2) - i\sigma^k\sin(\theta/2); \tag{2.2.83}$$

which may be directly inferred from eq. (2.2.35) by setting $\xi_j = 0$. Eq. (2.2.44), in turn, now reads

$$\begin{aligned} p_\mu \sigma^\mu &\to e^{-(i/2)\theta\sigma^k} p_0 e^{(i/2)\theta\sigma^k} + \left(\cos(\theta/2) - i\sigma^k \sin(\theta/2)\right) p_i \sigma^i \left(\cos(\theta/2) + i\sigma^k \sin(\theta/2)\right) \\ &= p_0 + p'_i \sigma^i. \end{aligned} \tag{2.2.84}$$

If $k = 1$, we have p_i rotated on the $(2,3)$ plane:

$$p'_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}_i^{\ j} p_j. \tag{2.2.85}$$

If $k = 2$, we have p_i rotated on the $(1,3)$ plane:

$$p'_i = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}_i^{\ j} p_j. \tag{2.2.86}$$

If $k = 3$, we have p_i rotated on the $(1,2)$ plane:

$$p'_i = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}_i^{\ j} p_j. \tag{2.2.87}$$

Spin Half Note that the presence of the generators of rotation, namely $\sigma^k/2$ in eq. (2.2.83), with eigenvalues $\pm 1/2$, confirms we are dealing with spin$-1/2$ systems.

Problem 2.21. Verify eq. (2.2.84) for any one of the $k = 1, 2, 3$. □

Boosts Next, we set $\vec{\theta} = 0$ and focus on the case

$$\xi_j \sigma^j \to \xi \sigma^k, \tag{2.2.88}$$

again for a fixed $k = 1, 2, 3$. Setting eq. (2.2.77), and remembering n is even,

$$L = \exp\left(\frac{1}{2}\xi\sigma^k\right) = \cosh(\xi/2) + \sigma^k \sinh(\xi/2). \tag{2.2.89}$$

Eq. (2.2.78) is now

$$\begin{aligned} p_\mu \sigma^\mu &\to \left(\cosh(\xi/2) + \sigma^k \sin(\xi/2)\right) p_\mu \sigma^\mu \left(\cosh(\xi/2) + \sigma^k \sin(\xi/2)\right) \\ &= p'_\mu \sigma^\mu. \end{aligned} \tag{2.2.90}$$

If $k = 1$, we have p_μ boosted in the $1-$direction:

$$p'_\mu = \begin{bmatrix} \cosh\xi & \sinh\xi & 0 & 0 \\ \sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_\mu^{\ \nu} p_\nu. \tag{2.2.91}$$

If $k = 2$, we have p_μ boosted in the 2−direction:

$$p'_\mu = \begin{bmatrix} \cosh\xi & 0 & \sinh\xi & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\xi & 0 & \cosh\xi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_\mu^{\ \ \nu} p_\nu. \tag{2.2.92}$$

If $k = 3$, we have p_μ boosted in the 3−direction:

$$p'_\mu = \begin{bmatrix} \cosh\xi & 0 & 0 & \sinh\xi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\xi & 0 & 0 & \cosh\xi \end{bmatrix}_\mu^{\ \ \nu} p_\nu. \tag{2.2.93}$$

Problem 2.22. Verify eq. (2.2.90) for any one of the $k = 1, 2, 3$. □

Boosts/Rotations & SL$_{2,\mathbb{C}}$ Spinors To summarize, we have discovered that the group of 2×2 matrices $\{L\}$ continuously connected to the identity obeying

$$\epsilon^{\text{AB}} L_{\text{A}}^{\ \ \text{I}} L_{\text{B}}^{\ \ \text{J}} = \epsilon^{\text{IJ}} \tag{2.2.94}$$

implements Lorentz transformations

$$L_{\text{A}}^{\ \ \text{I}} \overline{L_{\dot{\text{B}}}^{\ \ \dot{\text{J}}}} \sigma^\mu_{\ \ \text{I}\dot{\text{J}}} = \sigma^\nu_{\ \ \text{A}\dot{\text{B}}} \Lambda_\nu^{\ \ \mu}. \tag{2.2.95}$$

In terms of matrix multiplication,

$$L \sigma^\mu L^\dagger = \sigma^\nu \Lambda_\nu^{\ \ \mu}; \tag{2.2.96}$$

where the $\Lambda_\nu^{\ \ \mu}$ is the 4×4 Lorentz transformations parametrized by $\{\vec{\xi}, \vec{\theta}\}$ satisfying eq. (2.1.5).

Observe that we can take the complex conjugate of equations (2.2.94) and (2.2.96) to deduce that, for the same L in eq. (2.2.96) – L^* not only belongs to SL$_{2,\mathbb{C}}$, it also generates exactly the same Lorentz transformation $\Lambda_\nu^{\ \ \mu}$ in eq. (2.2.96).

$$\epsilon^{\text{AB}} \overline{L_{\text{A}}^{\ \ \text{I}} L_{\text{B}}^{\ \ \text{J}}} = \epsilon^{\text{IJ}}, \tag{2.2.97}$$

$$L^* (\sigma^\mu)^* (L^*)^\dagger = (\sigma^\nu)^* \Lambda_\nu^{\ \ \mu}. \tag{2.2.98}$$

For real p_μ, notice that $\det p_\mu \sigma^\mu = p^2 = \det p_\mu (\sigma^\mu)^*$. Despite generating the same Lorentz transformation, we shall see below, L and L^* are inequivalent representations of SL$_{2,\mathbb{C}}$ – i.e., there is no change-of-basis U such that $ULU^{-1} = L^*$.

Using the dotted and un-dotted index notation in eq. (2.2.57),

$$L_{\text{A}}^{\ \ \text{M}} \overline{L_{\dot{\text{B}}}^{\ \ \dot{\text{N}}}} p_{\text{M}\dot{\text{N}}} = (\sigma^\nu)_{\text{A}\dot{\text{B}}} \Lambda_\nu^{\ \ \mu} p_\mu \equiv p'_{\text{A}\dot{\text{B}}} \tag{2.2.99}$$

$$= \lambda_+ \xi'^{+}_{\text{A}} \overline{\xi'^{+}_{\dot{\text{B}}}} + \lambda_- \xi'^{-}_{\text{A}} \overline{\xi'^{-}_{\dot{\text{B}}}}; \tag{2.2.100}$$

where the 'new' but un-normalized eigenvectors and eigenvalues are

$$\xi'^{\pm}_{\text{A}} \left(p'_\mu = \Lambda_\mu^{\ \ \nu} p_\nu\right) = L_{\text{A}}^{\ \ \text{B}} \xi^{\pm}_{\text{B}} (p_\mu) \qquad \text{and} \qquad \overline{\xi'^{\pm}_{\text{A}}} \left(p'_\mu = \Lambda_\mu^{\ \ \nu} p_\nu\right) = \overline{L_{\dot{\text{A}}}^{\ \ \dot{\text{B}}}} \overline{\xi^{\pm}_{\dot{\text{B}}} (p_\mu)} \tag{2.2.101}$$

with the old eigenvalues

$$\lambda_\pm \equiv p_0 \pm |\vec{p}|. \tag{2.2.102}$$

Any 2-component object that transforms according to eq. (2.2.101), where the ${L_{\rm A}}^{\rm B}$ are $\text{SL}_{2,\mathbb{C}}$ matrices, is said to be a *spinor*. As already alluded to, in the context of $p_\mu \sigma^\mu$, they are also helicity eigenstates of $p_i \sigma^i$.

If we normalize the spinors to unity

$$\xi''^{\pm}_{\rm A} = \xi'^{\pm}_{\rm A} \left\{ \left(\xi'^{\pm}\right)^\dagger \xi'^{\pm} \right\}^{-\frac{1}{2}}; \tag{2.2.103}$$

then eq. (2.2.99) now reads

$${L_{\rm A}}^{\rm M} \overline{{L_{\dot{\rm B}}}^{\dot{\rm N}}} p_{\rm M\dot{N}} = p'_{\rm A\dot{B}} = \lambda'_+ \xi''^{+}_{\rm A} \overline{\xi''^{+}_{\dot{\rm B}}} + \lambda'_- \xi''^{-}_{\rm A} \overline{\xi''^{-}_{\dot{\rm B}}}; \tag{2.2.104}$$

with the new eigenvalues

$$\lambda'_\pm \equiv p'_0 \pm |\vec{p}'|. \tag{2.2.105}$$

*L vs. L** Furthermore, note that the L and its complex conjugate $\overline{L} = L^*$ are not equivalent transformations once Lorentz boosts are included; i.e., once $\vec{\xi} \neq 0$. To see this, we first recall, for any Taylor-expandable function f, $Uf(A)U^{-1} = f(UAU^{-1})$ for arbitrary operators A and (invertible) U. Remembering the form of L in (2.2.78), let us consider

$$UL^*U^{-1} = \exp\left(\frac{1}{2} U \left(\xi_j + i\theta_j\right) (\sigma^j)^* U^{-1}\right). \tag{2.2.106}$$

Suppose it is possible to find a change-of-basis such that L^* becomes L in eq. (2.2.78), that means we must have for a given j,

$$U \cdot \rho_j e^{-i\vartheta_j} (\sigma^j)^* U^{-1} = \rho_j e^{i\vartheta_j} \sigma^j, \tag{2.2.107}$$

$$\rho_j e^{i\vartheta_j} \equiv \xi_j - i\theta_j, \tag{2.2.108}$$

$$\rho_j = \sqrt{\xi_j^2 + \theta_j^2}, \qquad \tan\vartheta_j = -\frac{\theta_j}{\xi_j}. \tag{2.2.109}$$

Taking the determinant on both sides of the first line, for a fixed j,

$$\det\left[e^{-2i\vartheta_j} (\sigma^j)^*\right] = \det\left[\sigma^j\right] \tag{2.2.110}$$

$$e^{-4i\vartheta_j} \overline{\det\left[\sigma^j\right]} = -e^{-4i\vartheta_j} = \det\left[\sigma^j\right] = -1. \tag{2.2.111}$$

(We have used $\det\sigma^i = -1$.) The only situation L may be mapped to L^* and vice versa through a change-of-basis occurs when $\vartheta_j = 2\pi n/4 = \pi n/2$ for integer n. For even n, this corresponds to pure boosts, because

$$\xi_j - i\theta_j = \rho_j e^{i\frac{\pi}{2}n} = \pm\rho_j. \tag{2.2.112}$$

For odd n, this corresponds to pure rotations, because

$$\xi_j - i\theta_j = \rho_j e^{i\frac{\pi}{2}n} = \pm i\rho_j. \tag{2.2.113}$$

However, as we shall show below, there is no transformation U that could bring a pure boost L^* back to L:

$$U \cdot L[\vec{\theta} = \vec{0}]^* \cdot U^{-1} = U \cdot e^{\frac{1}{2}\xi_j (\sigma^j)^*} \cdot U^{-1} \neq e^{\frac{1}{2}\xi_j \sigma^j} = L[\vec{\theta} = \vec{0}]. \tag{2.2.114}$$

In other words, only the complex conjugate of a pure rotation may be mapped into the same pure rotation. In fact, using $\epsilon(\sigma^i)^*\epsilon^\dagger = -\sigma^i$ in eq. (2.2.26),

$$\epsilon \cdot \overline{L[\vec{\xi} = 0]} \cdot \epsilon^\dagger = \epsilon e^{+(i/2)\theta_j (\sigma^j)^*} \epsilon^\dagger = e^{+(i/2)\theta_j \epsilon (\sigma^j)^* \epsilon^\dagger} \tag{2.2.115}$$

$$= e^{-(i/2)\theta_j \sigma^j} = L[\vec{\xi} = 0]. \tag{2.2.116}$$

But – to reiterate – once $\vec{\xi} \neq 0$, there is no U such that $UL[\vec{\xi}, \vec{\theta}]^* U^{-1} = L[\vec{\xi}, \vec{\theta}]$.

Generators That L and L^* are generically inequivalent transformations is why the former corresponds to un-dotted indices and the latter to dotted ones – the notation helps distinguishes between them. At this point, let us write

$$L = \exp\left(-i\xi_j i\frac{\sigma^j}{2} - i\theta_j \frac{\sigma^j}{2}\right); \tag{2.2.117}$$

and by referring to generic Lorentz transformation in eq. (2.2.3), we may identify the boost and rotation generators as, respectively,

$$K^i_{\rm R} = i\frac{\sigma^i}{2} \qquad \text{and} \qquad J^i_{\rm R} = \frac{\sigma^i}{2}. \tag{2.2.118}$$

In this representation, therefore, the Lie algebra in equations (2.2.7) and (2.2.8) read

$$J^i_+ = \frac{1}{4}\left(\sigma^i + i^2\sigma^i\right) = 0 \tag{2.2.119}$$

$$J^i_- = \frac{1}{4}\left(\sigma^i - i^2\sigma^i\right) = \frac{\sigma^i}{2}. \tag{2.2.120}$$

The J^i_+ generators describe spin j_+ zero; whereas the J^i_- ones spin j_- one-half (since the Pauli matrices have eigenvalues ± 1). We therefore label this is as the $(j_+, j_-) = (0, 1/2)$ representation.

As for the L^*, we may express it as

$$L^* = \exp\left(-i\xi_j i\frac{(\sigma^j)^*}{2} - i\theta_j \frac{-(\sigma^j)^*}{2}\right) \tag{2.2.121}$$

and again referring to eq. (2.2.3),

$$K^i = i\frac{(\sigma^i)^*}{2} \qquad \text{and} \qquad J^i = -\frac{(\sigma^i)^*}{2}. \tag{2.2.122}$$

In this case, we may compute the Lie algebra in equations (2.2.7) and (2.2.8):

$$J_+^i = \frac{1}{4}\left(-(\sigma^i)^* + i^2(\sigma^i)^*\right) = -\frac{(\sigma^i)^*}{2} \tag{2.2.123}$$

$$J_-^i = \frac{1}{4}\left(-(\sigma^i)^* - i^2(\sigma^i)^*\right) = 0. \tag{2.2.124}$$

This is the $(j_+, j_-) = (1/2, 0)$ representation. We may also recall eq. (2.2.25) and discover that eq. (2.2.122) is equivalent to

$$K^i = \epsilon^\dagger \left(-\frac{i}{2}\sigma^i\right)\epsilon \qquad \text{and} \qquad J^i = \epsilon^\dagger \left(\frac{1}{2}\sigma^i\right)\epsilon; \tag{2.2.125}$$

which in turn implies we must also obtain an equivalent $(j_+, j_-) = (1/2, 0)$ representation using

$$K_{\text{L}}^i = -\frac{i}{2}\sigma^i \qquad \text{and} \qquad J_{\text{L}}^i = \frac{1}{2}\sigma^i. \tag{2.2.126}$$

At this point, eq. (2.2.125) applied to eq. (2.2.121) hands us

$$L^* = \epsilon^\dagger \exp\left(-\frac{1}{2}\left(\vec{\xi} + i\vec{\theta}\right)\cdot\vec{\sigma}\right)\epsilon \tag{2.2.127}$$

$$= \epsilon^\dagger (L^\dagger)^{-1}\epsilon, \tag{2.2.128}$$

where the second equality follows from the hermicity of the $\{\sigma^i\}$ and the fact that $(\exp(q_i\sigma^i))^{-1} = \exp(-q_i\sigma^i)$.

For later use, we employ the notation in eq. (2.2.21) and record here that eq. (2.2.126) may be obtained through

$$J_{\text{L}}^{\mu\nu} \equiv \frac{i}{4}\sigma^{[\mu}\bar{\sigma}^{\nu]}, \tag{2.2.129}$$

$$J_{\text{L}}^{0i} = \frac{i}{4}\left(\sigma^0(-)\sigma^i - \sigma^i\sigma^0\right) \tag{2.2.130}$$

$$= -\frac{i}{2}\sigma^i = K_{\text{L}}^i; \tag{2.2.131}$$

$$J_{\text{L}}^{ab} = \frac{i}{4}\left(\sigma^a(-)\sigma^b - \sigma^b(-)\sigma^a\right) \tag{2.2.132}$$

$$= -\frac{i}{4}[\sigma^a, \sigma^b] = \frac{1}{2}\epsilon^{abc}\sigma^c = \epsilon^{abc}J_{\text{L}}^c. \tag{2.2.133}$$

This is consistent with equations (2.2.1) and (2.2.2). Similarly, eq. (2.2.122) may be obtained through

$$J_{\text{R}}^{\mu\nu} \equiv \frac{i}{4}\bar{\sigma}^{[\mu}\sigma^{\nu]}, \tag{2.2.134}$$

$$J_{\text{R}}^{0i} = \frac{i}{4}\left(\sigma^0\sigma^i - (-)\sigma^i\sigma^0\right) \tag{2.2.135}$$

$$= +\frac{i}{2}\sigma^i = K_{\text{R}}^i; \tag{2.2.136}$$

$$J_{\mathrm{R}}^{ab} = \frac{i}{4}\left((-)\sigma^a \sigma^b - (-)\sigma^b \sigma^a\right) \tag{2.2.137}$$

$$= -\frac{i}{4}[\sigma^a, \sigma^b] = \frac{1}{2}\epsilon^{abc}\sigma^c = \epsilon^{abc} J_{\mathrm{R}}^c. \tag{2.2.138}$$

Summary

For the same set of real boost $\{\xi_j\}$ and rotation $\{\theta_j\}$ parameters, the $(j_+, j_-) = (0, 1/2)$ representation of $\mathrm{SL}_{2,\mathbb{C}}$ is provided by the transformation

$$L\left(\vec{\xi}, \vec{\theta}\right) = \exp\left(-i\vec{\xi} \cdot \vec{K}_{\mathrm{R}} - i\vec{\theta} \cdot \vec{J}_{\mathrm{R}}\right) = e^{\frac{1}{2}\left(\vec{\xi} - i\vec{\theta}\right)\cdot\vec{\sigma}}, \tag{2.2.139}$$

$$\vec{\xi} \cdot \vec{K}_{\mathrm{R}} \equiv \xi_j K_{\mathrm{R}}^j, \qquad \vec{\theta} \cdot \vec{J}_{\mathrm{R}} \equiv \theta_i J_{\mathrm{R}}^i, \tag{2.2.140}$$

$$K_{\mathrm{R}}^i = \frac{i}{2}\sigma^i = \frac{i}{4}\bar{\sigma}^{[0}\sigma^{i]}, \qquad J_{\mathrm{R}}^i = \frac{1}{2}\sigma^i = \frac{1}{2}\epsilon^{imn}\frac{i}{4}\bar{\sigma}^{[m}\sigma^{n]}; \tag{2.2.141}$$

whereas the inequivalent $(j_+, j_-) = (1/2, 0)$ representation of $\mathrm{SL}_{2,\mathbb{C}}$ is provided by its complex conjugate

$$\overline{L\left(\vec{\xi}, \vec{\theta}\right)} = \epsilon^\dagger \exp\left(-i\vec{\xi} \cdot \vec{K}_{\mathrm{L}} - i\vec{\theta} \cdot \vec{J}_{\mathrm{L}}\right)\epsilon = \epsilon^\dagger e^{-\frac{1}{2}\left(\vec{\xi} + i\vec{\theta}\right)\cdot\vec{\sigma}}\epsilon \tag{2.2.142}$$

$$= \epsilon^\dagger \left(L\left(\vec{\xi}, \vec{\theta}\right)^\dagger\right)^{-1} \epsilon = \epsilon^\dagger \left(L\left(\vec{\xi}, \vec{\theta}\right)^{-1}\right)^\dagger \epsilon, \tag{2.2.143}$$

$$\vec{\xi} \cdot \vec{K}_{\mathrm{L}} \equiv \xi_j K_{\mathrm{L}}^j, \qquad \vec{\theta} \cdot \vec{J}_{\mathrm{L}} \equiv \theta_i J_{\mathrm{L}}^i \tag{2.2.144}$$

$$K_{\mathrm{L}}^i = -\frac{i}{2}\sigma^i = \frac{i}{4}\sigma^{[0}\bar{\sigma}^{i]}, \qquad J_{\mathrm{L}}^i = \frac{1}{2}\sigma^i = \frac{1}{2}\epsilon^{imn}\frac{i}{4}\bar{\sigma}^{[m}\sigma^{n]}. \tag{2.2.145}$$

Problem 2.23. Consider the infinitesimal $\mathrm{SL}_{2,\mathbb{C}}$ transformation

$$L_{\mathrm{A}}{}^{\mathrm{B}} = \delta_{\mathrm{A}}{}^{\mathrm{B}} + \omega_{\mathrm{A}}{}^{\mathrm{B}}. \tag{2.2.146}$$

Show that, by viewing ϵ^{AB} and $\omega_{\mathrm{A}}{}^{\mathrm{B}}$ as matrices,

$$\epsilon \cdot \omega = (\epsilon \cdot \omega)^{\mathrm{T}}. \tag{2.2.147}$$

From this, deduce

$$\omega_{\mathrm{A}}{}^{\mathrm{B}} = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}, \tag{2.2.148}$$

where α, β, and γ are arbitrary complex parameters. Notice this yields 6 real parameters – in accordance to the 3 directions for boosts plus the 3 directions for rotations we uncovered in eq. (2.2.78). □

Problem 2.24. Check that the J^i and K^i in eq. (2.2.118), (2.2.122), and (2.2.126) satisfy the $\mathrm{SO}_{3,1}$ Lie Algebra (2.2.4), (2.2.5) and (2.2.6). □

Transformation of Chiral Spinors Even though we have defined spinors to be any 2 component object $\xi_{\rm A}$ that transforms as $\xi \to L\xi$ for all $L \in {\rm SL}_{2,\mathbb{C}}$, our discovery of two inequivalent representations demand that we sharpen this notion further.

Specifically, for the $L(\vec{\xi},\vec{\theta})$ in eq. (2.2.139), we would say that the a spinor transforming as

$$\xi'_{\rm A}\left(p'_\mu = \Lambda_\mu{}^\nu p_\nu\right) = L_{\rm A}{}^{\rm I}\xi_{\rm I}\left(p_\nu\right) \qquad \text{and} \tag{2.2.149}$$

$$\xi'_{\rm A}\left(p'_\mu = p_\nu \Lambda^\nu{}_\mu\right) = (L^{-1})_{\rm A}{}^{\rm I}\xi_{\rm I}\left(p_\nu\right), \tag{2.2.150}$$

is a $(j_+, j_-) = (0, 1/2)$ one; or "right-handed chiral spinor'. Whereas – recalling eq. (2.2.98) – for the same $L(\vec{\xi},\vec{\theta})$ in eq. (2.2.139), we would say that the spinor transforming as

$$\kappa'_{\rm A}\left(p'_\mu = \Lambda_\mu{}^\nu p_\nu\right) = \overline{L_{\rm A}{}^{\rm I}}\kappa_{\rm I}\left(p_\nu\right) \qquad \text{and} \tag{2.2.151}$$

$$\kappa'_{\rm A}\left(p'_\mu = p_\nu \Lambda^\nu{}_\mu\right) = \overline{(L^{-1})_{\rm A}{}^{\rm I}}\kappa_{\rm I}\left(p_\nu\right) \tag{2.2.152}$$

is a $(j_+, j_-) = (1/2, 0)$ one; or "left-handed chiral spinor'. We next turn to a different basis to express eq. (2.2.151).

Problem 2.25. Explain why

$$L = \epsilon^\dagger \cdot (L^{-1})^{\rm T} \cdot \epsilon = \epsilon^\dagger \cdot (L^{\rm T})^{-1} \cdot \epsilon, \tag{2.2.153}$$

$$L^* = \epsilon^\dagger \cdot (L^{-1})^\dagger \cdot \epsilon = \epsilon^\dagger \cdot (L^\dagger)^{-1} \cdot \epsilon. \tag{2.2.154}$$

(Hint: Recall eq. (**??**).) Since L^* is inequivalent to L, this shows that $(L^{-1})^\dagger$ is also inequivalent to L.

Then show that

$$(L^{-1})^\dagger \bar{\sigma}^\mu L^{-1} = \bar{\sigma}^\nu \Lambda_\nu{}^\mu; \tag{2.2.155}$$

followed by

$$L^\dagger \bar{\sigma}^\mu L = \Lambda^\mu{}_\nu \bar{\sigma}^\nu \tag{2.2.156}$$

We see from equations (2.2.98), (2.2.154) and (2.2.155) that, since $(L(\vec{\xi},\vec{\theta})^{-1})^\dagger$ is equivalent to $L(\vec{\xi},\vec{\theta})^*$, and since $L(\vec{\xi},\vec{\theta})^*$ implements the same Lorentz transformation $\Lambda_\nu{}^\mu$ as $L(\vec{\xi},\vec{\theta})$, the $(L(\vec{\xi},\vec{\theta})^\dagger)^{-1}$ also implements on the right-handed spinor the same $\Lambda_\nu{}^\mu$. Whereas, $L(\vec{\xi},\vec{\theta})^\dagger$ implements on the right handed spinor the inverse Lorentz transformation $\Lambda^\nu{}_\mu$. □

For the same $L(\vec{\xi},\vec{\theta})$ in eq. (2.2.139), we would say that the spinor transforming as

$$\eta'_{\rm A}\left(p'_\mu = \Lambda_\mu{}^\nu p_\nu\right) = \left((L^\dagger)^{-1}\right)_{\rm A}{}^{\rm I}\eta_{\rm I}\left(p_\nu\right) \qquad \text{and} \tag{2.2.157}$$

$$\eta'_{\rm A}\left(p'_\mu = p_\nu \Lambda^\nu{}_\mu\right) = \left(L^\dagger\right)_{\rm A}{}^{\rm I}\eta_{\rm I}\left(p_\nu\right) \tag{2.2.158}$$

is a $(j_+, j_-) = (1/2, 0)$ one; or "left-handed chiral spinor'; where the κ in eq. (2.2.151) and η in eq. (2.2.157) are related through the change-of-basis

$$\eta' = \epsilon \cdot \kappa' \qquad \text{and} \qquad \eta = \epsilon \cdot \kappa. \tag{2.2.159}$$

Because $\det p \cdot \bar{\sigma} = \det p_\mu \bar{\sigma}^\mu = p^2$, we see the spinor η obeying equations (2.2.157) and (2.2.158) must yield

$$\bar{p}_{\mathrm{A}\dot{\mathrm{B}}} \equiv p_\mu \bar{\sigma}^\mu{}_{\mathrm{A}\dot{\mathrm{B}}} = \lambda_+ \eta^+_{\mathrm{A}} \overline{\eta^+_{\dot{\mathrm{B}}}} + \lambda_- \eta^-_{\mathrm{A}} \overline{\eta^-_{\dot{\mathrm{B}}}}; \tag{2.2.160}$$

$$\lambda_\pm \equiv p_0 \pm |\vec{p}|. \tag{2.2.161}$$

Problem 2.26. $\mathrm{SL}_{2,\mathbb{C}}$ Covariant and Invariant Objects Suppose the spinor ξ is a right-handed spinor (i.e., subject to equations (2.2.149) and (2.2.150)) and q_μ is a Lorentz spacetime tensor that obeys

$$q'_\mu = \Lambda_\mu{}^\nu q_\nu; \tag{2.2.162}$$

show that

$$(\bar{\sigma}^\mu q'_\mu)(L\xi) = (L^\dagger)^{-1}(\bar{\sigma}^\mu q_\mu)\xi, \tag{2.2.163}$$

$$\epsilon \cdot (L\xi)^* = (L^\dagger)^{-1} \epsilon \cdot \xi^*. \tag{2.2.164}$$

Likewise, suppose u_μ is a Lorentz spacetime tensor that obeys

$$u'_\mu = u_\nu \Lambda^\nu{}_\mu; \tag{2.2.165}$$

show that show that

$$(\sigma^\mu u'_\mu)(L^\dagger \eta) = L^{-1}(\sigma^\mu u_\mu)\eta, \tag{2.2.166}$$

$$\epsilon \cdot (L^\dagger \eta)^* = L^{-1} \epsilon \cdot \eta^*. \tag{2.2.167}$$

Roughly speaking, $(\bar{\sigma} \cdot q)\xi$ and $\epsilon \cdot \xi^*$ transform like the left-handed spinor η; while $(\sigma \cdot u)\eta$ and $\epsilon^\dagger \eta^*$ transform like the right-handed spinor ξ.

Next, explain how

$$\xi^\dagger \bar{\sigma}^\mu \xi \qquad \text{and} \qquad \eta^\dagger \sigma^\mu \eta \tag{2.2.168}$$

transform under their relevant $\mathrm{SL}_{2,\mathbb{C}}$ transformations. Are

$$\xi^\dagger \xi \qquad \text{and} \qquad \eta^\dagger \eta \tag{2.2.169}$$

scalars under their relevant $\mathrm{SL}_{2,\mathbb{C}}$ transformations? Are

$$\xi^\dagger \eta \qquad \text{and} \qquad \eta^\dagger \xi \tag{2.2.170}$$

scalars under their relevant $\mathrm{SL}_{2,\mathbb{C}}$ transformations? □

PDEs for Spinors To form partial differential equations (PDEs) for spinor fields, the guiding principle is that they transform covariantly under Lorentz (i.e., $\mathrm{SL}_{2,\mathbb{C}}$) transformations, so they take the same form in all inertial frames.

Majorana Equations Firstly, recalling the momentum p_μ dependence in the transformation rule of ξ in eq. (2.2.149), we see that q_μ in eq. (2.2.163) may be replaced with it: $q_\mu = p_\mu$.

If ξ is now viewed as the Fourier coefficient of its position spacetime counterpart, we may now recognize

$$(\bar{\sigma}^\mu p_\mu)\xi\,(p)\,e^{-ip\cdot x} = i(\bar{\sigma}^\mu \partial_\mu)\left(\xi\,(p)\,e^{-ip\cdot x}\right). \tag{2.2.171}$$

Because the terms in equations (2.2.163) and (2.2.164) transform the same way, under $\xi' = L\xi$, we may immediate write down the $(0, 1/2)$ *Majorana equation* in position space:

$$i\bar{\sigma}^\mu \partial_\mu \xi(x) = m\ \epsilon \cdot \xi(x)^*. \tag{2.2.172}$$

The m here is of dimensions mass, because the left hand side involves a derivative, i.e., 1/length.

A similar discussion will let us write down the $(1/2, 0)$ counterpart from the terms in equations (2.2.166) and (2.2.167):

$$i\sigma^\mu \partial_\mu \eta(x) = m\ \epsilon \cdot \eta(x)^*. \tag{2.2.173}$$

Weyl Equations Setting $m = 0$ in equations (2.2.172) and (2.2.173) hands us the Weyl equations

$$i\bar{\sigma}^\mu \partial_\mu \xi = 0 \qquad \text{and} \qquad i\sigma^\mu \partial_\mu \eta = 0. \tag{2.2.174}$$

Dirac Equations Under the transformation $\xi' = L\xi$, eq. (2.2.163) transforms as $(\bar{\sigma}^\mu q'_\mu)\xi' = (L^\dagger)^{-1}(\bar{\sigma}^\mu q_\mu)\xi$, which thus transforms in the same manner as $\eta' = (L^\dagger)^{-1}\eta$. (Recall too, eq. (2.2.154) tells us $(L^\dagger)^{-1}$ is equivalent to L^*.) In a similar vein, under the transformation $\eta' = (L^\dagger)^{-1}\eta$, eq. (2.2.166) transforms as $(\sigma^\mu u'_\mu)\eta' = L(\sigma^\mu u_\mu)\eta$, which thus transforms in the same manner as $\xi' = L\xi$. Since L and L^* implement the same Lorentz transformation, we may write down the following pair of Lorentz covariant PDEs:

$$i\bar{\sigma}^\mu \partial_\mu \xi = m\ \eta \qquad \text{and} \qquad i\sigma^\mu \partial_\mu \eta = m\ \xi. \tag{2.2.175}$$

[37]The pair of PDEs in eq. (2.2.175) is known as the Dirac equation(s).

Completeness of $\{\sigma^\mu\}$: Spacetime vs. Spinor Indices Since the $\{\sigma^\mu\}$ form an orthonormal basis, they must admit some form of the completeness relation in eq. (**??**). Now, according to eq. (2.2.20), $c_\mu \sigma^\mu \Leftrightarrow c_\mu = (1/2)\text{Tr}\left[(c_\nu \sigma^\nu)\sigma^\mu\right]$ for any complex coefficients $\{c_\nu\}$. (We will not distinguish between dotted and un-dotted indices for now.) Consider

$$c_\mu (\sigma^\mu)_{\text{AB}} = \sum_{0\leq\mu\leq 3} \frac{1}{2}(\sigma^\mu)_{\text{AB}} \text{Tr}\left[(\sigma^\mu)^\dagger c_\nu \sigma^\nu\right] \tag{2.2.176}$$

$$= \sum_{1\leq \text{C,D}\leq 2} \left(\sum_{0\leq\mu\leq 3} \frac{1}{2}(\sigma^\mu)_{\text{AB}} \overline{(\sigma^\mu)^{\text{T}}}_{\text{DC}}\right) c_\nu (\sigma^\nu)_{\text{CD}} \tag{2.2.177}$$

$$= \sum_{1\leq \text{C,D}\leq 2} \left(\sum_{0\leq\mu\leq 3} \frac{1}{2}(\sigma^\mu)_{\text{AB}} \overline{(\sigma^\mu)}_{\text{CD}}\right) c_\nu (\sigma^\nu)_{\text{CD}}. \tag{2.2.178}$$

[37]That the same mass m appears in both equations, follows from the demand that the associated Lagrangian density $\mathcal{L}_{\text{Dirac mass}} = -m\left(\eta^\dagger \xi + \xi^\dagger \eta\right)$ (and hence its resulting contribution to the Hamiltonian) be Hermitian.

We may view the terms in the parenthesis on the last line as an operator that acts on the operator $c_\nu \sigma^\nu$. But since c_ν was arbitrary, it must act on each and every σ^ν to return σ^ν, since the left hand side is $c_\nu \sigma^\nu$. And because the $\{\sigma^\nu\}$ are the basis kets of the space of operators acting on a 2D complex vector space, the term in parenthesis must be the identity.

$$\sum_{0\leq\mu\leq 3} \frac{1}{2}(\sigma^\mu)_{\mathrm{AB}}\overline{(\sigma^\mu)}_{\mathrm{CD}} = \sum_{0\leq\mu\leq 3} \frac{1}{2}(\sigma^\mu)_{\mathrm{AB}}(\sigma^\mu)^{\mathrm{T}}_{\mathrm{CD}} = \delta^{\mathrm{C}}_{\mathrm{A}}\delta^{\mathrm{D}}_{\mathrm{B}} \tag{2.2.179}$$

In the second equality we have used the Hermitian nature of σ^μ to deduce $(\sigma^\mu)^\dagger_{\mathrm{AB}} = \overline{(\sigma^\mu)^{\mathrm{T}}}_{\mathrm{AB}} = (\sigma^\mu)_{\mathrm{AB}} \Leftrightarrow (\sigma^\mu)^{\mathrm{T}}_{\mathrm{AB}} = (\sigma^\mu)^*_{\mathrm{AB}}$. If we further employ $(\sigma^\mu)^* = \epsilon \cdot \bar{\sigma}_\mu \cdot \epsilon^\dagger = \epsilon \cdot \bar{\sigma}_\mu \cdot \epsilon^{\mathrm{T}}$ in eq. (2.2.30) within the leftmost expression,

$$\sum_{0\leq\mu\leq 3} \frac{1}{2}(\sigma^\mu)_{\mathrm{AB}}\overline{(\sigma^\mu)}_{\mathrm{CD}} = \sum_{0\leq\mu\leq 3} \frac{1}{2}(\sigma^\mu)_{\mathrm{AB}}\epsilon^{\mathrm{CM}}\epsilon^{\mathrm{DN}}(\bar{\sigma}_\mu)_{\mathrm{MN}}. \tag{2.2.180}$$

If we now restore the dotted notation on the right index, so that

$$\epsilon^{\mathrm{CM}}\epsilon^{\dot{\mathrm{D}}\dot{\mathrm{N}}}(\bar{\sigma}_\mu)_{\mathrm{M}\dot{\mathrm{N}}} \equiv (\bar{\sigma}_\mu)^{\mathrm{M}\dot{\mathrm{N}}}, \tag{2.2.181}$$

then eq. (2.2.179), with Einstein summation in force, becomes

$$\frac{1}{2}\sigma^\mu{}_{\mathrm{AB}}\bar{\sigma}_\mu{}^{\mathrm{CD}} = \delta^{\mathrm{C}}_{\mathrm{A}}\delta^{\mathrm{D}}_{\mathrm{B}} \tag{2.2.182}$$

with

$$\sigma_\mu{}^{\mathrm{CD}} \equiv (\sigma_\mu)_{\mathrm{EF}}\epsilon^{\mathrm{EC}}\epsilon^{\mathrm{FD}}. \tag{2.2.183}$$

Next, consider

$$(\sigma^\mu)_{\mathrm{M}\dot{\mathrm{N}}}(\sigma_\nu)^{\mathrm{M}\dot{\mathrm{N}}} = (\sigma^\mu)_{\mathrm{M}\dot{\mathrm{N}}}\epsilon^{\dot{\mathrm{M}}\dot{\mathrm{A}}}\epsilon^{\dot{\mathrm{N}}\dot{\mathrm{B}}}(\sigma_\mu)_{\mathrm{A}\dot{\mathrm{B}}} = (\sigma^\mu)^{\mathrm{T}}_{\dot{\mathrm{N}}\mathrm{M}}\epsilon^{\dot{\mathrm{M}}\dot{\mathrm{A}}}(\sigma_\mu)_{\mathrm{A}\dot{\mathrm{B}}}(\epsilon^{\mathrm{T}})^{\dot{\mathrm{B}}\dot{\mathrm{N}}} \tag{2.2.184}$$

$$= \mathrm{Tr}\left[(\sigma^\mu)^{\mathrm{T}} \cdot \epsilon \cdot \sigma_\mu \cdot \epsilon^\dagger\right] = \mathrm{Tr}\left[\epsilon^\dagger \cdot (\sigma^\mu)^{\mathrm{T}} \cdot \epsilon \cdot \sigma_\nu\right] \tag{2.2.185}$$

$$= \mathrm{Tr}\left[\bar{\sigma}^\mu \sigma_\nu\right] = \mathrm{Tr}\left[\sigma_\mu \sigma_\nu\right]. \tag{2.2.186}$$

Invoking the orthonormality of the $\{\sigma^\mu\}$ in eq. (2.2.20),

$$\frac{1}{2}(\sigma^\mu)_{\mathrm{M}\dot{\mathrm{N}}}(\sigma_\nu)^{\mathrm{M}\dot{\mathrm{N}}} = \delta^\mu{}_\nu. \tag{2.2.187}$$

Equation (2.2.187) tell us we may view the spacetime Lorentz index μ and the pair of spinor indices $\mathrm{A}\dot{\mathrm{B}}$ as different basis for describing tensors. For example, we may now switch between the momentum p_μ and $p_{\mathrm{A}\dot{\mathrm{B}}}$ via:

$$p_\mu \sigma^\mu{}_{\mathrm{A}\dot{\mathrm{B}}} = p_{\mathrm{A}\dot{\mathrm{B}}} \qquad \Leftrightarrow \qquad p_\mu = \frac{1}{2}\sigma_\mu{}^{\mathrm{A}\dot{\mathrm{B}}} p_{\mathrm{A}\dot{\mathrm{B}}}, \tag{2.2.188}$$

where the latter relation is a direct consequence of eq. (2.2.187),

$$p_\mu = \frac{1}{2}\sigma_\mu{}^{\mathrm{A}\dot{\mathrm{B}}}\sigma^\nu{}_{\mathrm{A}\dot{\mathrm{B}}} p_\nu = \delta^\nu_\mu p_\nu = p_\mu, \tag{2.2.189}$$

Levi-Civita as Spinor-Metric The Levi-Civita symbol may be viewed as the 'metric' for the both the dotted and un-dotted spinor. We will move the un-dotted indices as follows:

$$\xi_{\rm A} = \epsilon_{\rm AB}\xi^{\rm B} \qquad \text{and} \qquad \xi^{\rm A} = \xi_{\rm B}\epsilon^{\rm BA}. \tag{2.2.190}$$

Numerically, $\xi_1 = \epsilon_{12}\xi^2 = \xi^2$ while $\xi_2 = \epsilon_{21}\xi^1 = -\xi^1$. Notice we contract with the right index of ϵ when lowering the spinor index; but with the left index when raising. This is because the Levi-Civita symbol is anti-symmetric, and this distinction is necessary for consistency:

$$\xi_{\rm A} = \epsilon_{\rm AB}\xi^{\rm B} = \epsilon_{\rm AB}\epsilon^{\rm CB}\xi_{\rm C} = -\epsilon_{\rm AB}\epsilon^{\rm BC}\xi_{\rm C} \tag{2.2.191}$$
$$= \delta^{\rm C}_{\rm A}\xi_{\rm C} = \xi_{\rm A}. \tag{2.2.192}$$

Similarly,

$$\xi_{\dot{\rm A}} = \epsilon_{\dot{\rm A}\dot{\rm B}}\xi^{\dot{\rm B}} \qquad \text{and} \qquad \xi^{\dot{\rm A}} = \xi_{\dot{\rm B}}\epsilon^{\dot{\rm B}\dot{\rm A}}. \tag{2.2.193}$$

We may even move the indices of ϵ; for instance, keeping in mind $\epsilon^2 = -\mathbb{I}$,

$$\epsilon_{\rm AB} = \epsilon_{\rm AM}\epsilon_{\rm BN}\epsilon^{\rm MN} \tag{2.2.194}$$
$$= -\delta^{\rm N}_{\rm A}\epsilon_{\rm BN} = -\epsilon_{\rm BA}. \tag{2.2.195}$$

The primary reason why we may move these indices with ϵ and view the latter as a metric, is because the 'scalar product'

$$\xi \cdot \eta \equiv \epsilon^{\rm IJ}\xi_{\rm I}\eta_{\rm J} = \xi^{\rm J}\eta_{\rm J} = -\epsilon^{\rm JI}\eta_{\rm J}\xi_{\rm I} = -\eta\cdot\xi \tag{2.2.196}$$

is invariant under Lorentz $\mathrm{SL}_{2,\mathbb{C}}$ transformations. For, under the replacement

$$\xi_{\rm I} \to L_{\rm I}{}^{\rm A}\xi_{\rm A} \qquad \text{and} \qquad \eta_{\rm I} \to L_{\rm I}{}^{\rm A}\eta_{\rm A}, \tag{2.2.197}$$

the 'scalar product' transforms as

$$\xi\cdot\eta \to \epsilon^{\rm IJ}L_{\rm I}{}^{\rm A}L_{\rm J}{}^{\rm B}\xi_{\rm A}\eta_{\rm B} \tag{2.2.198}$$
$$= (\det L)\epsilon^{\rm AB}\xi_{\rm A}\eta_{\rm B} = \xi\cdot\eta. \tag{2.2.199}$$

The second equality is due to the defining condition of the $\mathrm{SL}_{2,\mathbb{C}}$ group, $\det L = 1$, as expressed in eq. (2.2.94). Likewise,

$$\epsilon^{\dot{\rm A}\dot{\rm B}}\xi_{\dot{\rm A}}\eta_{\dot{\rm B}} \to \epsilon^{\dot{\rm I}\dot{\rm J}}\overline{L_{\dot{\rm I}}{}^{\dot{\rm A}}L_{\dot{\rm J}}{}^{\dot{\rm B}}}\xi_{\dot{\rm A}}\eta_{\dot{\rm B}} = \epsilon^{\dot{\rm A}\dot{\rm B}}\xi_{\dot{\rm A}}\eta_{\dot{\rm B}}. \tag{2.2.200}$$

Note that the scalar product between a dotted and un-dotted spinor $\epsilon^{\rm AB}\xi_{\dot{\rm A}}\eta_{\rm B}$ would not, in general, be an invariant because its transformation will involve both L and L^*.

Since eq. (2.2.198) informs us that $\xi^{\rm I}\eta_{\rm I}$ is a $\mathrm{SL}_{2,\mathbb{C}}$ scalar, it must be that the upper index spinor transforms oppositely from its lower index counterpart. Let's see this explicitly.

$$\xi'^{\rm A} = \xi'_{\rm B}\epsilon^{\rm BA} = -\epsilon^{\rm AB}L_{\rm B}{}^{\rm C}\xi_{\rm C} \tag{2.2.201}$$
$$= -\epsilon^{\rm AB}L_{\rm B}{}^{\rm C}\epsilon_{\rm CD}\xi^{\rm D}. \tag{2.2.202}$$

Recalling eq. (**??**) and eq. (2.2.45),

$$\xi'^{\rm A} = ((L^{-1})^{\rm T})^{\rm A}{}_{\rm D}\xi^{\rm D} = \xi^{\rm D}(L^{-1})_{\rm D}{}^{\rm A}. \tag{2.2.203}$$

Parity in 2D $SL_{2,\mathbb{C}}$ We will now demonstrate that the parity operator does not exist within the $SL_{2,\mathbb{C}}$ representations we have been studying. This has important consequences for constructing the parity covariant Dirac equation, for instance. Now, by parity, we mean the transformation $P \in SL_{2,\mathbb{C}}$ that would – for arbitrary p_μ – flip the sign of its spatial components, namely

$$P(p_0\mathbb{I} + \vec{p}\cdot\vec{\sigma})P^\dagger = P(p_\mu\sigma^\mu)P^\dagger = p_0\mathbb{I} - \vec{p}\cdot\vec{\sigma} = p_\mu\bar{\sigma}^\mu. \tag{2.2.204}$$

In fact, since this is for arbitrary p_μ, we may put $p_0 = 0$, $p_i = \delta_i^j$ (for fixed j), and see that

$$P\sigma^j P^\dagger = -\sigma^j, \qquad j \in \{1,2,3\}. \tag{2.2.205}$$

[38]We may also set $p_i = 0$ in eq. (2.2.204) and observe that P needs to be unitary if it is to be a representation of $SL_{2,\mathbb{C}}$:

$$P\left(p_0\mathbb{I}\right)P^\dagger = p_0\mathbb{I} \qquad \Leftrightarrow \qquad P\cdot P^\dagger = \mathbb{I}. \tag{2.2.207}$$

Remember eq. (2.2.35) is in fact the most general form of an $SL_{2,\mathbb{C}}$ transformation. We may therefore take its dagger and set it equal to its inverse in (2.2.37).

$$\cos\left(\frac{|\vec{\psi}|^*}{2}\right) + i\frac{\vec{\psi}^*\cdot\vec{\sigma}}{|\vec{\psi}|^*}\sin\left(\frac{|\vec{\psi}|^*}{2}\right) = \cos\left(\frac{|\vec{\psi}|}{2}\right) + i\frac{\vec{\psi}\cdot\vec{\sigma}}{|\vec{\psi}|}\sin\left(\frac{|\vec{\psi}|}{2}\right). \tag{2.2.208}$$

[39]Since the Pauli matrices are linearly independent and orthogonal to the identity, we must have $\vec{\psi}$ real; i.e., the most general unitary operator that is also an $SL_{2,\mathbb{C}}$ transformation is thus nothing but the rotation operator

$$P = \exp\left(-\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}\right), \qquad \vec{\theta} \in \mathbb{R}. \tag{2.2.209}$$

Returning to eq. (2.2.204), and recalling it must hold for arbitrary $\vec{p}$, we may now set $p_j = \theta_j$:

$$P\left(p_0 + \vec{\theta}\cdot\vec{\sigma}\right)P^\dagger = p_0 + \exp\left(-\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}\right)(\vec{\theta}\cdot\vec{\sigma})\exp\left(+\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}\right) \tag{2.2.210}$$

$$= p_0 + \exp\left(-\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}\right)\exp\left(+\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}\right)(\vec{\theta}\cdot\vec{\sigma}) = p_0 + \vec{\theta}\cdot\vec{\sigma}. \tag{2.2.211}$$

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[38]It is, of course, possible to find the parity operator that works for a given $\vec{p}$; it is given by

$$P = \left[\begin{array}{cc} \xi^+ & \xi^- \end{array}\right]\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]\left[\begin{array}{cc} \xi^+ & \xi^- \end{array}\right]^\dagger, \qquad P\xi^\pm = \xi^\mp, \qquad P(\vec{p}\cdot\vec{\sigma})P^\dagger = P(\vec{p}\cdot\vec{\sigma})P = -\vec{p}\cdot\vec{\sigma}. \tag{2.2.206}$$

Here, the $\xi^\pm$ are the helicity eigenstates in equations (2.2.49) and (2.2.51). But since this is a $\vec{p}$ specific operator, that would not be a parity operation on the whole space $\vec{x} \to -\vec{x}$; for that to be true we need it to be applicable for all $\vec{p}$ and, from eq. (2.2.205), therefore $\vec{p}$-independent.

[39]For a function like sine or cosine which may be Taylor expanded on the complex plane, $f(z)^* = f(z^*)$.

In other words, it is impossible to construct a parity operator, for otherwise we would find the end result to be $p_0 - \vec{\theta}\cdot\vec{\sigma}$.

Before we moving on, let us prove the statement alluded to earlier, that the complex conjugate of pure boost, i.e., $\exp((1/2)\vec{\xi}\cdot\vec{\sigma}^*)$ (with real $\vec{\xi}$), cannot be equivalent to the pure boost itself. (The proof is very similar to the one we just delineated for the non-existence of the parity operator.) Suppose a U existed, such that

$$U \exp\left(\frac{1}{2}\vec{\xi}\cdot\vec{\sigma}^*\right) U^{-1} = \exp\left(\frac{1}{2}\vec{\xi}\cdot\vec{\sigma}\right). \tag{2.2.212}$$

According to equations (2.2.23) and (2.2.25), we may convert this into

$$(U\cdot\epsilon)\exp\left(-\frac{1}{2}\vec{\xi}\cdot\vec{\sigma}\right)(U\cdot\epsilon)^{-1} = \exp\left(-\frac{1}{2}(U\cdot\epsilon)(\vec{\xi}\cdot\vec{\sigma})(U\cdot\epsilon)^{-1}\right) = \exp\left(\frac{1}{2}\vec{\xi}\cdot\vec{\sigma}\right). \tag{2.2.213}$$

Since $\vec{\xi}$ is arbitrary we must have

$$U'\sigma^j U'^{-1} = -\sigma^j, \qquad U' \equiv U\cdot\epsilon. \tag{2.2.214}$$

Now, if $\det U' \neq 1$, we may define $U'' \equiv U'/(\det U')^{1/2} \Rightarrow \det U'' = 1$, i.e., $U'' \in \mathrm{SL}_{2,\mathbb{C}}$, so that

$$U'\sigma^j U'^{-1} = (\det U')^{1/2} U''\sigma^j U''^{-1} (\det U')^{-1/2} \tag{2.2.215}$$

$$= U''\sigma^j U''^{-1} = -\sigma^j. \tag{2.2.216}$$

Since U'' is a $\mathrm{SL}_{2,\mathbb{C}}$ transformation, we may use its form in eq. (2.2.35) and its inverse in eq. (2.2.37). If we first contract eq. (2.2.216) with the same $\vec{\psi}$ in eq. (2.2.35), we arrive at the following contradiction:

$$\exp\left(-\frac{i}{2}\vec{\psi}\cdot\vec{\sigma}\right)\left(\vec{\psi}\cdot\vec{\sigma}\right)\exp\left(+\frac{i}{2}\vec{\psi}\cdot\vec{\sigma}\right) = \exp\left(-\frac{i}{2}\vec{\psi}\cdot\vec{\sigma}\right)\exp\left(+\frac{i}{2}\vec{\psi}\cdot\vec{\sigma}\right)\left(\vec{\psi}\cdot\vec{\sigma}\right) \tag{2.2.217}$$

$$= \vec{\psi}\cdot\vec{\sigma} = -\vec{\psi}\cdot\vec{\sigma}. \tag{2.2.218}$$

To discuss parity for spinors, we therefore need to go beyond these 2 component ones.

Parity & Clifford Algebra

2.3 Orthonormal Frames; Timelike, Spacelike vs. Null Vectors; Gravitational Time Dilation

Curved Spacetime, Spacetime Volume The generalization of the 'distance-squared' between x^μ to $x^\mu + \mathrm{d}x^\mu$, from the Minkowski to the curved case, is the following "line element":

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{2.3.1}$$

where x is simply shorthand for the spacetime coordinates $\{x^\mu\}$, which we emphasize may no longer be Cartesian. We also need to demand that $g_{\mu\nu}$ be real, symmetric, and has 1 positive eigenvalue associated with the one 'time' coordinate and $(d-1)$ negative ones for the spatial coordinates. The infinitesimal spacetime volume continues to take the form

$$\mathrm{d}(\text{vol.}) = \mathrm{d}^d x \sqrt{|g(x)|}, \tag{2.3.2}$$

where $|g(x)| = |\det g_{\mu\nu}(x)|$ is now the absolute value of the determinant of the metric $g_{\mu\nu}$.

Orthonormal Basis Cartesian coordinates play a basic but special role in interpreting physics in both flat Euclidean space δ_{ij} and flat Minkowski spacetime $\eta_{\mu\nu}$: they parametrize time durations and spatial distances in orthogonal directions – i.e., every increasing tick mark along a given Cartesian axis corresponds directly to a measurement of increasing length or time in that direction. This is generically not so, say, for coordinates in curved space(time) because the notion of what constitutes a 'straight line' is significantly more subtle there; or even spherical coordinates $(r \geq 0, 0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi)$ in flat 3D space – for the latter, only the radial coordinate r corresponds to actual distance (from the origin).

Therefore, just like the curved space case, to interpret physics in the neighborhood of some spacetime location x^μ, we introduce an orthonormal basis $\{\varepsilon^{\widehat{\mu}}{}_\alpha\}$ through the 'diagonalization' process:

$$g_{\mu\nu}(x) = \eta_{\alpha\beta}\varepsilon^{\widehat{\alpha}}{}_\mu(x)\varepsilon^{\widehat{\beta}}{}_\nu(x). \tag{2.3.3}$$

By defining $\varepsilon^{\widehat{\alpha}} \equiv \varepsilon^{\widehat{\alpha}}{}_\mu \mathrm{d}x^\mu$, the analog to achieving a Cartesian-like expression for the spacetime metric is

$$\mathrm{d}s^2 = \left(\varepsilon^{\widehat{0}}\right)^2 - \sum_{i=1}^{D}\left(\varepsilon^{\widehat{i}}\right)^2 = \eta_{\mu\nu}\varepsilon^{\widehat{\mu}}\varepsilon^{\widehat{\nu}}. \tag{2.3.4}$$

This means under a local Lorentz transformation – i.e., for all

$$\Lambda^\mu{}_\alpha(x)\Lambda^\nu{}_\beta(x)\eta_{\mu\nu} = \eta_{\alpha\beta}, \tag{2.3.5}$$

$$\varepsilon'^{\widehat{\mu}}(x) = \Lambda^\mu{}_\alpha(x)\varepsilon'^{\widehat{\alpha}}(x) \tag{2.3.6}$$

– the metric remains the same:

$$\mathrm{d}s^2 = \eta_{\mu\nu}\varepsilon^{\widehat{\mu}}\varepsilon^{\widehat{\nu}} = \eta_{\mu\nu}\varepsilon'^{\widehat{\mu}}\varepsilon'^{\widehat{\nu}}. \tag{2.3.7}$$

By viewing $\widehat{\varepsilon}$ as the matrix with the αth row and μth column given by $\varepsilon^{\widehat{\alpha}}{}_\mu$, the determinant of the metric $g_{\mu\nu}$ can be written as

$$\det g_{\mu\nu}(x) = (\det\widehat{\varepsilon})^2 \det\eta_{\mu\nu}. \tag{2.3.8}$$

The infinitesimal spacetime volume in eq. (2.3.2) now can be expressed as

$$\mathrm{d}^d x\sqrt{|g(x)|} = \mathrm{d}^d x \det\widehat{\varepsilon} \tag{2.3.9}$$

$$= \varepsilon^{\widehat{0}} \wedge \varepsilon^{\widehat{1}} \wedge \cdots \wedge \varepsilon^{\widehat{d-1}}. \tag{2.3.10}$$

The second equality follows because

$$\begin{aligned}\varepsilon^{\widehat{0}} \wedge \dots \wedge \varepsilon^{\widehat{d-1}} &= \varepsilon^{\widehat{0}}{}_{\mu_1} \mathrm{d}x^{\mu_1} \wedge \dots \wedge \varepsilon^{\widehat{0}}{}_{\mu_d} \mathrm{d}x^{\mu_d} \\ &= \epsilon_{\mu_1 \dots \mu_d} \varepsilon^{\widehat{0}}{}_{\mu_1} \dots \varepsilon^{\widehat{d-1}}{}_{\mu_d} \mathrm{d}x^0 \wedge \dots \wedge \mathrm{d}x^{d-1} = (\det \widehat{\varepsilon}) \mathrm{d}^d x. \qquad (2.3.11)\end{aligned}$$

Of course, that $g_{\mu\nu}$ may be 'diagonalized' follows from the fact that $g_{\mu\nu}$ is a real symmetric matrix:

$$g_{\mu\nu} = \sum_{\alpha,\beta} O^\alpha{}_\mu \lambda_\alpha \eta_{\alpha\beta} O^\beta{}_\nu = \sum_{\alpha,\beta} \varepsilon^{\widehat{\alpha}}{}_\mu \eta_{\alpha\beta} \varepsilon^{\widehat{\beta}}{}_\nu, \qquad (2.3.12)$$

where all $\{\lambda_\alpha\}$ are positive by assumption, so we may take their positive root:

$$\varepsilon^{\widehat{\alpha}}{}_\mu = \sqrt{\lambda_\alpha} O^\alpha{}_\mu, \qquad \{\lambda_\alpha > 0\}, \qquad (\text{No sum over } \alpha). \qquad (2.3.13)$$

That $\varepsilon^{\widehat{0}}{}_\mu$ acts as 'standard clock' and $\{\varepsilon^{\widehat{i}}{}_\mu | i = 1, 2, \dots, D\}$ act as 'standard rulers' is because they are of unit length:

$$g^{\mu\nu} \varepsilon^{\widehat{\alpha}}{}_\mu \varepsilon^{\widehat{\beta}}{}_\nu = \eta^{\alpha\beta}. \qquad (2.3.14)$$

The $\widehat{\cdot}$ on the index indicates it is to be moved with the flat metric, namely

$$\varepsilon^{\widehat{\alpha}}{}_\mu = \eta^{\alpha\beta} \varepsilon_{\widehat{\beta}\mu} \qquad \text{and} \qquad \varepsilon_{\widehat{\alpha}\mu} = \eta_{\alpha\beta} \varepsilon^{\widehat{\beta}}{}_\mu; \qquad (2.3.15)$$

while the spacetime index is to be moved with the spacetime metric

$$\varepsilon^{\widehat{\alpha}\mu} = g^{\mu\nu} \varepsilon^{\widehat{\alpha}}{}_\nu \qquad \text{and} \qquad \varepsilon^{\widehat{\alpha}}{}_\mu = g_{\mu\nu} \varepsilon^{\widehat{\alpha}\nu}. \qquad (2.3.16)$$

In other words, we view $\varepsilon_{\widehat{\alpha}}{}^\mu$ as the μth spacetime component of the αth vector field in the basis set $\{\varepsilon_{\widehat{\alpha}}{}^\mu | \alpha = 0, 1, 2, \dots, D \equiv d - 1\}$. We may elaborate on the interpretation that $\{\varepsilon^{\widehat{\alpha}}{}_\mu\}$ act as 'standard clock/rulers' as follows. For a test (scalar) function $f(x)$ defined throughout spacetime, the rate of change of f along $\varepsilon_{\widehat{0}}$ is

$$\langle \mathrm{d}f | \varepsilon_{\widehat{0}} \rangle = \varepsilon_{\widehat{0}}{}^\mu \partial_\mu f \equiv \frac{\mathrm{d}f}{\mathrm{d}y^0}; \qquad (2.3.17)$$

whereas that along $\varepsilon_{\widehat{i}}$ is

$$\langle \mathrm{d}f | \varepsilon_{\widehat{i}} \rangle = \varepsilon_{\widehat{i}}{}^\mu \partial_\mu f \equiv \frac{\mathrm{d}f}{\mathrm{d}y^i}; \qquad (2.3.18)$$

where y^0 and $\{y^i\}$ are to be viewed as 'time' and 'spatial' parameters along the integral curves of $\{\varepsilon_{\widehat{\mu}}{}^\alpha\}$. That these are Cartesian-like can now be expressed as

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}y^\mu} \middle| \frac{\mathrm{d}}{\mathrm{d}y^\nu} \right\rangle = \varepsilon_{\widehat{\mu}}{}^\alpha \varepsilon_{\widehat{\nu}}{}^\beta \langle \partial_\alpha | \partial_\beta \rangle = \varepsilon_{\widehat{\mu}}{}^\alpha \varepsilon_{\widehat{\nu}}{}^\beta g_{\alpha\beta} = \eta_{\mu\nu}. \qquad (2.3.19)$$

It is worth reiterating that the first equalities of eq. (2.3.12) are really assumptions, in that the definitions of curved spaces include assuming all the eigenvalues of the metric are positive whereas that of curved spacetimes include assuming all but one eigenvalue is negative.[40]

Commutators & Coordinates Note that the $\{\mathrm{d}/\mathrm{d}y^\mu\}$ in eq. (2.3.19) do not, generically, commute. For instance, acting on a scalar function,

$$\left[\frac{\mathrm{d}}{\mathrm{d}y^\mu}, \frac{\mathrm{d}}{\mathrm{d}y^\nu}\right] f(x) = \left(\frac{\mathrm{d}}{\mathrm{d}y^\mu}\frac{\mathrm{d}}{\mathrm{d}y^\nu} - \frac{\mathrm{d}}{\mathrm{d}y^\nu}\frac{\mathrm{d}}{\mathrm{d}y^\mu}\right) f(x) \tag{2.3.20}$$

$$= \left(\varepsilon_{\widehat{\mu}}{}^{\alpha}\partial_\alpha \varepsilon_{\widehat{\nu}}{}^{\beta} - \varepsilon_{\widehat{\nu}}{}^{\alpha}\partial_\alpha \varepsilon_{\widehat{\mu}}{}^{\beta}\right)\partial_\beta f(x) \neq 0. \tag{2.3.21}$$

More generally, for any two vector fields V^μ and W^μ, their commutator is

$$[V,W]^\mu = V^\sigma \nabla_\sigma W^\mu - W^\sigma \nabla_\sigma V^\mu \tag{2.3.22}$$

$$= V^\sigma \partial_\sigma W^\mu - W^\sigma \partial_\sigma V^\mu. \tag{2.3.23}$$

(Can you explain why the covariant derivatives can be replaced with partial ones?) A theorem in differential geometry[41] tells us:

> A set of $1 < N \leq d$ vector fields $\{\mathrm{d}/\mathrm{d}\xi^\mu\}$ form a coordinate basis in the $d-$dimensional space(time) they inhabit, if and only if they commute.

To elaborate: if these N vector fields commute, we may integrate them to find a $N-$dimensional coordinate grid within the $d-$dimensional spacetime. Conversely, we are already accustomed to the fact that the partial derivatives with respect to the coordinates of space(time) do, of course, commute amongst themselves. When $N = d$, and if $[\mathrm{d}/\mathrm{d}y^\mu, \mathrm{d}/\mathrm{d}y^\nu] = 0$ in eq. (2.3.19), we would not only have found coordinates $\{y^\mu\}$ for our spacetime, we would have found this spacetime is a flat one.

What are coordinates? It is perhaps important to clarify what a coordinate system is. In 2D, for instance, if we had $[\mathrm{d}/\mathrm{d}y^0, \mathrm{d}/\mathrm{d}y^1] \neq 0$, this means it is not possible to vary the 'coordinate' y^0 (i.e., along the integral curve of $\mathrm{d}/\mathrm{d}y^0$) without holding the 'coordinate' y^1 fixed; or, it is not possible to hold y^0 fixed while moving along the integral curve of $\mathrm{d}/\mathrm{d}y^1$.

Problem 2.27. Example: Schutz [2] Exercise 2.1 In 2D flat space, starting from Cartesian coordinates x^i, we may convert to cylindrical coordinates

$$(x^1, x^2) = r(\cos\phi, \sin\phi). \tag{2.3.24}$$

The pair of vector fields $(\partial_r, \partial_\phi)$ do form a coordinate basis – it is possible to hold r fixed while going along the integral curve of ∂_ϕ and vice versa. However, show via a direct calculation that the following commutator involving the unit vector fields $\widehat{r}$ and $\widehat{\phi}$ is not zero:

$$\left[\widehat{r}, \widehat{\phi}\right] f(r,\phi) \neq 0; \tag{2.3.25}$$

[40] In $d-$spacetime dimensions, with our sign convention in place, if there were n 'time' directions and $(d-n)$ 'spatial' ones, then this carries with it the assumption that $g_{\mu\nu}$ has n positive eigenvalues and $(d-n)$ negative ones.

[41] See, for instance, Schutz [2] for a pedagogical discussion.

where

$$\widehat{r} \equiv \cos(\phi)\partial_{x^1} + \sin(\phi)\partial_{x^2}, \tag{2.3.26}$$

$$\widehat{\phi} \equiv -\sin(\phi)\partial_{x^1} + \cos(\phi)\partial_{x^2}. \tag{2.3.27}$$

Therefore $\widehat{r}$ and $\widehat{\phi}$ do not form a coordinate basis. □

Timelike, Spacelike, and Null Distances/Vectors A fundamental difference between (curved) space versus spacetime, is that the former involves strictly positive distances while the latter – because of the $\eta_{00} = +1$ for orthonormal 'time' versus $\eta_{ii} = -1$ for the ith orthonormal space component – involves positive, zero, and negative distances.

With our 'mostly minus' sign convention (cf. eq. (2.1.1)), a vector v^μ is:

- *Time-like* if $v^2 \equiv \eta_{\mu\nu} v^{\widehat{\mu}} v^{\widehat{\nu}} > 0$. We have seen in §(2.1): if $v^2 > 0$, it is always possible to find a Lorentz transformation Λ (cf. eq. (2.1.5)) such that $\Lambda^\mu{}_\alpha v^{\widehat{\alpha}} = (v'^{\widehat{0}}, \vec{0})$. In flat spacetime, if $\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu > 0$ then this result indicates it is always possible to find an inertial frame where $\mathrm{d}s^2 = \mathrm{d}t'^2$: hence the phrase 'timelike'.

  More generally, for a timelike trajectory $z^\mu(\lambda)$ in curved spacetime – i.e., $g_{\mu\nu}(\mathrm{d}z^\mu/\mathrm{d}\lambda)(\mathrm{d}z^\nu/\mathrm{d}\lambda) > 0$, we may identify

$$\mathrm{d}\tau \equiv \mathrm{d}\lambda \sqrt{g_{\mu\nu}(z(\lambda)) \frac{\mathrm{d}z^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}z^\nu}{\mathrm{d}\lambda}} \tag{2.3.28}$$

  as the (infinitesimal) *proper time*, the time read by the watch of an observer whose worldline is $z^\mu(\lambda)$. (As a check: when $g_{\mu\nu} = \eta_{\mu\nu}$ and the observer is at rest, namely $\mathrm{d}\vec{z} = 0$, then $\mathrm{d}\tau = \mathrm{d}t$.) Using orthonormal frame fields in eq. (2.3.12),

$$\mathrm{d}\tau = \mathrm{d}\lambda \sqrt{\eta_{\alpha\beta} \frac{\mathrm{d}z^{\widehat{\alpha}}}{\mathrm{d}\lambda} \frac{\mathrm{d}z^{\widehat{\beta}}}{\mathrm{d}\lambda}}, \qquad \frac{\mathrm{d}z^{\widehat{\alpha}}}{\mathrm{d}\lambda} \equiv \varepsilon^{\widehat{\alpha}}{}_\mu \frac{\mathrm{d}z^\mu}{\mathrm{d}\lambda}. \tag{2.3.29}$$

  Furthermore, since $v^{\widehat{\mu}} \equiv \mathrm{d}z^{\widehat{\mu}}/\mathrm{d}\lambda$ is assumed to be timelike, it must be possible to find a local Lorentz transformation $\Lambda^\mu{}_\nu(z)$ such that $\Lambda^\mu{}_\nu v^{\widehat{\nu}} = (v'^{\widehat{0}}, \vec{0})$; assuming $\mathrm{d}\lambda > 0$,

$$\begin{aligned} \mathrm{d}\tau &= \mathrm{d}\lambda \sqrt{\eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \frac{\mathrm{d}z^{\widehat{\alpha}}}{\mathrm{d}\lambda} \frac{\mathrm{d}z^{\widehat{\beta}}}{\mathrm{d}\lambda}}, \\ &= \mathrm{d}\lambda \sqrt{\left(\frac{\mathrm{d}z'^{\widehat{0}}}{\mathrm{d}\lambda}\right)^2} = |\mathrm{d}z'^{\widehat{0}}|. \end{aligned} \tag{2.3.30}$$

- *Space-like* if $v^2 \equiv \eta_{\mu\nu} v^{\widehat{\mu}} v^{\widehat{\nu}} < 0$. We have seen in §(2.1): if $v^2 < 0$, it is always possible to find a Lorentz transformation Λ such that $\Lambda^\mu{}_\alpha v^{\widehat{\alpha}} = (0, v'^{\widehat{i}})$. In flat spacetime, if $\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu < 0$ then this result indicates it is always possible to find an inertial frame where $\mathrm{d}s^2 = -\mathrm{d}\vec{x}'^2$: hence the phrase 'spacelike'.

More generally, for a spacelike trajectory $z^\mu(\lambda)$ in curved spacetime – i.e., $g_{\mu\nu}(\mathrm{d}z^\mu/\mathrm{d}\lambda)(\mathrm{d}z^\nu/\mathrm{d}\lambda) < 0$, we may identify

$$\mathrm{d}\ell \equiv \mathrm{d}\lambda\sqrt{\left|g_{\mu\nu}(z(\lambda))\frac{\mathrm{d}z^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}z^\nu}{\mathrm{d}\lambda}\right|} \tag{2.3.31}$$

as the (infinitesimal) *proper length*, the distance read off some measuring rod whose trajectory is $z^\mu(\lambda)$. (As a check: when $g_{\mu\nu} = \eta_{\mu\nu}$ and $\mathrm{d}t = 0$, i.e., the rod is lying on the constant$-t$ surface, then $\mathrm{d}\ell = |\mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}|^{1/2}$.) Using the orthonormal frame fields in eq. (2.3.12),

$$\mathrm{d}\ell = \mathrm{d}\lambda\sqrt{\left|\eta_{\alpha\beta}\frac{\mathrm{d}z^{\widehat{\alpha}}}{\mathrm{d}\lambda}\frac{\mathrm{d}z^{\widehat{\beta}}}{\mathrm{d}\lambda}\right|}, \qquad \frac{\mathrm{d}z^{\widehat{\alpha}}}{\mathrm{d}\lambda} \equiv \varepsilon^{\widehat{\alpha}}{}_\mu\frac{\mathrm{d}z^\mu}{\mathrm{d}\lambda}. \tag{2.3.32}$$

Furthermore, since $v^{\widehat{\mu}} \equiv \mathrm{d}z^{\widehat{\mu}}/\mathrm{d}\lambda$ is assumed to be spacelike, it must be possible to find a local Lorentz transformation $\Lambda^\mu{}_\nu(z)$ such that $\Lambda^\mu{}_\nu v^{\widehat{\nu}} = (0, v'^{\widehat{i}})$; assuming $\mathrm{d}\lambda > 0$,

$$\mathrm{d}\ell = \mathrm{d}\lambda\sqrt{\eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\frac{\mathrm{d}z^{\widehat{\alpha}}}{\mathrm{d}\lambda}\frac{\mathrm{d}z^{\widehat{\beta}}}{\mathrm{d}\lambda}} = |\mathrm{d}\vec{z}'|\,; \tag{2.3.33}$$

$$\mathrm{d}\vec{z}'^{\widehat{i}} \equiv \Lambda^i{}_\mu\varepsilon^{\widehat{\mu}}{}_\nu\mathrm{d}z^\nu. \tag{2.3.34}$$

- *Null* if $v^2 \equiv \eta_{\mu\nu}v^{\widehat{\mu}}v^{\widehat{\nu}} = 0$. We have already seen, in flat spacetime, if $\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = 0$ then $|\mathrm{d}\vec{x}|/\mathrm{d}x^0 = |\mathrm{d}\vec{x}'|/\mathrm{d}x'^0 = 1$ in all inertial frames.

It is physically important to reiterate: one of the reasons why it is important to make such a distinction between vectors, is because it is *not possible* to find a Lorentz transformation that would linearly transform one of the above three types of vectors into another different type – for e.g., it is not possible to Lorentz transform a null vector into a time-like one (a photon has no 'rest frame'); or a time-like vector into a space-like one; etc. This is because their Lorentzian 'norm-squared'

$$v^2 \equiv \eta_{\mu\nu}v^{\widehat{\mu}}v^{\widehat{\nu}} = \eta_{\alpha\beta}\Lambda^\alpha{}_\mu\Lambda^\beta{}_\nu v^{\widehat{\mu}}v^{\widehat{\nu}} = \eta_{\alpha\beta}v'^{\widehat{\alpha}}v'^{\widehat{\beta}} \tag{2.3.35}$$

has to be invariant under all Lorentz transformations $v'^{\widehat{\alpha}} \equiv \Lambda^\alpha{}_\mu v^{\widehat{\mu}}$. This in turn teaches us: if v^2 were positive, it has to remain so; likewise, if it were zero or negative, a Lorentz transformation cannot alter this attribute.

Problem 2.28. Orthonormal Frames in Kerr-Schild Spacetimes A special class of geometries, known as *Kerr-Schild* spacetimes, take the following form.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + Hk_\mu k_\nu \tag{2.3.36}$$

Many of the known black hole spacetimes can be put in this form; and in such a context, $\bar{g}_{\mu\nu}$ usually refers to flat or de Sitter spacetime.[42] The k_μ is null with respect to $\bar{g}_{\mu\nu}$, i.e.,

$$\bar{g}_{\alpha\beta}k^\alpha k^\beta = 0, \tag{2.3.37}$$

[42] See Gibbons et al. [7] arXiv: hep-th/0404008. The special property of Kerr-Schild coordinates is that Einstein's equations become *linear* in these coordinates.

and we shall move its indices with $\bar{g}_{\mu\nu}$.

Verify that the inverse metric is

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - H k^\mu k^\nu, \tag{2.3.38}$$

where $\bar{g}^{\mu\sigma}$ is the inverse of $\bar{g}_{\mu\sigma}$, namely $\bar{g}^{\mu\sigma}\bar{g}_{\sigma\nu} \equiv \delta^\mu{}_\nu$. Suppose the orthonormal frame fields are known for $\bar{g}_{\mu\nu}$, namely

$$\bar{g}_{\mu\nu} = \eta_{\alpha\beta}\bar{\varepsilon}^{\widehat{\alpha}}{}_\mu \bar{\varepsilon}^{\widehat{\beta}}{}_\nu; \tag{2.3.39}$$

verify that the orthonormal frame fields are

$$\varepsilon^{\widehat{\alpha}}{}_\mu = \bar{\varepsilon}^{\widehat{\alpha}}{}_\sigma \left(\delta^\sigma{}_\mu + \frac{1}{2} H k^\sigma k_\mu\right). \tag{2.3.40}$$

Can you explain why k^μ is also null with respect to the full metric $g_{\mu\nu}$? □

Proper times and Gravitational Time Dilation Consider two observers sweeping out their respective timelike worldlines in spacetime, $y^\mu(\lambda)$ and $z^\mu(\lambda)$. If we use the time coordinate of the geometry to parameterize their trajectories, their proper times – i.e., the time read by their watches – are given by

$$\mathrm{d}\tau_y \equiv \mathrm{d}t\sqrt{g_{\mu\nu}(y(t))\dot{y}^\mu \dot{y}^\nu}, \qquad \dot{y}^\mu \equiv \frac{\mathrm{d}y^\mu}{\mathrm{d}t}; \tag{2.3.41}$$

$$\mathrm{d}\tau_z \equiv \mathrm{d}t\sqrt{g_{\mu\nu}(z(t))\dot{z}^\mu \dot{z}^\nu}, \qquad \dot{z}^\mu \equiv \frac{\mathrm{d}z^\mu}{\mathrm{d}t}. \tag{2.3.42}$$

In flat spacetime, clocks that are synchronized in one frame are no longer synchronized in a different frame – chronology is not a Lorentz invariant. We see that, in curved spacetime, the infinitesimal *passage* of proper time measured by observers at the same 'coordinate time' t depends on their spacetime locations:

$$\frac{\mathrm{d}\tau_y}{\mathrm{d}\tau_z} = \sqrt{\frac{g_{\mu\nu}(y(t))\dot{y}^\mu \dot{y}^\nu}{g_{\alpha\beta}(z(t))\dot{y}^\alpha \dot{y}^\beta}}. \tag{2.3.43}$$

Physically speaking, eq. (2.3.43) does not, in general, yield the ratio of proper times measured by observers at two different locations. (Drawing a spacetime diagram here helps.) To do so, one would have to specify the trajectories of both $y^\mu(\lambda_1 \leq \lambda \leq \lambda_2)$ and $z^\mu(\lambda'_1 \leq \lambda' \leq \lambda'_2)$, before the integrals $\Delta\tau_1 \equiv \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda\sqrt{g_{\mu\nu}\dot{y}^\mu \dot{y}^\nu}$ and $\Delta\tau_2 \equiv \int_{\lambda'_1}^{\lambda'_2} \mathrm{d}\lambda'\sqrt{g_{\mu\nu}\dot{z}^\mu \dot{z}^\nu}$ are evaluated and compared.

Problem 2.29. Example The spacetime geometry around the Earth itself can be approximated by the line element

$$\mathrm{d}s^2 = \left(1 - \frac{r_{s,E}}{r}\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - r_{s,E}/r} - r^2\left(\mathrm{d}\theta^2 + \sin(\theta)^2 \mathrm{d}\phi^2\right), \tag{2.3.44}$$

where t is the time coordinate and (r, θ, ϕ) are analogs of the spherical coordinates. Whereas $r_{s,E}$ is known as the Schwarzschild radius of the Earth, and depends on the Earth's mass M_{E} through the expression

$$r_{s,E} \equiv 2G_{\mathrm{N}}M_{\mathrm{E}}. \tag{2.3.45}$$

Find the $4-$beins of the geometry in eq. (2.3.44). Then find the numerical value of $r_{s,E}$ in eq. (2.3.45) and take the ratio $r_{s,E}/R_{\rm E}$, where $R_{\rm E}$ is the radius of the Earth. Explain why this means we may – for practical purposes – expand the metric in eq. (2.3.45) as

$$\begin{aligned} \mathrm{d}s^2 = \left(1 - \frac{r_{s,E}}{r}\right)\mathrm{d}t^2 - \mathrm{d}r^2\left(1 + \frac{r_{s,E}}{r} + \left(\frac{r_{s,E}}{r}\right)^2 + \left(\frac{r_{s,E}}{r}\right)^3 + \dots\right) \\ - r^2\left(\mathrm{d}\theta^2 + \sin(\theta)^2\mathrm{d}\phi^2\right). \end{aligned} \tag{2.3.46}$$

Since we are not in flat spacetime, the (t, r, θ, ϕ) are no longer subject to the same interpretation. However, use your computation of $r_{s,E}/R_{\rm E}$ to *estimate* the error incurred if we do continue to interpret t and r as though they measured time and radial distances, with respect to a frame centered at the Earth's core.

Consider placing one clock at the base of the Taipei 101 tower and another at its tip. Denoting the time elapsed at the base of the tower as $\Delta\tau_{\rm B}$; that at the tip as $\Delta\tau_{\rm T}$; and assuming for simplicity the Earth is a perfect sphere – show that eq. (2.3.43) translates to

$$\frac{\Delta\tau_{\rm B}}{\Delta\tau_{\rm T}} = \sqrt{\frac{g_{00}(R_{\rm E})}{g_{00}(R_{\rm E}+h_{101})}} \approx 1 + \frac{1}{2}\left(\frac{r_{s,E}}{R_{\rm E}+h_{101}} - \frac{r_{s,E}}{R_{\rm E}}\right). \tag{2.3.47}$$

Here, $R_{\rm E}$ is the radius of the Earth and h_{101} is the height of the Taipei 101 tower. Notice the right hand side is related to the difference in the Newtonian gravitational potentials at the top and bottom of the tower.

In actuality, both clocks are in motion, since the Earth is rotating. Can you estimate what is the error incurred from assuming they are at rest? First arrive at eq. (2.3.47) analytically, then plug in the relevant numbers to compute the numerical value of $\Delta\tau_{\rm B}/\Delta\tau_{\rm T}$. Does the clock at the base of Taipei 101 or that on its tip tick more slowly?

This gravitational time dilation is an effect that needs to be accounted for when setting up a network of Global Positioning Satellites (GPS); for details, see Ashby [5]. □

2.4 Connections, Curvature, Geodesics

Connections & Christoffel Symbols The partial derivative on a scalar φ is a rank-1 tensor, so we shall simply define the covariant derivative acting on φ to be

$$\nabla_\alpha\varphi = \partial_\alpha\varphi. \tag{2.4.1}$$

Because the partial derivative itself cannot yield a tensor once it acts on tensor, we need to introduce a connection $\Gamma^\mu{}_{\alpha\beta}$, i.e.,

$$\nabla_\sigma V^\mu = \partial_\sigma V^\mu + \Gamma^\mu{}_{\sigma\rho}V^\rho. \tag{2.4.2}$$

Under a coordinate transformation of the partial derivatives and V^μ, say going from x to x',

$$\partial_\sigma V^\mu + \Gamma^\mu{}_{\sigma\rho}V^\rho = \frac{\partial x'^\lambda}{\partial x^\sigma}\frac{\partial x^\mu}{\partial x'^\nu}\partial_{\lambda'}V^{\nu'} + \left(\frac{\partial x'^\lambda}{\partial x^\sigma}\frac{\partial^2 x^\mu}{\partial x'^\lambda x'^\nu} + \Gamma^\mu{}_{\sigma\rho}\frac{\partial x^\rho}{\partial x'^\nu}\right)V^{\nu'}. \tag{2.4.3}$$

On the other hand, if $\nabla_\sigma V^\mu$ were to transform as a tensor,

$$\partial_\sigma V^\mu + \Gamma^\mu{}_{\sigma\rho} V^\rho = \frac{\partial x'^\lambda}{\partial x^\sigma}\frac{\partial x^\mu}{\partial x'^\nu}\partial_{\lambda'} V^{\nu'} + \frac{\partial x'^\lambda}{\partial x^\sigma}\frac{\partial x^\mu}{\partial x'^\tau}\Gamma^{\tau'}{}_{\lambda'\nu'} V^{\nu'}. \tag{2.4.4}$$

[43]Since $V^{\nu'}$ is an arbitrary vector, we may read off its coefficient on the right hand sides of equations (2.4.3) and (2.4.4), and deduce the connection has to transform as

$$\frac{\partial x'^\lambda}{\partial x^\sigma}\frac{\partial^2 x^\mu}{\partial x'^\lambda x'^\nu} + \Gamma^\mu{}_{\sigma\rho}(x)\frac{\partial x^\rho}{\partial x'^\nu} = \frac{\partial x'^\lambda}{\partial x^\sigma}\frac{\partial x^\mu}{\partial x'^\tau}\Gamma^{\tau'}{}_{\lambda'\nu'}(x'). \tag{2.4.5}$$

Moving all the Jacobians onto the connection written in the $\{x^\mu\}$ frame,

$$\Gamma^{\tau'}{}_{\kappa'\nu'}(x') = \frac{\partial x'^\tau}{\partial x^\mu}\frac{\partial^2 x^\mu}{\partial x'^\kappa x'^\nu} + \frac{\partial x'^\tau}{\partial x^\mu}\Gamma^\mu{}_{\sigma\rho}(x)\frac{\partial x^\sigma}{\partial x'^\kappa}\frac{\partial x^\rho}{\partial x'^\nu}. \tag{2.4.6}$$

All connections have to satisfy this non-tensorial transformation law. On the other hand, if we found an object that transforms according to eq. (2.4.6), and if one employs it in eq. (2.4.2), then the resulting $\nabla_\alpha V^\mu$ would transform as a tensor.

Product rule Because covariant derivatives should reduce to partial derivatives in flat Cartesian coordinates, it is natural to require the former to obey the usual product rule. For any two tensors T_1 and T_2, and suppressing all indices,

$$\nabla(T_1 T_2) = (\nabla T_1)T_2 + T_1(\nabla T_2). \tag{2.4.7}$$

Problem 2.30. Covariant Derivative on 1-form Let us take the covariant derivative of a 1-form:

$$\nabla_\alpha V_\mu = \partial_\alpha V_\mu + \Gamma'^\sigma{}_{\alpha\mu} V_\sigma. \tag{2.4.8}$$

Can you prove that this connection is negative of the vector one in eq. (2.4.2)?

$$\Gamma'^\sigma{}_{\alpha\mu} = -\Gamma^\sigma{}_{\alpha\mu}, \tag{2.4.9}$$

where $\Gamma^\sigma{}_{\alpha\mu}$ is the connection in eq. (2.4.2) – if we define the covariant derivative of a scalar to be simply the partial derivative acting on the same, i.e.,

$$\nabla_\alpha (V^\mu W_\mu) = \partial_\alpha (V^\mu W_\mu)? \tag{2.4.10}$$

You should assume the product rule holds, namely $\nabla_\alpha (V^\mu W_\mu) = (\nabla_\alpha V^\mu) W_\mu + V^\mu (\nabla_\alpha W_\mu)$. Expand these covariant derivatives in terms of the connections and argue why this leads to eq. (2.4.9). □

Suppose we found two such connections, ${}_{(1)}\Gamma^\tau{}_{\kappa\nu}(x)$ and ${}_{(2)}\Gamma^\tau{}_{\kappa\nu}(x)$. Notice their difference does transform as a tensor because the first term on the right hand side involving the Hessian $\partial^2 x/\partial x'\partial x'$ cancels out:

$${}_{(1)}\Gamma^{\tau'}{}_{\kappa'\nu'}(x') - {}_{(2)}\Gamma^{\tau'}{}_{\kappa'\nu'}(x') = \frac{\partial x'^\tau}{\partial x^\mu}\left({}_{(1)}\Gamma^\mu{}_{\sigma\rho}(x) - {}_{(2)}\Gamma^\mu{}_{\sigma\rho}(x)\right)\frac{\partial x^\sigma}{\partial x'^\kappa}\frac{\partial x^\rho}{\partial x'^\nu}. \tag{2.4.11}$$

[43]All un-primed indices represent tensor components in the x-system; while all primed indices those in the x' system.

Now, any connection can be decomposed into its symmetric and antisymmetric parts in the following sense:

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2}\Gamma^{\mu}{}_{\{\alpha\beta\}} + \frac{1}{2}\Gamma^{\mu}{}_{[\alpha\beta]}. \tag{2.4.12}$$

This is, of course, mere tautology. However, let us denote

$$_{(1)}\Gamma^{\mu}{}_{\alpha\beta} \equiv \frac{1}{2}\Gamma^{\mu}{}_{\alpha\beta} \qquad \text{and} \qquad _{(2)}\Gamma^{\mu}{}_{\alpha\beta} \equiv \frac{1}{2}\Gamma^{\mu}{}_{\beta\alpha}; \tag{2.4.13}$$

so that

$$\frac{1}{2}\Gamma^{\mu}{}_{[\alpha\beta]} = {}_{(1)}\Gamma^{\mu}{}_{\alpha\beta} - {}_{(2)}\Gamma^{\mu}{}_{\alpha\beta} \equiv T^{\mu}{}_{\alpha\beta}. \tag{2.4.14}$$

We then see that this anti-symmetric part of the connection is in fact a tensor. It is the symmetric part $(1/2)\Gamma^{\mu}{}_{\{\alpha\beta\}}$ that does not transform as a tensor. *For the rest of these notes, by $\Gamma^{\mu}{}_{\alpha\beta}$ we shall always mean a symmetric connection.* This means our covariant derivative would now read

$$\nabla_{\alpha} V^{\mu} = \partial_{\alpha} V^{\mu} + \Gamma^{\mu}{}_{\alpha\beta} V^{\beta} + T^{\mu}{}_{\alpha\beta} V^{\beta}. \tag{2.4.15}$$

As is common within the physics literature, we proceed to set to zero the torsion term: $T^{\mu}{}_{\alpha\beta} \to 0$. If we further impose the metric compatibility condition,

$$\nabla_{\mu} g_{\alpha\beta} = 0, \tag{2.4.16}$$

then we have already seen in §(1) this implies

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} g^{\mu\sigma} \left(\partial_{\alpha} g_{\beta\sigma} + \partial_{\beta} g_{\alpha\sigma} - \partial_{\sigma} g_{\alpha\beta} \right). \tag{2.4.17}$$

Parallel Transport & Riemann Tensor Along a curve $z^{\mu}(\lambda)$ such that one end is $z^{\mu}(\lambda = \lambda_1) = x'^{\mu}$ and the other end is $z^{\mu}(\lambda = \lambda_2) = x^{\mu}$, we may parallel transport some vector V^{α} from x' to x by exponentiating the covariant derivative along $z^{\mu}(\lambda)$. If $V^{\alpha}(x' \to x)$ is the result of this parallel transport, we have

$$V^{\alpha}(x' \to x) = \exp\left[(\lambda_2 - \lambda_1) \dot{z}^{\mu}(\lambda_1) \nabla_{\mu} \right] V^{\alpha}(x'). \tag{2.4.18}$$

This is the covariant derivative analog of the Taylor expansion of a scalar function – where, translation by a constant spacetime vector a^{μ} may be implemented as

$$f(x^{\mu} + a^{\mu}) = \exp\left(a^{\nu} \partial_{\nu} \right) f(x^{\mu}). \tag{2.4.19}$$

To elucidate the definition of geometric curvature as the failure of tensors to remain invariant under parallel transport, we may now attempt to parallel transport a vector V^{α} around a closed parallelogram defined by the tangent vectors A and B. We shall soon see how the Riemann tensor itself emerges from such an analysis.

Let the 4 sides of this parallelogram have infinitesimal affine parameter length ϵ. We will now start from one of its 4 corners, which we will denote as x. V^{α} will be parallel transported from x to $x + \epsilon A$; then to $x + \epsilon A + \epsilon B$; then to $x + \epsilon A + \epsilon B - \epsilon A = x + \epsilon B$; and finally back to $x+\epsilon B-\epsilon B=x$. Let us first work out the parallel transport along the 'side' A using eq. (2.4.18). Denoting $\nabla_A \equiv A^\mu \nabla_\mu$, $\nabla_B \equiv B^\mu \nabla_\mu$, etc.,

$$\begin{aligned} V^\alpha(x \to x+\epsilon A) &= \exp(\epsilon \nabla_A) V^\alpha(x), \\ &= V^\alpha(x) + \epsilon \nabla_A V^\alpha(x) + \frac{\epsilon^2}{2} \nabla_A^2 V^\alpha(x) + \mathcal{O}\left(\epsilon^3\right). \end{aligned} \tag{2.4.20}$$

We then parallel transport this result from $x+\epsilon A$ to $x+\epsilon A+\epsilon B$.

$$\begin{aligned} & V^\alpha(x \to x+\epsilon A \to x+\epsilon A+\epsilon B) \\ &= \exp(\epsilon \nabla_B) \exp(\epsilon \nabla_A) V^\alpha(x), \\ &= V^\alpha(x) + \epsilon \nabla_A V^\alpha(x) + \frac{\epsilon^2}{2} \nabla_A^2 V^\alpha(x) \\ &\quad + \epsilon \nabla_B V^\alpha(x) + \epsilon^2 \nabla_B \nabla_A V^\alpha(x) \\ &\quad + \frac{\epsilon^2}{2} \nabla_B^2 V^\alpha(x) + \mathcal{O}\left(\epsilon^3\right) \\ &= V^\alpha(x) + \epsilon \left(\nabla_A + \nabla_B\right) V^\alpha(x) + \frac{\epsilon^2}{2} \left(\nabla_A^2 + \nabla_B^2 + 2\nabla_B \nabla_A\right) V^\alpha(x) + \mathcal{O}\left(\epsilon^3\right). \end{aligned} \tag{2.4.21}$$

Pressing on, we now parallel transport this result from $x+\epsilon A+\epsilon B$ to $x+\epsilon B$.

$$\begin{aligned} & V^\alpha(x \to x+\epsilon A \to x+\epsilon A+\epsilon B \to x+\epsilon B) \\ &= \exp(-\epsilon \nabla_A) \exp(\epsilon \nabla_B) \exp(\epsilon \nabla_A) V^\alpha(x), \\ &= V^\alpha(x) + \epsilon \left(\nabla_A + \nabla_B\right) V^\alpha(x) + \frac{\epsilon^2}{2} \left(\nabla_A^2 + \nabla_B^2 + 2\nabla_B \nabla_A\right) V^\alpha(x) \\ &\quad - \epsilon \nabla_A V^\alpha(x) - \epsilon^2 \left(\nabla_A^2 + \nabla_A \nabla_B\right) V^\alpha(x) \\ &\quad + \frac{\epsilon^2}{2} \nabla_A^2 V^\alpha(x) + \mathcal{O}\left(\epsilon^3\right) \\ &= V^\alpha(x) + \epsilon \nabla_B V^\alpha(x) + \epsilon^2 \left(\frac{1}{2} \nabla_B^2 + \nabla_B \nabla_A - \nabla_A \nabla_B\right) V^\alpha(x) + \mathcal{O}\left(\epsilon^3\right). \end{aligned} \tag{2.4.22}$$

Finally, we parallel transport this back to $x+\epsilon B-\epsilon B=x$.

$$\begin{aligned} & V^\alpha(x \to x+\epsilon A \to x+\epsilon A+\epsilon B \to x+\epsilon B \to x) \\ &= \exp(-\epsilon \nabla_B) \exp(-\epsilon \nabla_A) \exp(\epsilon \nabla_B) \exp(\epsilon \nabla_A) V^\alpha(x), \\ &= V^\alpha(x) + \epsilon \nabla_B V^\alpha(x) + \epsilon^2 \left(\frac{1}{2} \nabla_B^2 + \nabla_B \nabla_A - \nabla_A \nabla_B\right) V^\alpha(x) \\ &\quad - \epsilon \nabla_B V^\alpha(x) - \epsilon^2 \nabla_B^2 V^\alpha(x) \\ &\quad + \frac{\epsilon^2}{2} \nabla_B^2 V^\alpha(x) + \mathcal{O}\left(\epsilon^3\right) \\ &= V^\alpha(x) + \epsilon^2 \left(\nabla_B \nabla_A - \nabla_A \nabla_B\right) V^\alpha(x) + \mathcal{O}\left(\epsilon^3\right). \end{aligned} \tag{2.4.23}$$

We have arrived at the central characterization of *local* geometric curvature. By parallel transporting a vector around an infinitesimal parallelogram, we see the parallel transported vector differs from the original one by the commutator of covariant derivatives with respect to the two tangent vectors defining the parallelogram. In the same vein, their difference is also proportional to the area of this parallogram, i.e., it scales as $\mathcal{O}(\epsilon^2)$ for infinitesimal ϵ.

$$V^\alpha(x \to x + \epsilon A \to x + \epsilon A + \epsilon B \to x + \epsilon B \to x) - V^\alpha(x) \tag{2.4.24}$$
$$= \epsilon^2 [\nabla_B, \nabla_A] V^\alpha(x) + \mathcal{O}\left(\epsilon^3\right),$$
$$[\nabla_B, \nabla_A] \equiv \nabla_B \nabla_A - \nabla_A \nabla_B. \tag{2.4.25}$$

We shall proceed to calculate the commutator in a coordinate basis.

$$\begin{aligned}[]
[\nabla_A, \nabla_B] V^\mu &\equiv A^\sigma \nabla_\sigma \left(B^\rho \nabla_\rho V^\mu\right) - B^\sigma \nabla_\sigma \left(A^\rho \nabla_\rho V^\mu\right) \\
&= \left(A^\sigma \nabla_\sigma B^\rho - B^\sigma \nabla_\sigma A^\rho\right) \nabla_\rho V^\mu + A^\sigma B^\rho [\nabla_\sigma, \nabla_\rho] V^\mu.
\end{aligned} \tag{2.4.26}$$

Let us tackle the two groups separately. Firstly,

$$\begin{aligned}[]
[A, B]^\rho \nabla_\rho V^\mu &\equiv \left(A^\sigma \nabla_\sigma B^\rho - B^\sigma \nabla_\sigma A^\rho\right) \nabla_\rho V^\mu \\
&= \left(A^\sigma \partial_\sigma B^\rho + \Gamma^\rho{}_{\sigma\lambda} A^\sigma B^\lambda - B^\sigma \partial_\sigma A^\rho - \Gamma^\rho{}_{\sigma\lambda} B^\sigma A^\lambda\right) \nabla_\rho V^\mu \\
&= \left(A^\sigma \partial_\sigma B^\rho - B^\sigma \partial_\sigma A^\rho\right) \nabla_\rho V^\mu.
\end{aligned} \tag{2.4.27}$$

Next, we need $A^\sigma B^\rho [\nabla_\sigma, \nabla_\rho] V^\mu = A^\sigma B^\rho (\nabla_\sigma \nabla_\rho - \nabla_\rho \nabla_\sigma) V^\mu$. The first term is

$$\begin{aligned}
A^\sigma B^\rho \nabla_\sigma \nabla_\rho V^\mu &= A^\sigma B^\rho \left(\partial_\sigma \nabla_\rho V^\mu - \Gamma^\lambda{}_{\sigma\rho} \nabla_\lambda V^\mu + \Gamma^\mu{}_{\sigma\lambda} \nabla_\rho V^\lambda\right) \\
&= A^\sigma B^\rho \left(\partial_\sigma \left(\partial_\rho V^\mu + \Gamma^\mu{}_{\rho\lambda} V^\lambda\right) - \Gamma^\lambda{}_{\sigma\rho} \left(\partial_\lambda V^\mu + \Gamma^\mu{}_{\lambda\omega} V^\omega\right) + \Gamma^\mu{}_{\sigma\lambda} \left(\partial_\rho V^\lambda + \Gamma^\lambda{}_{\rho\omega} V^\omega\right)\right) \\
&= A^\sigma B^\rho \Big\{\partial_\sigma \partial_\rho V^\mu + \partial_\sigma \Gamma^\mu{}_{\rho\lambda} V^\lambda + \Gamma^\mu{}_{\rho\lambda} \partial_\sigma V^\lambda - \Gamma^\lambda{}_{\sigma\rho} \left(\partial_\lambda V^\mu + \Gamma^\mu{}_{\lambda\omega} V^\omega\right) \\
&\qquad + \Gamma^\mu{}_{\sigma\lambda} \left(\partial_\rho V^\lambda + \Gamma^\lambda{}_{\rho\omega} V^\omega\right) \Big\}.
\end{aligned} \tag{2.4.28}$$

Swapping $(\sigma \leftrightarrow \rho)$ within the parenthesis $\{\dots\}$ and subtract the two results, we gather

$$\begin{aligned}
A^\sigma B^\rho [\nabla_\sigma, \nabla_\rho] V^\mu &= A^\sigma B^\rho \Big\{\partial_{[\sigma} \Gamma^\mu{}_{\rho]\lambda} V^\lambda + \Gamma^\mu{}_{\lambda[\rho} \partial_{\sigma]} V^\lambda - \Gamma^\lambda{}_{[\sigma\rho]} \left(\partial_\lambda V^\mu + \Gamma^\mu{}_{\lambda\omega} V^\omega\right) \\
&\qquad + \Gamma^\mu{}_{\lambda[\sigma} \partial_{\rho]} V^\lambda + \Gamma^\mu{}_{\lambda[\sigma} \Gamma^\lambda{}_{\rho]\omega} V^\omega \Big\}
\end{aligned} \tag{2.4.29}$$
$$= A^\sigma B^\rho \left(\partial_{[\sigma} \Gamma^\mu{}_{\rho]\omega} + \Gamma^\mu{}_{\lambda[\sigma} \Gamma^\lambda{}_{\rho]\omega}\right) V^\omega. \tag{2.4.30}$$

Notice we have used the symmetry of the Christoffel symbols $\Gamma^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\beta\alpha}$ to arrive at this result. Since A and B are arbitrary, let us observe that the commutator of covariant derivatives acting on a vector field is not a different operator, but rather an algebraic operation:

$$[\nabla_\mu, \nabla_\nu] V^\alpha = R^\alpha{}_{\beta\mu\nu} V^\beta, \tag{2.4.31}$$
$$R^\alpha{}_{\beta\mu\nu} \equiv \partial_{[\mu} \Gamma^\alpha{}_{\nu]\beta} + \Gamma^\alpha{}_{\sigma[\mu} \Gamma^\sigma{}_{\nu]\beta} \tag{2.4.32}$$
$$= \partial_\mu \Gamma^\alpha{}_{\nu\beta} - \partial_\nu \Gamma^\alpha{}_{\mu\beta} + \Gamma^\alpha{}_{\sigma\mu} \Gamma^\sigma{}_{\nu\beta} - \Gamma^\alpha{}_{\sigma\nu} \Gamma^\sigma{}_{\mu\beta}. \tag{2.4.33}$$

Inserting the results in equations (2.4.27) and (2.4.30) into eq. (2.4.26) – we gather, for arbitrary vector fields A and B:

$$\left([\nabla_A, \nabla_B] - \nabla_{[A,B]}\right) V^\mu = R^\mu{}_{\nu\alpha\beta} V^\nu A^\alpha B^\beta. \tag{2.4.34}$$

Moreover, we may return to eq. (2.4.24) and re-express it as

$$V^\alpha(x \to x+\epsilon A \to x+\epsilon A+\epsilon B \to x+\epsilon B \to x) - V^\alpha(x) \tag{2.4.35}$$
$$= \epsilon^2 \left(R^\alpha{}_{\beta\mu\nu}(x) V^\beta(x) B^\mu(x) A^\nu(x) + \nabla_{[B,A]} V^\alpha(x) \right) + \mathcal{O}\left(\epsilon^3\right). \tag{2.4.36}$$

When $A = \partial_\mu$ and $B = \partial_\nu$ are coordinate basis vectors themselves, $[A,B] = [\partial_\mu, \partial_\nu] = 0$, and eq. (2.4.34) then coincides with eq. (2.4.31). Earlier, we have already mentioned: if $[A,B]=0$, the vector fields A and B can be integrated to form a local 2D coordinate system; while if $[A,B] \neq 0$, they cannot form a good coordinate system. Hence the failure of parallel transport invariance due to the $\nabla_{[A,B]}$ term in eq. (2.4.35) is really a measure of the coordinate-worthiness of A and B; whereas it is the Riemann tensor term that appears to tell us something about the intrinsic local curvature of the geometry itself.

Problem 2.31. Symmetries of the Riemann tensor Explain why, if a tensor $\Sigma_{\alpha\beta}$ is antisymmetric in one coordinate system, it has to be anti-symmetric in any other coordinate system. Similarly, explain why, if $\Sigma_{\alpha\beta}$ is symmetric in one coordinate system, it has to be symmetric in any other coordinate system. Compute the Riemann tensor in a locally flat coordinate system[44] and show that

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(\partial_\beta \partial_{[\mu} g_{\nu]\alpha} - \partial_\alpha \partial_{[\mu} g_{\nu]\beta} \right). \tag{2.4.37}$$

From this result, argue that Riemann has the following symmetries:

$$R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}, \qquad R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta}, \qquad R_{\mu\nu\alpha\beta} = -R_{\mu\nu\beta\alpha}. \tag{2.4.38}$$

This indicates the components of the Riemann tensor are not all independent. Below, we shall see there are additional differential relations (aka "Bianchi identities") between various components of the Riemann tensor.

Finally, use these symmetries to show that

$$[\nabla_\alpha, \nabla_\beta] V_\nu = -R^\mu{}_{\nu\alpha\beta} V_\mu. \tag{2.4.39}$$

Hint: Start with $[\nabla_\alpha, \nabla_\beta](g_{\nu\sigma} V^\sigma)$. □

Ricci tensor and scalar Because of the symmetries of Riemann in eq. (2.4.38), we have $g^{\alpha\beta} R_{\alpha\beta\mu\nu} = -g^{\alpha\beta} R_{\beta\alpha\mu\nu} = -g^{\beta\alpha} R_{\beta\alpha\mu\nu} = 0$; and likewise, $R_{\alpha\beta\mu}{}^{\mu} = 0$. In fact, the Ricci tensor is defined as the sole distinct and non-zero contraction of Riemann:

$$R_{\mu\nu} \equiv R^\sigma{}_{\mu\sigma\nu}. \tag{2.4.40}$$

This is a symmetric tensor, $R_{\mu\nu} = R_{\nu\mu}$, because of eq. (2.4.38); for,

$$R_{\mu\nu} = g^{\sigma\rho} R_{\sigma\mu\rho\nu} = g^{\rho\sigma} R_{\rho\nu\sigma\mu} = R_{\nu\mu}. \tag{2.4.41}$$

Its contraction yields the Ricci scalar

$$\mathcal{R} \equiv g^{\mu\nu} R_{\mu\nu}. \tag{2.4.42}$$

[44] See equations (2.5.6) through (2.5.8) below.

Problem 2.32. Commutator of covariant derivatives on higher rank tensor Prove that

$$\begin{aligned}&[\nabla_\mu, \nabla_\nu] T^{\alpha_1 \dots \alpha_N}{}_{\beta_1 \dots \beta_M} \\ &= R^{\alpha_1}{}_{\sigma\mu\nu} T^{\sigma\alpha_2\dots\alpha_N}{}_{\beta_1\dots\beta_M} + R^{\alpha_2}{}_{\sigma\mu\nu} T^{\alpha_1\sigma\alpha_3\dots\alpha_N}{}_{\beta_1\dots\beta_M} + \dots + R^{\alpha_N}{}_{\sigma\mu\nu} T^{\alpha_1\dots\alpha_{N-1}\sigma}{}_{\beta_1\dots\beta_M} \\ &- R^{\sigma}{}_{\beta_1\mu\nu} T^{\alpha_1\dots\alpha_N}{}_{\sigma\beta_2\dots\beta_M} - R^{\sigma}{}_{\beta_2\mu\nu} T^{\alpha_1\dots\alpha_N}{}_{\beta_1\sigma\beta_3\dots\beta_M} - \dots - R^{\sigma}{}_{\beta_M\mu\nu} T^{\alpha_1\dots\alpha_N}{}_{\beta_1\dots\beta_{M-1}\sigma}.\end{aligned} \tag{2.4.43}$$

Also verify that

$$[\nabla_\alpha, \nabla_\beta]\varphi = 0, \tag{2.4.44}$$

where φ is a scalar. □

Problem 2.33. Differential Bianchi identities I Show that

$$R^{\mu}{}_{[\alpha\beta\delta]} = 0. \tag{2.4.45}$$

Hint: Use the Riemann tensor expressed in a locally flat coordinate system. □

Problem 2.34. Differential Bianchi identities II If $[A, B] \equiv AB - BA$, can you show that the differential operator

$$[\nabla_\alpha, [\nabla_\beta, \nabla_\delta]] + [\nabla_\beta, [\nabla_\delta, \nabla_\alpha]] + [\nabla_\delta, [\nabla_\alpha, \nabla_\beta]] \tag{2.4.46}$$

is actually zero? (Hint: Just expand out the commutators.) Why does that imply

$$\nabla_{[\alpha} R^{\mu\nu}{}_{\beta\delta]} = 0? \tag{2.4.47}$$

Using this result, show that

$$\nabla_\sigma R^{\sigma\beta}{}_{\mu\nu} = \nabla_{[\mu} R^{\beta}{}_{\nu]}. \tag{2.4.48}$$

The *Einstein tensor* is defined as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R}. \tag{2.4.49}$$

From eq. (2.4.48) can you show the divergence-less property of the Einstein tensor, i.e.,

$$\nabla^\mu G_{\mu\nu} = \nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} \right) = 0? \tag{2.4.50}$$

This will be an important property when discussing Einstein's equations for General Relativity. □

Geodesics As already noted, even in flat spacetime, $\mathrm{d}s^2$ is not positive-definite (cf. (2.1.1)), unlike its purely spatial counterpart. Therefore, when computing the distance along a line in spacetime $z^\mu(\lambda)$, with boundary values $z(\lambda_1) \equiv x'$ and $z(\lambda_2) \equiv x$, we need to take the square root of its absolute value:

$$s = \int_{\lambda_1}^{\lambda_2} \left| g_{\mu\nu}\left(z(\lambda)\right) \frac{\mathrm{d}z^\mu(\lambda)}{\mathrm{d}\lambda} \frac{\mathrm{d}z^\nu(\lambda)}{\mathrm{d}\lambda} \right|^{1/2} \mathrm{d}\lambda. \tag{2.4.51}$$

A geodesic in curved spacetime that joins two points x and x' is a path that extremizes the distance between them. Using an affine parameter to describe the geodesic, i.e., using a λ such that $\sqrt{|g_{\mu\nu}\dot{z}^\mu\dot{z}^\nu|}$ = constant, this amounts to imposing the principle of stationary action on Synge's world function:

$$\sigma(x, x') \equiv \frac{1}{2}(\lambda_2 - \lambda_1)\int_{\lambda_1}^{\lambda_2} g_{\alpha\beta}(z(\lambda))\frac{\mathrm{d}z^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}z^\beta}{\mathrm{d}\lambda}\mathrm{d}\lambda, \tag{2.4.52}$$

$$z^\mu(\lambda_1) = x'^\mu, \qquad z^\mu(\lambda_2) = x^\mu. \tag{2.4.53}$$

When evaluated on geodesics, eq. (2.4.52) is half the square of the geodesic distance between x and x'. The curved spactime geodesic equation in affine-parameter form which follows from eq. (2.4.52), is

$$\frac{D^2 z^\mu}{\mathrm{d}\lambda^2} \equiv \frac{\mathrm{d}^2 z^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu{}_{\alpha\beta}\frac{\mathrm{d}z^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}z^\beta}{\mathrm{d}\lambda} = 0. \tag{2.4.54}$$

The Lagragian associated with eq. (2.4.52),

$$L_{\mathrm{g}} \equiv \frac{1}{2}g_{\mu\nu}(z(\lambda))\dot{z}^\mu\dot{z}^\nu, \qquad \dot{z}^\mu \equiv \frac{\mathrm{d}z^\mu}{\mathrm{d}\lambda}, \tag{2.4.55}$$

not only oftentimes provides a more efficient means of computing the Christoffel symbols, it is a constant of motion. Unlike the curved space case, however, this Lagrangian L_{g} can now be positive, zero, or negative.

- If $\dot{z}^\mu$ is timelike, then by choosing the affine parameter to be proper time $\mathrm{d}\lambda\sqrt{g_{\mu\nu}\dot{z}^\mu\dot{z}^\nu} = \mathrm{d}\tau$, we see that the Lagrangian is then set to $L_{\mathrm{g}} = 1/2$.
- If $\dot{z}^\mu$ is spacelike, then by choosing the affine parameter to be proper length $\mathrm{d}\lambda\sqrt{|g_{\mu\nu}\dot{z}^\mu\dot{z}^\nu|} = \mathrm{d}\ell$, we see that the Lagrangian is then set to $L_{\mathrm{g}} = -1/2$.
- If $\dot{z}^\mu$ is null, then the Lagrangian is zero: $L_{\mathrm{g}} = 0$.

Formal solution to geodesic equation We may re-write eq. (2.4.54) into an integral equation by simply integrating both sides with respect to the affine parameter λ:

$$v^\mu(\lambda) = v^\mu(\lambda_1) - \int_{z(\lambda_1)}^{z(\lambda)} \Gamma^\mu{}_{\alpha\beta}v^\alpha \mathrm{d}z^\beta; \tag{2.4.56}$$

where $v^\mu \equiv \mathrm{d}z^\mu/\mathrm{d}\lambda$; the lower limit is $\lambda = \lambda_1$; and we have left the upper limit indefinite. The integral on the right hand side can be viewed as an integral operator acting on the tangent vector at $v^\alpha(z(\lambda))$. By iterating this equation infinite number of times – akin to the Born series expansion in quantum mechanics – it is possible to arrive at a formal (as opposed to explicit) solution to the geodesic equation.

Problem 2.35. Synge's World Function In Minkowski Verify that Synge's world function (cf. (2.4.52)) in Minkowski spacetime is

$$\bar{\sigma}(x, x') = \frac{1}{2}(x - x')^2 \equiv \frac{1}{2}\eta_{\mu\nu}(x - x')^\mu(x - x')^\nu, \tag{2.4.57}$$

$$(x - x')^\mu \equiv x^\mu - x'^\mu. \tag{2.4.58}$$

Hint: If we denote the geodesic $z^\mu(0 \leq \lambda \leq 1)$ joining x' to x in Minkowski spacetime, verify that the solution is

$$z^\mu(0 \leq \lambda \leq 1) = x'^\mu + \lambda(x - x')^\mu. \tag{2.4.59}$$

□

Problem 2.36. Show that eq. (2.4.54) takes the same form under re-scaling and constant shifts of the parameter λ. That is, if

$$\lambda = a\lambda' + b, \tag{2.4.60}$$

for constants a and b, then eq. (2.4.54) becomes

$$\frac{D^2 z^\mu}{\mathrm{d}\lambda'^2} \equiv \frac{\mathrm{d}^2 z^\mu}{\mathrm{d}\lambda'^2} + \Gamma^\mu{}_{\alpha\beta}\frac{\mathrm{d}z^\alpha}{\mathrm{d}\lambda'}\frac{\mathrm{d}z^\beta}{\mathrm{d}\lambda'} = 0. \tag{2.4.61}$$

For the timelike and spacelike cases, this is telling us that proper time and proper length are respectively only defined up to an overall re-scaling and an additive shift. In other words, both the base units and its 'zero' may be altered at will. □

Problem 2.37. Let $v^\mu(x)$ be a vector field defined throughout a given spacetime. Show that the geodesic equation (2.4.54) follows from

$$v^\sigma \nabla_\sigma v^\mu = 0, \tag{2.4.62}$$

i.e., v^μ is parallel transported along itself – provided we recall the 'velocity flow' interpretation of a vector field:

$$v^\mu\left(z(s)\right) = \frac{\mathrm{d}z^\mu}{\mathrm{d}s}. \tag{2.4.63}$$

Parallel transport preserves norm-squared The metric compatibility condition in eq. (2.4.16) obeyed by the covariant derivative ∇_α can be thought of as the requirement that the norm-squared $v^2 \equiv g_{\mu\nu}v^\mu v^\nu$ of a geodesic vector (v^μ subject to eq. (2.4.62)) be preserved under parallel transport. Can you explain this statement using the appropriate equations?

Non-affine form of geodesic equation Suppose instead

$$v^\sigma \nabla_\sigma v^\mu = \kappa v^\mu. \tag{2.4.64}$$

This is the more general form of the geodesic equation, where the parameter λ is not an affine one. Nonetheless, by considering the quantity $v^\sigma \nabla_\sigma (v^\mu/(v_\nu v^\nu)^p)$, for some real number p, show how eq. (2.4.64) can be transformed into the form in eq. (2.4.62); that is, identify an appropriate v'^μ such that

$$v'^\sigma \nabla_\sigma v'^\mu = 0. \tag{2.4.65}$$

You should comment on how this re-scaling fails when v^μ is null.

Starting from the finite distance integral

$$s \equiv \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \sqrt{|g_{\mu\nu}(z(\lambda))\dot{z}^\mu \dot{z}^\nu|}, \qquad \dot{z}^\mu \equiv \frac{\mathrm{d}z^\mu}{\mathrm{d}\lambda}, \tag{2.4.66}$$

$$z^\mu(\lambda_1) = x', \qquad z^\mu(\lambda_2) = x; \tag{2.4.67}$$

show that demanding s be extremized leads to the non-affine geodesic equation

$$\ddot{z}^\mu + \Gamma^\mu{}_{\alpha\beta}\dot{z}^\alpha \dot{z}^\beta = \dot{z}^\mu \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \sqrt{g_{\alpha\beta}\dot{z}^\alpha \dot{z}^\beta}. \tag{2.4.68}$$

□

Problem 2.38. Null Geodesics & Weyl Transformations Suppose two geometries $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ are related via a Weyl transformation

$$g_{\mu\nu}(x) = \Omega(x)^2 \bar{g}_{\mu\nu}(x). \tag{2.4.69}$$

Consider the null geodesic equation in the geometry $g_{\mu\nu}(x)$,

$$k'^\sigma \nabla_\sigma k'^\mu = 0, \qquad g_{\mu\nu}k'^\mu k'^\nu = 0 \tag{2.4.70}$$

where ∇ is the covariant derivative with respect to $g_{\mu\nu}$; as well as the null geodesic equation in $\bar{g}_{\mu\nu}(x)$,

$$k^\sigma \overline{\nabla}_\sigma k^\mu = 0, \qquad \bar{g}_{\mu\nu}k^\mu k^\nu = 0; \tag{2.4.71}$$

where $\overline{\nabla}$ is the covariant derivative with respect to $\bar{g}_{\mu\nu}$. Show that

$$k^\mu = \Omega^2 \cdot k'^\mu. \tag{2.4.72}$$

Hint: First show that the Christoffel symbol $\overline{\Gamma}^\mu{}_{\alpha\beta}[\bar{g}]$ built solely out of $\bar{g}_{\mu\nu}$ is related to $\Gamma^\mu{}_{\alpha\beta}[g]$ built out of $g_{\mu\nu}$ through the relation

$$\Gamma^\mu{}_{\alpha\beta}[g] = \bar{\Gamma}^\mu{}_{\alpha\beta}[\bar{g}] + \delta^\mu_{\{\beta}\overline{\nabla}_{\alpha\}} \ln \Omega - \bar{g}_{\alpha\beta}\overline{\nabla}^\mu \ln \Omega. \tag{2.4.73}$$

Then remember to use the constraint $g_{\mu\nu}k'^\mu k'^\nu = 0 = \bar{g}_{\mu\nu}k^\mu k^\nu$.

A spacetime is said to be conformally flat if it takes the form

$$g_{\mu\nu}(x) = \Omega(x)^2 \eta_{\mu\nu}. \tag{2.4.74}$$

Solve the null geodesic equation explicitly in such a spacetime. □

Problem 2.39. Light Deflection Due To Static Mass Monopole in 4D In General Relativity the weak field metric generated by an isolated system, of total mass M, is dominated by its mass monopole and hence goes as $1/r$ (i.e., its Newtonian potential)

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\Phi\delta_{\mu\nu} = \eta_{\mu\nu} - \frac{r_s}{r}\delta_{\mu\nu}, \tag{2.4.75}$$

where we assume $|\Phi| = r_s/r \ll 1$ and

$$r_s \equiv 2G_{\rm N}M. \tag{2.4.76}$$

Now, the metric of an isolated static non-rotating black hole – i.e., the Schwarzschild black hole – in isotropic coordinates is

$$\mathrm{d}s^2 = \left(\frac{1-\frac{r_s}{4r}}{1+\frac{r_s}{4r}}\right)^2 \mathrm{d}t^2 - \left(1+\frac{r_s}{4r}\right)^4 \mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}, \qquad\qquad r \equiv \sqrt{\vec{x}\cdot\vec{x}}. \tag{2.4.77}$$

The $r_s \equiv 2G_{\rm N}M$ here is the Schwarzschild radius; any object falling behind $r < r_s$ will not be able to return to the $r > r_s$ region unless it is able to travel faster than light.

Expand this metric in eq. (2.4.77) up to first order r_s/r and verify this yields eq. (2.4.75). We may therefore identify eq. (2.4.75) as either the metric due to the monopole moment of some static mass density $\rho(\vec{x})$ or the far field limit $r_s/r \ll 1$ of the Schwarzschild black hole.

Statement of Problem: Now consider shooting a beam of light from afar, and by solving the appropriate null geodesic equations, figure out how much angular deflection $\Delta\varphi$ it suffers due to the presence of a mass monopole. Express the answer in terms of the coordinate radius of closest approach r_0.

Hints: First, write down the affine-parameter form of the Lagrangian L_g for geodesic motion in eq. (2.4.75) in spherical coordinates

$$\vec{x} = r\left(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)\right). \tag{2.4.78}$$

Because of the spherical symmetry of the problem, we may always assume that all geodesic motion takes place on the equatorial plane:

$$\theta = \frac{\pi}{2}. \tag{2.4.79}$$

Proceed to argue one may always choose the affine parameter λ such that

$$\dot{t} = \left(1 - \frac{r_s}{r}\right)^{-1}; \tag{2.4.80}$$

such that when $r_s \to 0$, the coordinate time t becomes proper time. Next, show that angular momentum conservation $-\partial L_g/\partial\dot{\phi} \equiv \ell$ (constant) yields

$$\dot{\phi} = \frac{\ell}{r^2}\left(1+\frac{r_s}{r}\right)^{-1}. \tag{2.4.81}$$

We are primarily interested in the trajectory as a function of angle, so we may eliminate all $\dot{r} \equiv \mathrm{d}r/\mathrm{d}\lambda$ as

$$\dot{r} = \frac{\mathrm{d}\phi}{\mathrm{d}\lambda}r'(\phi) = \frac{\ell}{r^2}\left(1+\frac{r_s}{r}\right)^{-1} r'(\phi), \tag{2.4.82}$$

where eq. (2.4.81) was employed in the second equality. At this point, by utilizing equations (2.4.79), (2.4.80), (2.4.81) and (2.4.82), verify that the geodesic Lagrangian now takes the form

$$L_g = \frac{1}{2}\left(\frac{r}{r-r_s} - \frac{\ell^2}{r^2(1+r_s/r)}\left(1+\left(\frac{r'(\phi)}{r}\right)^2\right)\right). \tag{2.4.83}$$

Remember that null geodesics render $L_g = 0$. If r_0 is the coordinate radius of closest approach, which we shall assume is appreciably larger than the Schwarzschild radius $r_0 \gg r_s$, that means $r'(\phi) = 0$ when $r = r_0$. Show that

$$\ell = r_0 \sqrt{\frac{r_0 + r_s}{r_0 - r_s}}. \tag{2.4.84}$$

Working to first order in r_s, proceed to show that

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{1}{\sqrt{r^2 - r_0^2}} \left(\frac{r_0}{r} + \frac{r_s}{r + r_0} \right) + \mathcal{O}\left(r_s^2\right). \tag{2.4.85}$$

By integrating from infinity $r = \infty$ to closest approach $r = r_0$ and then out to infinity again $r = \infty$, show that the angular deflection is

$$\Delta\varphi = \frac{2r_s}{r_0}. \tag{2.4.86}$$

Even though r_0 is the coordinate radius of closest approach, in a weakly curved spacetime dominated by the monopole moment of the central object, estimate the error incurred if we set r_0 to be the *physical* radius of closest approach. What is the angular deflection due to the Sun, if a beam of light were to just graze its surface?

Note that, if the photon were undeflected, the total change in angle $(\int_{r=\infty}^{r_0} \mathrm{d}r + \int_{r_0}^{\infty} \mathrm{d}r)(\mathrm{d}\phi/\mathrm{d}r)$ would be π. Therefore, the total *deflection* angle is

$$\Delta\varphi = 2 \left| \int_{r=\infty}^{r_0} \frac{\mathrm{d}\phi}{\mathrm{d}r} \mathrm{d}r \right| - \pi. \tag{2.4.87}$$

For further help on this problem, consult §8.5 of Weinberg [1]. □

2.5 Equivalence Principles, Geometry-Induced Tidal Forces, Isometries & Geometric Tensors

Weak Equivalence Principle, "Free-Fall" & Gravity as a Non-Force The universal nature of gravitation – how it appears to act in the same way upon all material bodies independent of their internal composition – is known as the Weak Equivalence Principle. As we will see, the basic reason why the weak equivalence principle holds is because *everything* inhabits the same spacetime $g_{\mu\nu}$.

Within non-relativistic physics, the acceleration of some mass M_1 located at $\vec{x}_1$, due to the Newtonian gravitational 'force' exerted by some other mass M_2 at $\vec{x}_2$, is given by

$$M_1 \frac{\mathrm{d}^2 \vec{x}_1}{\mathrm{d}t^2} = -\widehat{n} \frac{G_{\mathrm{N}} M_1 M_2}{|\vec{x}_1 - \vec{x}_2|^2}, \qquad \widehat{n} \equiv \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|}. \tag{2.5.1}$$

Strictly speaking the M_1 on the left hand side is the 'inertial mass', a characterization of the resistance – so to speak – of any material body to being accelerated by an external force. While the M_1 on the right hand side is the 'gravitational mass', describing the strength to which the material body interacts with the gravitational 'force'. Viewed from this perspective, the equivalence principle is the assertion that the inertial and gravitational masses are the same, so that the resulting motion does not depend on them:

$$\frac{\mathrm{d}^2\vec{x}_1}{\mathrm{d}t^2} = -\widehat{n}\frac{G_{\mathrm{N}}M_2}{|\vec{x}_1-\vec{x}_2|^2}. \tag{2.5.2}$$

Similarly, the acceleration of body 2 due to the gravitational force exerted by body 1 is independent of M_2:

$$\frac{\mathrm{d}^2\vec{x}_2}{\mathrm{d}t^2} = +\widehat{n}\frac{G_{\mathrm{N}}M_1}{|\vec{x}_1-\vec{x}_2|^2}. \tag{2.5.3}$$

This Weak Equivalence Principle[45] is one of the primary motivations that led Einstein to recognize gravitation as the manifestation of curved spacetime. The reason why inertial mass appears to be equal to its gravitational counterpart, is because material bodies now follow (timelike) geodesics $z^\mu(\tau)$ in curved spacetimes:

$$a^\mu \equiv \frac{D^2 z^\mu}{\mathrm{d}\tau^2} \equiv \frac{\mathrm{d}^2 z^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\alpha\beta}\frac{\mathrm{d}z^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}z^\beta}{\mathrm{d}\tau} = 0; \qquad g_{\mu\nu}\left(z(\lambda)\right)\frac{\mathrm{d}z^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}z^\nu}{\mathrm{d}\tau} > 0; \tag{2.5.4}$$

so that their motion only depends on the curved geometry itself and does not depend on their own mass.[46] From this point of view, gravity is no longer a force.

Note that, strictly speaking, this "gravity-induced-dynamics-as-geodesics" is actually an idealization that applies for material bodies with no internal structure and whose proper sizes are very small compared to the length scale(s) associated with the geometric curvature itself. In reality, all physical systems have internal structure – non-trivial quadrupole moments, spin/rotation, etc. – and may furthermore be large enough that their full dynamics require detailed analysis to understand properly.

Newton vs. Einstein Observe that the Newtonian gravity of eq. (2.5.1) in an instantaneous force, in that the force on body 1 due to body 2 (or, vice versa) changes immediately when body 2 starts changing its position $\vec{x}_2$ – even though it is located at a finite distance away. However, Special Relativity tells us there ought to be an ultimate speed limit in Nature, i.e., no physical effect/information can travel faster than c. This apparent inconsistency between Newtonian gravity and Einstein's Special Relativity is of course a driving motivation that led Einstein to General Relativity. As we shall see shortly, by postulating that the effects of gravitation are in fact the result of residing in a curved spacetime, the Lorentz symmetry responsible for Special Relativity is recovered in any local "freely-falling" frame.

Massless particles Finally, this dynamics-as-geodesics also led Einstein to realize – if gravitation does indeed apply universally – that massless particles such as photons, i.e., electromagnetic waves, must also be influenced by the gravitational field too. This is a significant departure from Newton's law of gravity in eq. (2.5.1), which may lead one to suspect otherwise, since $M_{\text{photon}} = 0$. It is possible to justify this statement in detail, but we shall simply assert

[45] See Will [6] arXiv: 1403.7377 for a review on experimental tests of various versions of the Equivalence Principle and other aspects of General Relativity.

[46] If there *were* an external non-gravitational force f^μ, then the covariant Newton's second law for a system of mass M would read: $MD^2z^\mu/\mathrm{d}\tau^2 = f^\mu$.

here – to leading order in the JWKB approximation, photons in fact sweep out *null* geodesics $z^\mu(\lambda)$ in curved spacetimes:

$$a^\mu \equiv \frac{D^2 z^\mu}{\mathrm{d}\lambda^2} = 0, \qquad g_{\mu\nu}\left(z(\lambda)\right) \frac{\mathrm{d}z^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}z^\nu}{\mathrm{d}\lambda} = 0. \tag{2.5.5}$$

Locally flat coordinates, Einstein Equivalence Principle & Symmetries We now come to one of the most important features of curved spacetimes. In the neighborhood of a timelike geodesic $y^\mu = (s, \vec{y})$, one may choose *Fermi normal coordinates* $x^\mu \equiv (s, \vec{x})$ such that spacetime appears flat up to distances of $\mathcal{O}(1/|\max R_{\mu\nu\alpha\beta}(y = (s, \vec{y}))|^{1/2})$; namely, $g_{\mu\nu} = \eta_{\mu\nu}$ plus corrections that begin at quadratic order in the displacement $\vec{x} - \vec{y}$:

$$g_{00}(x) = 1 - R_{0a0b}(s) \cdot (x^a - y^a)(x^b - y^b) + \mathcal{O}\left((x-y)^3\right), \tag{2.5.6}$$

$$g_{0i}(x) = -\frac{2}{3} R_{0aib}(s) \cdot (x^a - y^a)(x^b - y^b) + \mathcal{O}\left((x-y)^3\right), \tag{2.5.7}$$

$$g_{ij}(x) = \eta_{ij} - \frac{1}{3} R_{iajb}(s) \cdot (x^a - y^a)(x^b - y^b) + \mathcal{O}\left((x-y)^3\right). \tag{2.5.8}$$

Here $x^0 = s$ is the time coordinate, and is also the proper time of the observer with the trajectory $y^\mu(s) = (s, \vec{y})$. (The $\vec{y}$ are fixed spatial coordinates; i.e., they do not depend on s.) Suppose you were placed inside a closed box, so you cannot tell what's outside. Then provided the box is small enough, you will not be able to distinguish between being in "free-fall" in a gravitational field versus being in a completely empty Minkowski spacetime.

As already alluded to in the "Newton vs. Einstein" discussion above, just as the rotation and translation symmetries of flat Euclidean space carried over to a small enough region of curved spaces – the FNC expansion of equations (2.5.6) through (2.5.8) indicates that, within the spacetime neighborhood of a freely-falling observer, any curved spacetime is Lorentz and spacetime-translation symmetric. To sum:

> Physically speaking, in a freely falling frame $\{x^\mu\}$ – i.e., centered along a timelike geodesic at $x = y$ – physics in a curved spacetime is the same as that in flat Minkowski spacetime up to corrections that go at least as
>
> $$\epsilon_{\mathrm{E}} \equiv \frac{\text{Length or inverse mass scale of system}}{\text{Length scale of the spacetime geometric curvature}}. \tag{2.5.9}$$

This is the essence of the equivalence principle that lead Einstein to recognize curved spacetime to be the setting to formulate his General Theory of Relativity. As a simple example, the geodesic y^μ itself obeys the free-particle version of Newton's 2nd law: $\mathrm{d}^2 y^\mu/\mathrm{d}s^2 = 0$.

Problem 2.40. In this problem, we will understand why we may always choose the frame where the spatial components $\vec{y}$ are time (i.e., $s-$)independent.

First use the geodesic equation obeyed by y^α to conclude $\mathrm{d}y^\alpha/\mathrm{d}s$ are constants. If s refers to the proper time of the freely falling observer at $y^\alpha(s)$, then explain why

$$\eta_{\alpha\beta} \frac{\mathrm{d}y^\alpha}{\mathrm{d}s}\frac{\mathrm{d}y^\beta}{\mathrm{d}s} = 1. \tag{2.5.10}$$

Since this is a Lorentz invariant condition, $\{y^\alpha\}$ can be Lorentz boosted $y^\alpha \to \Lambda^\alpha{}_\mu y^\mu$ to the rest frame such that

$$\frac{\mathrm{d}y^\alpha}{\mathrm{d}s} \to \Lambda^\alpha{}_\mu \frac{\mathrm{d}y^\mu}{\mathrm{d}s} = \left(1, \vec{0}\right). \tag{2.5.11}$$

To sum: in the co-moving frame of the freely falling observer $y^\alpha(s)$, the only s dependence in equations (2.5.6), (2.5.7) and (2.5.8) occur in the Riemann tensor. □

Problem 2.41. Verify that the coefficients in front of the Riemann tensor in equations (2.5.6), (2.5.7) and (2.5.8) are independent of the spacetime dimension. That is, starting with

$$g_{00}(x) = 1 - A \cdot R_{0a0b}(s) \cdot (x-y)^a (x-y)^b + \mathcal{O}\left((x-y)^3\right), \tag{2.5.12}$$

$$g_{0i}(x) = -B \cdot R_{0aib}(s) \cdot (x-y)^a (x-y)^b + \mathcal{O}\left((x-y)^3\right), \tag{2.5.13}$$

$$g_{ij}(x) = \eta_{ij} - C \cdot R_{iajb}(s) \cdot (x-y)^a (x-y)^b + \mathcal{O}\left((x-y)^3\right), \tag{2.5.14}$$

where A, B, C are unknown constants, recover the Riemann tensor at $x = y$. Hint: the calculation of R_{0ijk} and R_{abij} may require the Bianchi identity $R_{0[ijk]} = 0$.

Note: This problem is not meant to be a derivation of the Fermi normal expansion in equations (2.5.6), (2.5.7), and (2.5.8) – for that, see Poisson [?] §1.6 – but merely a consistency check. □

Problem 2.42. Gravitational force in a weak gravitational field Consider the following metric:

$$g_{\mu\nu}(t, \vec{x}) = \eta_{\mu\nu} + 2\Phi(\vec{x})\delta_{\mu\nu}, \tag{2.5.15}$$

where $\Phi(\vec{x})$ is time-independent. Assume this is a weak gravitational field, in that $|\Phi| \ll 1$ everywhere in spacetime, and there are no non-gravitational forces. (Linearized General Relativity reduce to the familiar Poisson equation $\vec{\nabla}^2\Phi = 4\pi G_{\mathrm{N}}\rho$, where $\rho(\vec{x})$ is the mass/energy density of matter.) Starting from the non-affine form of the action principle

$$\begin{aligned} -Ms &= -M \int_{t_1}^{t_2} \mathrm{d}t \sqrt{g_{\mu\nu}\dot{z}^\mu \dot{z}^\nu}, & \dot{z}^\mu &\equiv \frac{\mathrm{d}z^\mu}{\mathrm{d}t} \\ &= -M \int_{t_1}^{t_2} \mathrm{d}t \sqrt{1 - \vec{v}^2 + 2\Phi(1+\vec{v}^2)}, & \vec{v}^2 &\equiv \delta_{ij}\dot{z}^i \dot{z}^j; \end{aligned} \tag{2.5.16}$$

expand this action to lowest order in $\vec{v}^2$ and Φ and work out the geodesic equation of a 'test mass' M sweeping out some worldline z^μ in such a spacetime. (You should find something very familiar from Classical Mechanics.) Show that, in this non-relativistic limit, Newton's law of gravitation is recovered:

$$\frac{\mathrm{d}^2 z^i}{\mathrm{d}t^2} = -\partial_i \Phi. \tag{2.5.17}$$

We see that, in the weakly curved spacetime of eq. (2.5.15), Φ may indeed be identified as the Newtonian potential. □

Geodesic Deviation & Tidal Forces We now turn to the derivation of the *geodesic deviation* equation. Consider two geodesics that are infinitesimally close-by. Let both of them be parametrized by λ, so that we may connect one geodesic to the other at the same λ via an infinitesimal vector ξ^μ. We will denote the tangent vector to one of geodesics to be U^μ, such that

$$U^\sigma \nabla_\sigma U^\mu = 0. \tag{2.5.18}$$

Furthermore, we will assume that $[U, \xi] = 0$, i.e., U and ξ may be integrated to form a 2D coordinate system in the neighborhood of this pair of geodesics. Then

$$U^\alpha U^\beta \nabla_\alpha \nabla_\beta \xi^\mu = \nabla_U \nabla_U \xi^\mu = -R^\mu{}_{\nu\alpha\beta} U^\nu \xi^\alpha U^\beta. \tag{2.5.19}$$

As its name suggests, this equation tells us how the deviation vector ξ^μ joining two infinitesimally displaced geodesics is accelerated by the presence of spacetime curvature through the Riemann tensor. If spacetime were flat, the acceleration will be zero: two initially parallel geodesics will remain so.

For a macroscopic system, if U^μ is a timelike vector tangent to, say, the geodesic trajectory of its center-of-mass, the geodesic deviation equation (2.5.19) then describes *tidal forces* acting on it. In other words, the relative acceleration between the 'particles' that comprise the system – induced by spacetime curvature – would compete with the system's internal forces.[47]

Derivation of eq. (2.5.19) Starting with the geodesic equation $U^\sigma \nabla_\sigma U^\mu = 0$, we may take its derivative along ξ.

$$\begin{aligned}
\xi^\alpha \nabla_\alpha \left(U^\beta \nabla_\beta U^\mu\right) &= 0, \\
\left(\xi^\alpha \nabla_\alpha U^\beta - U^\alpha \nabla_\alpha \xi^\beta\right) \nabla_\beta U^\mu + U^\beta \nabla_\beta \xi^\alpha \nabla_\alpha U^\mu + \xi^\alpha U^\beta \nabla_\alpha \nabla_\beta U^\mu &= 0 \\
[\xi, U]^\beta \nabla_\beta U^\mu + U^\beta \nabla_\beta (\xi^\alpha \nabla_\alpha U^\mu) - U^\beta \xi^\alpha \nabla_\beta \nabla_\alpha U^\mu + \xi^\alpha U^\beta \nabla_\alpha \nabla_\beta U^\mu &= 0 \\
U^\beta \nabla_\beta (U^\alpha \nabla_\alpha \xi^\mu) &= -\xi^\alpha U^\beta [\nabla_\alpha, \nabla_\beta] U^\mu \\
U^\beta \nabla_\beta (U^\alpha \nabla_\alpha \xi^\mu) &= -\xi^\alpha U^\beta R^\mu{}_{\nu\alpha\beta} U^\nu.
\end{aligned}$$

We have repeatedly used $[\xi, U] = 0$ to state, for example, $\nabla_U \xi^\rho = U^\sigma \nabla_\sigma \xi^\rho = \xi^\sigma \nabla_\sigma U^\rho = \nabla_\xi U^\rho$. It is also possible to use a more elegant notation to arrive at eq. (2.5.19).

$$\nabla_U U^\mu = 0 \tag{2.5.20}$$

$$\nabla_\xi \nabla_U U^\mu = 0 \tag{2.5.21}$$

$$\nabla_U \underbrace{\nabla_\xi U^\mu}_{=\nabla_U \xi^\mu} + [\nabla_\xi, \nabla_U] U^\mu = 0 \tag{2.5.22}$$

$$\nabla_U \nabla_U \xi^\mu = -R^\mu{}_{\nu\alpha\beta} U^\nu \xi^\alpha U^\beta \tag{2.5.23}$$

On the last line, we have exploited the assumption that $[U, \xi] = 0$ to say $[\nabla_\xi, \nabla_U] U^\mu = ([\nabla_\xi, \nabla_U] - \nabla_{[\xi,U]}) U^\mu$ – recall eq. (2.4.34).

[47] The first gravitational wave detectors were in fact based on measuring the tidal squeezing and stretching of solid bars of aluminum. They are known as "Weber bars", named after their inventor Joseph Weber.

Problem 2.43. Geodesic Deviation & FNC Argue that all the Christoffel symbols $\Gamma^\alpha{}_{\mu\nu}$ evaluated along the free-falling geodesic in equations (2.5.6)-(2.5.8), namely when $x = y$, vanish. Then argue that all the time derivatives of the Christoffel symbols vanish along y too: $\partial_s^{n\geq 1}\Gamma^\alpha{}_{\mu\nu} = 0$. (Hints: Recall from Problem (1.9) that, specifying the first derivatives of the metric is equivalent to specifying the Christoffel symbols. Why is $\partial_s^{n\geq 1} g_{\alpha\beta}(x = y) = 0$? Why is $\partial_s^{n\geq 1}\partial_i g_{\alpha\beta}(x = y) = 0$?) Why does this imply, denoting $U^\mu \equiv \mathrm{d}y^\mu/\mathrm{d}s$, the geodesic equation

$$U^\nu \nabla_\nu U^\mu = \frac{\mathrm{d}U^\mu}{\mathrm{d}s} = 0? \tag{2.5.24}$$

Next, evaluate the geodesic deviation equation in these Fermi Normal Coordinates (FNC) system. Specifically, show that

$$U^\alpha U^\beta \nabla_\alpha \nabla_\beta \xi^\mu = \frac{\mathrm{d}^2\xi^\mu}{\mathrm{d}s^2} = -R^\mu{}_{0\nu 0}\xi^\nu. \tag{2.5.25}$$

Why does this imply, *if* the deviation vector is purely spatial at a given $s = s_0$, specifically $\xi^0(s_0) = \mathrm{d}\xi^0/\mathrm{d}s_0 = 0$, then it remains so for all time? (Hint: In an FNC system and on the world line of the free-falling observer, $R^0{}_{0\alpha\beta} = R_{00\alpha\beta}$. What do the (anti)symmetries of the Riemann tensor say about the right hand side?) □

Problem 2.44. Tidal forces due to mass monopole of isolated body In this problem we will consider sprinkling test masses initially at rest on the surface of an imaginary sphere of very small radius r_ϵ, whose center is located far from that of a static isolated body whose stress tensor is dominated by its mass density $\rho(\vec{x})$. We will examine how these test masses will respond to the gravitational tidal forces exerted by ρ.

Assume that the weak field metric generated by ρ is given by eq. (2.5.15); it is possible to justify this statement by using the linearized Einstein's equations. Show that the vector field

$$U^\mu(t, \vec{x}) \equiv \delta^\mu_0(1 - \Phi(\vec{x})) - t\delta^\mu_i \partial_i \Phi(\vec{x}) \tag{2.5.26}$$

is a timelike geodesic up to linear order in the Newtonian potential Φ. This U^μ may be viewed as the tangent vector to the worldline of the observer who was released from rest in the $(t, \vec{x})$ coordinate system at $t = 0$. (To ensure this remains a valid perturbative solution we shall also assume $t/r \ll 1$.) Let $\xi^\mu = (\xi^0, \vec{\xi})$ be the deviation vector whose spatial components we wish to interpret as the small displacement vector joining the center of the imaginary sphere to its surface. Use the above U^α to show that – up to first order in Φ – the right hand sides of its geodesic deviation equations are

$$U^\alpha U^\beta \nabla_\alpha \nabla_\beta \xi^0 = 0, \tag{2.5.27}$$

$$U^\alpha U^\beta \nabla_\alpha \nabla_\beta \xi^i = R_{i0j0}\xi^j; \tag{2.5.28}$$

where the linearized Riemann tensor reads

$$R_{i0j0} = -\partial_i \partial_j \Phi(\vec{x}). \tag{2.5.29}$$

Assuming that the monopole contribution dominates,

$$\Phi(\vec{x}) \approx \Phi(r) = -\frac{G_\mathrm{N} M}{r} = -\frac{r_s}{2r}, \tag{2.5.30}$$

show that these tidal forces have strengths that scale as $1/r^3$ as opposed to the $1/r^2$ forces of Newtonian gravity itself – specifically, you should find

$$R_{i0j0} \approx -\left(\delta^{ij} - \widehat{r}^i\widehat{r}^j\right)\frac{\Phi'(r)}{r} - \widehat{r}^i\widehat{r}^j\Phi''(r), \qquad \widehat{r}^i \equiv \frac{x^i}{r}, \tag{2.5.31}$$

so that the result follows simply from counting the powers of $1/r$ from $\Phi'(r)/r$ and $\Phi''(r)$. By setting $\vec{\xi}$ to be (anti-)parallel and perpendicular to the radial direction $\widehat{r}$, argue that the test masses lying on the radial line emanating from the body centered at $\vec{x} = \vec{0}$ will be *stretched apart* while the test masses lying on the plane perpendicular to $\widehat{r}$ will be *squeezed together*. (Hint: You should be able to see that $\delta^{ij} - \widehat{r}^i\widehat{r}^j$ is the Euclidean space orthogonal to $\widehat{r}$.)

The shape of the Earth's ocean tides can be analyzed in this manner by viewing the Earth as 'falling' in the gravitational fields of the Moon and the Sun. □

Interlude Let us pause to summarize the physics we have revealed thus far.

> In a curved spacetime, the collective motion of a system of mass M sweeps out a timelike geodesic – recall equations (2.4.54), (2.4.62), and (2.4.68) – whose dynamics is actually independent of M as long as its internal structure can be neglected. In the co-moving frame of an observer situated within this same system, physical laws appear to be the same as that in Minkowski spacetime up to distances of order $1/|\max R_{\widehat{\alpha}\widehat{\beta}\widehat{\mu}\widehat{\nu}}|^{1/2}$. However, once the finite size of the physical system is taken into account, one would find tidal forces exerted upon it due to spacetime curvature itself – this is described by the geodesic deviation eq. (2.5.25).

Killing Vectors A geometry is said to enjoy an isometry – or, symmetry – when we perform the following infinitesimal displacement

$$x^\mu \to x^\mu + \xi^\mu(x) \tag{2.5.32}$$

and find that the geometry is unchanged

$$g_{\mu\nu}(x) \to g_{\mu\nu}(x) + \mathcal{O}\left(\xi^2\right). \tag{2.5.33}$$

Generically, under the infinitesimal transformation of eq. (2.5.32),

$$g_{\mu\nu}(x) \to g_{\mu\nu}(x) + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu. \tag{2.5.34}$$

where

$$\nabla_{\{\mu}\xi_{\nu\}} = \xi^\sigma\partial_\sigma g_{\mu\nu} + g_{\sigma\{\mu}\partial_{\nu\}}\xi^\sigma. \tag{2.5.35}$$

If an isometry exists along the integral curve of ξ^μ, it has to obey Killing's equation – recall equations (1.2.42) and (1.2.43) –

$$\nabla_{\{\mu}\xi_{\nu\}} = \xi^\sigma\partial_\sigma g_{\mu\nu} + \partial_{\{\mu}\xi^\sigma g_{\nu\}\sigma} = 0. \tag{2.5.36}$$

In fact, by exponentiating the infinitesimal coordinate transformation, it is possible to show that – if ξ^μ is a Killing vector (i.e., it satisfies eq. (2.5.36)), then an isometry exists along its integral curve. In other words, A spacetime geometry enjoys an isometry (aka symmetry) along the integral curve of ξ^μ iff it obeys $\nabla_{\{\mu}\xi_{\nu\}} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$.

In a $d-$dimensional spacetime, there are at most $d(d+1)/2$ Killing vectors. A spacetime that has $d(d+1)/2$ Killing vectors is called *maximally symmetric*. (See Weinberg [1] for a discussion.)

Problem 2.45. Conserved quantities along geodesics (I of II) ∘ If p_μ denotes the 'momentum' variable of a geodesic

$$p_\mu \equiv \frac{\partial L_{\rm g}}{\partial \dot{z}^\mu}, \tag{2.5.37}$$

where $L_{\rm g}$ is defined in eq. (2.4.55), and if ξ^μ is a Killing vector of the same geometry $\nabla_{\{\alpha}\xi_{\beta\}} = 0$, show that

$$\xi^\mu(z(\lambda)) p_\mu(\lambda) \tag{2.5.38}$$

is a constant along the geodesic $z^\mu(\lambda)$. Hints: You should be able to show, if you perturb the coordinates by the Killing vector ξ^μ, namely $x^\mu \to x^\mu + \xi^\mu$, then you should obtain to first order in ξ,

$$\dot{z}^\mu \to \dot{z}^\mu + \dot{z}^\sigma \partial_\sigma \xi^\mu, \tag{2.5.39}$$

$$L_{\rm g} \to L_{\rm g}; \tag{2.5.40}$$

i.e., the Lagrangian is invariant if you recall eq. (2.5.36). On the other hand, varying the Lagrangian to first order yields

$$\delta L_{\rm g} = \frac{\partial L_{\rm g}}{\partial \dot{z}^\sigma}\dot{\xi}^\sigma + \frac{\partial L_{\rm g}}{\partial z^\sigma}\xi^\sigma + \mathcal{O}\left(\xi^2\right). \tag{2.5.41}$$

(II of II) ∘ The vector field version of this result goes as follows.

If the geodesic equation $v^\sigma \nabla_\sigma v^\mu = 0$ holds, and if ξ^μ is a Killing vector, then $\xi_\nu v^\nu$ is conserved along the integral curve of v^μ.

Can you demonstrate the validity of this statement? □

Second Derivatives of Killing Vectors Now let us also consider the second derivatives of ξ^μ. In particular, we will now explain why

$$\nabla_\alpha \nabla_\beta \xi_\delta = R^\lambda{}_{\alpha\beta\delta}\xi_\lambda. \tag{2.5.42}$$

Consider

$$0 = \nabla_\delta \nabla_{\{\alpha}\xi_{\beta\}} \tag{2.5.43}$$

$$= [\nabla_\delta, \nabla_\alpha]\xi_\beta + \nabla_\alpha\nabla_\delta\xi_\beta + [\nabla_\delta, \nabla_\beta]\xi_\alpha + \nabla_\beta\nabla_\delta\xi_\alpha \tag{2.5.44}$$

$$= -R^\lambda{}_{\beta\delta\alpha}\xi_\lambda - \nabla_\alpha\nabla_\beta\xi_\delta - R^\lambda{}_{\alpha\delta\beta}\xi_\lambda - \nabla_\beta\nabla_\alpha\xi_\delta \tag{2.5.45}$$

Because Bianchi says $0 = R^\lambda{}_{[\alpha\beta\delta]} \Rightarrow R^\lambda{}_{\alpha\beta\delta} = R^\lambda{}_{\beta\alpha\delta} + R^\lambda{}_{\delta\beta\alpha}$.

$$0 = -R^\lambda{}_{\beta\delta\alpha}\xi_\lambda - \nabla_\alpha\nabla_\beta\xi_\delta + \left(R^\lambda{}_{\beta\alpha\delta} + R^\lambda{}_{\delta\beta\alpha}\right)\xi_\lambda - \nabla_\beta\nabla_\alpha\xi_\delta \tag{2.5.46}$$

$$0 = -2R^{\lambda}{}_{\beta\delta\alpha}\xi_{\lambda} - \nabla_{\{\beta}\nabla_{\alpha\}}\xi_{\delta} - [\nabla_{\beta}, \nabla_{\alpha}]\xi_{\delta} \tag{2.5.47}$$

$$0 = -2R^{\lambda}{}_{\beta\delta\alpha}\xi_{\lambda} - 2\nabla_{\beta}\nabla_{\alpha}\xi_{\delta} \tag{2.5.48}$$

This proves eq. (2.5.42).

Commutators of Killing Vectors Next, we will show that

The commutator of 2 Killing vectors is also a Killing vector.

Let U and V be Killing vectors. If $\xi \equiv [U, V]$, we need to verify that

$$\nabla_{\{\alpha}\xi_{\beta\}} = \nabla_{\{\alpha}[U, V]_{\beta\}} = 0. \tag{2.5.49}$$

More explicitly, let us compute:

$$\begin{aligned}
&\nabla_{\alpha}(U^{\mu}\nabla_{\mu}V_{\beta} - V^{\mu}\nabla_{\mu}U_{\beta}) + (\alpha \leftrightarrow \beta) \\
&= \nabla_{\alpha}U^{\mu}\nabla_{\mu}V_{\beta} - \nabla_{\alpha}V^{\mu}\nabla_{\mu}U_{\beta} + U^{\mu}\nabla_{\alpha}\nabla_{\mu}V_{\beta} - V^{\mu}\nabla_{\alpha}\nabla_{\mu}U_{\beta} + (\alpha \leftrightarrow \beta) \\
&= -\nabla_{\mu}U_{\alpha}\nabla^{\mu}V_{\beta} + \nabla_{\mu}V_{\alpha}\nabla^{\mu}U_{\beta} + U^{\mu}\nabla_{[\alpha}\nabla_{\mu]}V_{\beta} + U^{\mu}\nabla_{\mu}\nabla_{\alpha}V_{\beta} - V^{\mu}\nabla_{[\alpha}\nabla_{\mu]}U_{\beta} - V^{\mu}\nabla_{\mu}\nabla_{\alpha}U_{\beta} + (\alpha \leftrightarrow \beta) \\
&= -U^{\mu}R^{\sigma}{}_{\beta\alpha\mu}V_{\sigma} + V^{\mu}R^{\sigma}{}_{\beta\alpha\mu}U_{\sigma} + (\alpha \leftrightarrow \beta) \\
&= -U^{[\mu}V^{\sigma]}R_{\sigma\{\beta\alpha\}\mu} = 0.
\end{aligned}$$

The $(\alpha \leftrightarrow \beta)$ means we are taking all the terms preceding it and swapping $\alpha \leftrightarrow \beta$. Moreover, we have repeatedly used the Killing equations $\nabla_{\alpha}U_{\beta} = -\nabla_{\beta}U_{\alpha}$ and $\nabla_{\alpha}V_{\beta} = -\nabla_{\beta}V_{\alpha}$.

Problem 2.46. Killing Vectors in Minkowski In Minkowski spacetime $g_{\mu\nu} = \eta_{\mu\nu}$, with Cartesian coordinates $\{x^{\mu}\}$, use eq. (2.5.42) to argue that the most general Killing vector takes the form

$$\xi_{\mu} = \ell_{\mu} + \omega_{\mu\nu}x^{\nu}, \tag{2.5.50}$$

for constant ℓ_{μ} and $\omega_{\mu\nu}$. (Hint: Think about Taylor expansions; use eq. (2.5.42) to show that the 2nd and higher partial derivatives of ξ_{δ} are zero.) Then use the Killing equation (2.5.36) to infer that

$$\omega_{\mu\nu} = -\omega_{\nu\mu}. \tag{2.5.51}$$

The ℓ_{μ} corresponds to infinitesimal spacetime translation and the $\omega_{\mu\nu}$ to infinitesimal Lorentz boosts and rotations. Explain why this implies the following are the Killing vectors of flat spacetime:

$$\partial_{\mu} \qquad \text{(Generators of spacetime translations)} \tag{2.5.52}$$

and

$$x^{[\mu}\partial^{\nu]} \qquad \text{(Generators of Lorentz boosts or rotations).} \tag{2.5.53}$$

There are d distinct ∂_{μ}'s and (due to their antisymmetry) $(1/2)(d^2 - d)$ distinct $x^{[\mu}\partial^{\nu]}$'s. Therefore there are a total of $d(d+1)/2$ Killing vectors in Minkowski – i.e., it is maximally symmetric. □

It might be instructive to check our understanding of rotation and boosts against the 2D case we have worked out earlier via different means. Up to first order in the rotation angle θ, the 2D rotation matrix in eq. (2.1.59) reads

$$\widehat{R}^i{}_j(\theta) = \begin{bmatrix} 1 & -\theta \\ \theta & 1 \end{bmatrix} + \mathcal{O}\left(\theta^2\right). \tag{2.5.54}$$

In other words, $\widehat{R}^i{}_j(\theta) = \delta_{ij} - \theta\epsilon_{ij}$, where ϵ_{ij} is the Levi-Civita symbol in 2D with $\epsilon_{12} \equiv 1$. Applying a rotation of the 2D Cartesian coordinates x^i upon a test (scalar) function f,

$$f(x^i) \to f\left(\widehat{R}^i{}_j x^j\right) = f\left(x^i - \theta\epsilon_{ij}x^j + \mathcal{O}\left(\theta^2\right)\right) \tag{2.5.55}$$

$$= f(\vec{x}) - \theta\epsilon_{ij}x^j\partial_i f(\vec{x}) + \mathcal{O}\left(\theta^2\right). \tag{2.5.56}$$

Since θ is arbitrary, the basic differential operator that implements an infinitesimal rotation of the coordinate system on any Minkowski scalar is

$$-\epsilon_{ij}x^j\partial_i = x^1\partial_2 - x^2\partial_1. \tag{2.5.57}$$

This is the 2D version of eq. (2.5.53) for rotations. As for 2D Lorentz boosts, eq. (2.1.58) tells us

$$\Lambda^\mu{}_\nu(\xi) = \begin{bmatrix} 1 & \xi \\ \xi & 1 \end{bmatrix} + \mathcal{O}\left(\xi^2\right). \tag{2.5.58}$$

(This ξ is known as *rapidity*.) Here, we have $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \xi \cdot \epsilon^\mu{}_\nu$, where $\epsilon_{\mu\nu}$ is the Levi-Civita tensor in 2D Minkowski with $\epsilon_{01} \equiv 1$. Therefore, to implement an infinitesimal Lorentz boost on the Cartesian coordinates within a test (scalar) function $f(x^\mu)$, we do

$$f(x^\mu) \to f\left(\Lambda^\mu{}_\nu x^\nu\right) = f\left(x^\mu + \xi\epsilon^\mu{}_\nu x^\nu + \mathcal{O}\left(\xi^2\right)\right) \tag{2.5.59}$$

$$= f(x) - \xi\epsilon_{\nu\mu}x^\nu\partial^\mu f(x) + \mathcal{O}\left(\xi^2\right). \tag{2.5.60}$$

Since ξ is arbitrary, to implement a Lorentz boost of the coordinate system on any Minkowski scalar, the appropriate differential operator is

$$\epsilon_{\mu\nu}x^\mu\partial^\nu = x^0\partial^1 - x^1\partial^0; \tag{2.5.61}$$

which again is encoded within eq. (2.5.53).

Problem 2.47. Verify that Lie Algebra of $\text{SO}_{D,1}$ in (2.1.100) is recovered if we exploit eq. (2.5.53) to define

$$J^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu), \tag{2.5.62}$$

where $\partial^\mu \equiv \eta^{\mu\nu}\partial_\nu$. This tells us, under a Lorentz boost or rotation $f(x) \to \exp(-(i/2)\omega_{\mu\nu}J^{\mu\nu})f(x)$. □

Problem 2.48. Co-moving Observers & Rulers In Cosmology We live in a universe that, at the very largest length scales, is described by the following spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a(t)^2 \mathrm{d}\vec{x} \cdot \mathrm{d}\vec{x}; \tag{2.5.63}$$

where $a(t)$ describes the relative size of the universe. Enumerate as many constants-of-motion as possible of this geometry. (Hint: Focus on the spatial part of the metric and try to draw a connection with the previous problem.)

In this cosmological context, a co-moving observer is one that does not move spatially, i.e., $\mathrm{d}\vec{x} = 0$. Solve the geodesic swept out by such an observer.

Galaxies A and B are respectively located at $\vec{x}$ and $\vec{x}'$ at a fixed cosmic time t. What is their spatial distance on this constant t slice of spacetime? □

Problem 2.49. Killing identities involving Ricci Prove the following results. If ξ^μ is a Killing vector and $R_{\alpha\beta}$ and $\mathcal{R}$ are the Ricci tensor and scalar respectively, then

$$\xi^\alpha \nabla^\beta R_{\alpha\beta} = 0 \qquad \text{and} \qquad \xi^\alpha \nabla_\alpha \mathcal{R} = 0. \tag{2.5.64}$$

Hints: First use eq. (2.5.42) to show that

$$\Box \xi_\delta = -R^\lambda{}_\delta \xi_\lambda, \tag{2.5.65}$$

$$\Box \equiv g^{\alpha\beta} \nabla_\alpha \nabla_\beta = \nabla_\alpha \nabla^\alpha. \tag{2.5.66}$$

Then take the divergence on both sides, and commute the covariant derivatives until you obtain the term $\Box \nabla^\delta \xi_\delta$ – what is $\nabla^\delta \xi_\delta$ equal to? Argue why $\xi^\alpha \nabla^\beta R_{\alpha\beta} = \nabla^\beta (\xi^\alpha R_{\alpha\beta})$. You may also need to employ the Einstein tensor Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ to infer that $\xi^\alpha \nabla_\alpha \mathcal{R} = 0$. □

Problem 2.50. In d spacetime dimensions, show that

$$\partial_{[\alpha_1} \left(J^\mu \widetilde{\epsilon}_{\alpha_2 \dots \alpha_d]\mu} \right) \tag{2.5.67}$$

is proportional to $\nabla_\sigma J^\sigma$. What is the proportionality factor? (This discussion provides a differential forms based language to write $\mathrm{d}^d x \sqrt{|g|} \nabla_\sigma J^\sigma$.) If $\nabla_\sigma J^\sigma = 0$, what does the Poincaré lemma tell us about eq. (2.5.67)? Find the dual of your result and argue there must an antisymmetric tensor $\Sigma^{\mu\nu}$ such that

$$J^\mu = \nabla_\nu \Sigma^{\mu\nu}. \tag{2.5.68}$$

Hint: For the first step, explain why eq. (2.5.67) is proportional to the Levi-Civita symbol $\epsilon_{\alpha_1 \dots \alpha_d}$. □

Problem 2.51. Gauge-covariant derivative Let ψ be a vector under *group* transformations. By this we mean that, if $\psi^{\check{a}}$ corresponds to the ath component of ψ, then given some matrix $U^{\check{a}}{}_{\check{b}}$, ψ transforms as

$$\psi^{\check{a}'} = U^{\check{a}'}{}_{\check{b}} \psi^{\check{b}} \qquad\qquad (\text{or, } \psi' = U\psi). \tag{2.5.69}$$

Compare eq. (2.5.69) to how a spacetime vector transforms under coordinate transformations:

$$V^{\mu'}(x') = \mathcal{J}^{\mu'}{}_{\sigma} V^{\sigma}(x), \qquad \mathcal{J}^{\mu}{}_{\sigma} \equiv \frac{\partial x'^{\mu}}{\partial x^{\sigma}}. \tag{2.5.70}$$

Now, let us consider taking the gauge-covariant derivative $\check{D}$ of ψ such that it still transforms 'covariantly' under group transformations, namely

$$\check{D}_\alpha \psi' = \check{D}_\alpha (U\psi) = U(\check{D}_\alpha \psi). \tag{2.5.71}$$

Crucially:

> *We shall now demand that the gauge-covariant derivative transforms covariantly – i.e., eq.* (2.5.71) *holds – even when the group transformation $U(x)$ depends on spacetime coordinates.*

First check that, the spacetime-covariant derivative cannot be equal to the gauge-covariant derivative in general, i.e.,

$$\nabla_\alpha \psi' \neq \check{D}_\alpha \psi', \tag{2.5.72}$$

by showing that eq. (2.5.71) is not satisfied.

Just as the spacetime-covariant derivative was built from the partial derivative by adding a Christoffel symbol, $\nabla = \partial + \Gamma$, we may build a gauge-covariant derivative by adding to the spacetime-covariant derivative a *gauge potential*:

$$(\check{D}_\mu)^{\check{a}}{}_{\check{b}} \equiv \delta^a_b \nabla_\mu + (A_\mu)^{\check{a}}{}_{\check{b}}. \tag{2.5.73}$$

Or, in gauge-index-free notation,

$$\check{D}_\mu \equiv \nabla_\mu + A_\mu. \tag{2.5.74}$$

With the definition in eq. (2.5.73), how must the gauge potential A_μ (or, equivalently, $(A_\mu)^{\check{a}}{}_{\check{b}}$) transform so that eq. (2.5.71) is satisfied? Compare the answer to the transformation properties of the Christoffel symbol in eq. (2.4.6). (Since the answer can be found in most Quantum Field Theory textbooks, make sure you verify the covariance explicitly!)

Bonus: Here, we have treated ψ as a spacetime scalar and the gauge-covariant derivative $\check{D}_\alpha$ itself as a scalar under group transformations. Can you generalize the analysis here to the higher-rank tensor case? □

2.6 Special Topic: Gravitational Perturbation Theory

Carrying out perturbation theory about some fixed 'background' geometry $\bar{g}_{\mu\nu}$ has important physical applications. As such, in this section, we will in fact proceed to set up a general and systematic perturbation theory involving the metric:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{2.6.1}$$

where $\bar{g}_{\mu\nu}$ is an arbitrary 'background' metric and $h_{\mu\nu}$ is a small deviation. I will also take the opportunity to discuss the transformation properties of $h_{\mu\nu}$ under infinitesimal coordinate transformations, i.e., the gauge transformations of gravitons.

Metric inverse, Determinant Whenever performing a perturbative analysis, we shall agree to move all tensor indices – including that of $h_{\mu\nu}$ – with the $\bar{g}_{\alpha\beta}$. For example,

$$h^\alpha{}_\beta \equiv \bar{g}^{\alpha\sigma} h_{\sigma\beta}, \qquad \text{and} \qquad h^{\alpha\beta} \equiv \bar{g}^{\alpha\sigma}\bar{g}^{\beta\rho} h_{\sigma\rho}. \tag{2.6.2}$$

With this convention in place, let us note that the inverse metric is a geometric series. Firstly,

$$g_{\mu\nu} = \bar{g}_{\mu\sigma}\left(\delta^\sigma_\nu + h^\sigma{}_\nu\right) \doteq \bar{g} \cdot (\mathbb{I} + \mathsf{h}). \tag{2.6.3}$$

(Here, h is a matrix, whose μth row and νth column is $h^\mu{}_\nu \equiv \bar{g}^{\mu\sigma} h_{\sigma\nu}$.) Remember that, for invertible matrices A and B, we have $(A \cdot B)^{-1} = B^{-1} A^{-1}$. Therefore

$$g^{-1} = (\mathbb{I} + \mathsf{h})^{-1} \cdot \bar{g}^{-1}. \tag{2.6.4}$$

If we were dealing with numbers instead of matrices, the geometric series $1/(1+z) = \sum_{\ell=0}^{\infty} (-)^\ell z^\ell$ may come to mind. You may directly verify that this prescription, in fact, still works.

$$g^{\mu\nu} = \left(\delta^\mu{}_\lambda + \sum_{\ell=1}^{\infty} (-)^\ell h^\mu{}_{\sigma_1} h^{\sigma_1}{}_{\sigma_2} \dots h^{\sigma_{\ell-2}}{}_{\sigma_{\ell-1}} h^{\sigma_{\ell-1}}{}_\lambda \right) \bar{g}^{\lambda\nu} \tag{2.6.5}$$

$$= \bar{g}^{\mu\nu} + \sum_{\ell=1}^{\infty} (-)^\ell h^\mu{}_{\sigma_1} h^{\sigma_1}{}_{\sigma_2} \dots h^{\sigma_{\ell-2}}{}_{\sigma_{\ell-1}} h^{\sigma_{\ell-1}\nu} \tag{2.6.6}$$

$$= \bar{g}^{\mu\nu} - h^{\mu\nu} + h^\mu{}_{\sigma_1} h^{\sigma_1\nu} - h^\mu{}_{\sigma_1} h^{\sigma_1}{}_{\sigma_2} h^{\sigma_2\nu} + \dots. \tag{2.6.7}$$

The square root of the determinant of the metric can be computed order-by-order in perturbation theory via the following formula. For any matrix A,

$$\det A = \exp\left[\mathrm{Tr}\left[\ln A\right]\right], \tag{2.6.8}$$

where Tr is the matrix trace; for e.g., $\mathrm{Tr}\left[\mathsf{h}\right] = h^\sigma{}_\sigma$. Taking the determinant of both sides of eq. (2.6.3), and using the property $\det[A \cdot B] = \det A \cdot \det B$,

$$\det g_{\alpha\beta} = \det \bar{g}_{\alpha\beta} \cdot \det\left[\mathbb{I} + \mathsf{h}\right], \tag{2.6.9}$$

so that eq. (2.6.8) can be employed to state

$$\sqrt{|g|} = \sqrt{|\bar{g}|} \cdot \exp\left[\frac{1}{2}\mathrm{Tr}\left[\ln[\mathbb{I} + \mathsf{h}]\right]\right]. \tag{2.6.10}$$

The first few terms read

$$\sqrt{|g|} = \sqrt{|\bar{g}|}\Bigg(1 + \frac{1}{2}h + \frac{1}{8}h^2 - \frac{1}{4}h^{\sigma\rho}h_{\sigma\rho} + \frac{1}{48}h^3 - \frac{1}{8}h \cdot h^{\sigma\rho}h_{\sigma\rho} + \frac{1}{6}h^{\sigma\rho}h_{\rho\kappa}h^\kappa{}_\sigma + \mathcal{O}[h^4]\Bigg) \tag{2.6.11}$$

$$h \equiv h^\sigma{}_\sigma. \tag{2.6.12}$$

Covariance, Covariant Derivatives, Geometric Tensors Under a coordinate transformation $x \equiv x(x')$, the full metric of course transforms as a tensor. The full metric $g_{\alpha'\beta'}$ in this new x' coordinate system reads

$$g_{\alpha'\beta'}(x') = (\bar{g}_{\mu\nu}(x(x')) + h_{\mu\nu}(x(x'))) \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta}. \tag{2.6.13}$$

If we define the 'background metric' to transform covariantly; namely

$$\bar{g}_{\alpha'\beta'}(x') \equiv \bar{g}_{\mu\nu}(x(x')) \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta}; \tag{2.6.14}$$

then, from eq. (2.6.13), the perturbation itself can be treated as a tensor

$$h_{\alpha'\beta'}(x') = h_{\mu\nu}(x(x')) \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta}. \tag{2.6.15}$$

These will now guide us to construct the geometric tensors – the full Riemann tensor, Ricci tensor and Ricci scalar – using the covariant derivative $\overline{\nabla}$ with respect to the 'background metric' $\bar{g}_{\mu\nu}$ and its associated geometric tensors. Let's begin by considering this background covariant derivative acting on the full metric in eq. (2.6.1):

$$\overline{\nabla}_\alpha g_{\mu\nu} = \overline{\nabla}_\alpha \left(\bar{g}_{\mu\nu} + h_{\mu\nu}\right) = \overline{\nabla}_\alpha h_{\mu\nu}. \tag{2.6.16}$$

On the other hand, the usual rules of covariant differentiation tell us

$$\overline{\nabla}_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \overline{\Gamma}^\sigma{}_{\alpha\mu} g_{\sigma\nu} - \overline{\Gamma}^\sigma{}_{\alpha\nu} g_{\mu\sigma}; \tag{2.6.17}$$

where the Christoffel symbols here are built out of the 'background metric',

$$\overline{\Gamma}^\sigma{}_{\alpha\mu} = \frac{1}{2} \bar{g}^{\sigma\lambda} \left(\partial_\alpha \bar{g}_{\mu\lambda} + \partial_\mu \bar{g}_{\alpha\lambda} - \partial_\lambda \bar{g}_{\mu\alpha}\right). \tag{2.6.18}$$

Problem 2.52. Relation between 'background' and 'full' Christoffel Show that equations (2.6.16) and (2.6.17) can be used to deduce that the full Christoffel symbol

$$\Gamma^\alpha{}_{\mu\nu}[g] = \frac{1}{2} g^{\alpha\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \tag{2.6.19}$$

can be related to that of its background counterpart through the relation

$$\Gamma^\alpha{}_{\mu\nu}[g] = \overline{\Gamma}^\alpha{}_{\mu\nu}[\bar{g}] + \delta\Gamma^\alpha{}_{\mu\nu}. \tag{2.6.20}$$

Here,

$$\delta\Gamma^\alpha{}_{\mu\nu} \equiv \frac{1}{2} g^{\alpha\sigma} H_{\sigma\mu\nu}, \tag{2.6.21}$$

$$H_{\sigma\mu\nu} \equiv \overline{\nabla}_\mu h_{\nu\sigma} + \overline{\nabla}_\nu h_{\mu\sigma} - \overline{\nabla}_\sigma h_{\mu\nu}. \tag{2.6.22}$$

Notice the difference between the 'full' and 'background' Christoffel symbols, namely $\Gamma^\mu{}_{\alpha\beta} - \overline{\Gamma}^\mu{}_{\alpha\beta}$, is a tensor. □

Problem 2.53. Geometric tensors With the result in eq. (2.6.20), show that for an arbitrary 1-form V_β,

$$\nabla_\alpha V_\beta = \overline{\nabla}_\alpha V_\beta - \delta\Gamma^\sigma{}_{\alpha\beta} V_\sigma. \tag{2.6.23}$$

Use this to compute $[\nabla_\alpha, \nabla_\beta] V_\lambda$ and proceed to show that the exact Riemann tensor is

$$R^\alpha{}_{\beta\mu\nu}[g] = \bar{R}^\alpha{}_{\beta\mu\nu}[\bar{g}] + \delta R^\alpha{}_{\beta\mu\nu}, \tag{2.6.24}$$

$$\delta R^\alpha{}_{\beta\mu\nu} \equiv \overline{\nabla}_{[\mu} \delta\Gamma^\alpha{}_{\nu]\beta} + \delta\Gamma^\alpha{}_{\sigma[\mu} \delta\Gamma^\sigma{}_{\nu]\beta} \tag{2.6.25}$$

$$= \frac{1}{2}\overline{\nabla}_\mu \left(g^{\alpha\lambda} H_{\lambda\nu\beta}\right) - \frac{1}{2}\overline{\nabla}_\nu \left(g^{\alpha\lambda} H_{\lambda\mu\beta}\right) + \frac{1}{4} g^{\alpha\lambda} g^{\sigma\rho} \left(H_{\lambda\mu\sigma} H_{\rho\beta\nu} - H_{\lambda\nu\sigma} H_{\rho\beta\mu}\right), \tag{2.6.26}$$

where $\bar{R}^\alpha{}_{\beta\mu\nu}[\bar{g}]$ is the Riemann tensor built entirely out of the background metric $\bar{g}_{\alpha\lambda}$. □

From eq. (2.6.24), the Ricci tensor and scalars can be written down:

$$R_{\mu\nu}[g] = R^\sigma{}_{\mu\sigma\nu} \qquad \text{and} \qquad \mathcal{R}[g] = g^{\mu\nu} R_{\mu\nu}. \tag{2.6.27}$$

From these formulas, perturbation theory can now be carried out. The primary reason why these geometric tensors admit an infinite series is because of the geometric series of the full inverse metric eq. (2.6.6). I find it helpful to remember, when one multiplies two infinite series which do not have negative powers of the expansion object $h_{\mu\nu}$, the terms that contain precisely n powers of $h_{\mu\nu}$ is a discrete convolution: for instance, such an nth order piece of the Ricci scalar is

$$\delta_n \mathcal{R} = \sum_{\ell=0}^{n} \delta_\ell g^{\mu\nu} \delta_{n-\ell} R_{\mu\nu}, \tag{2.6.28}$$

where $\delta_\ell g^{\mu\nu}$ is the piece of the full inverse metric containing exactly ℓ powers of $h_{\mu\nu}$ and $\delta_{n-\ell} R_{\mu\nu}$ is that containing precisely $n - \ell$ powers of the same.

Problem 2.54. Linearized geometric tensors The Riemann tensor that contains up to one power of $h_{\mu\nu}$ can be obtained readily from eq. (2.6.24). The H^2 terms begin at order h^2, so we may drop them; and since H is already linear in h, the g^{-1} contracted into it can be set to the background metric.

$$R^\alpha{}_{\beta\mu\nu}[g] = \bar{R}^\alpha{}_{\beta\mu\nu}[\bar{g}] + \frac{1}{2}\overline{\nabla}_{[\mu} \left(\overline{\nabla}_{\nu]} h_\beta{}^\alpha + \overline{\nabla}_{|\beta|} h_{\nu]}^\alpha - \overline{\nabla}^\alpha h_{\nu]\beta}\right) + \mathcal{O}(h^2) \tag{2.6.29}$$

$$= \bar{R}^\alpha{}_{\beta\mu\nu}[\bar{g}] + \frac{1}{2}\left([\overline{\nabla}_\mu, \overline{\nabla}_\nu] h_\beta{}^\alpha + \overline{\nabla}_\mu \overline{\nabla}_\beta h_\nu{}^\alpha - \overline{\nabla}_\nu \overline{\nabla}_\beta h_\mu{}^\alpha - \overline{\nabla}_\mu \overline{\nabla}^\alpha h_{\nu\beta} + \overline{\nabla}_\nu \overline{\nabla}^\alpha h_{\mu\beta}\right) + \mathcal{O}(h^2).$$

(The $|\beta|$ on the first line indicates the β is not to be antisymmetrized.) Starting from the linearized Riemann tensor in eq. (2.6.29), let us work out the linearized Ricci tensor, Ricci scalar, and Einstein tensor.

Specifically, show that one contraction of eq. (2.6.29) yields the linearized Ricci tensor:

$$R_{\beta\nu} = \overline{R}_{\beta\nu} + \delta_1 R_{\beta\nu} + \mathcal{O}(h^2), \tag{2.6.30}$$

$$\delta_1 R_{\beta\nu} \equiv \frac{1}{2}\left(\overline{\nabla}^\mu \overline{\nabla}_{\{\beta} h_{\nu\}\mu} - \overline{\nabla}_\nu \overline{\nabla}_\beta h - \overline{\nabla}^\mu \overline{\nabla}_\mu h_{\beta\nu}\right). \tag{2.6.31}$$

Contracting this Ricci tensor result with the full inverse metric, verify that the linearized Ricci scalar is

$$\mathcal{R} = \overline{\mathcal{R}} + \delta_1 \mathcal{R} + \mathcal{O}(h^2), \tag{2.6.32}$$

$$\delta_1 \mathcal{R} \equiv -h^{\beta\nu}\bar{R}_{\beta\nu} + \left(\overline{\nabla}^{\mu}\overline{\nabla}^{\nu} - \bar{g}^{\mu\nu}\overline{\nabla}^{\sigma}\overline{\nabla}_{\sigma}\right) h_{\mu\nu}. \tag{2.6.33}$$

Now, let us define the variable $\bar{h}_{\mu\nu}$ through the relation

$$h_{\mu\nu} \equiv \bar{h}_{\mu\nu} - \frac{\bar{g}_{\mu\nu}}{d-2}\bar{h}, \qquad \bar{h} \equiv \bar{h}^{\sigma}{}_{\sigma}. \tag{2.6.34}$$

First explain why this is equivalent to

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{\bar{g}_{\mu\nu}}{2}h. \tag{2.6.35}$$

(Hint: First calculate the trace of $\bar{h}$ in terms of h.) In (3+1)D this $\bar{h}_{\mu\nu}$ is often dubbed the "trace-reversed" perturbation – can you see why? Then show that the linearized Einstein tensor is

$$G_{\mu\nu} = \bar{G}_{\mu\nu}[\bar{g}] + \delta_1 G_{\mu\nu} + \mathcal{O}(\bar{h}^2), \tag{2.6.36}$$

where

$$\begin{aligned}\delta_1 G_{\mu\nu} \equiv &-\frac{1}{2}\left(\overline{\Box}\bar{h}_{\mu\nu} + \bar{g}_{\mu\nu}\overline{\nabla}_{\sigma}\overline{\nabla}_{\rho}\bar{h}^{\sigma\rho} - \overline{\nabla}_{\{\mu}\overline{\nabla}^{\sigma}\bar{h}_{\nu\}\sigma}\right) \\ &+ \frac{1}{2}\left(\bar{g}_{\mu\nu}\bar{h}^{\rho\sigma}\bar{R}_{\rho\sigma} + \bar{h}_{\{\mu}{}^{\sigma}\bar{R}_{\nu\}\sigma} - \bar{h}_{\mu\nu}\bar{\mathcal{R}} - 2\bar{h}^{\rho\sigma}\bar{R}_{\mu\rho\nu\sigma}\right).\end{aligned} \tag{2.6.37}$$

Cosmology, Kerr/Schwarzschild black holes, and Minkowski spacetimes are three physically important geometries. This result may be used to study linear perturbations about them. □

Second order Ricci For later purposes, we collect the second order Ricci tensor – see, for e.g., equation 35.58b of [4]:[48]

$$\begin{aligned}\delta_2 R_{\mu\nu} = \frac{1}{2}\Bigg\{&\frac{1}{2}\overline{\nabla}_{\mu}h_{\alpha\beta}\overline{\nabla}_{\nu}h^{\alpha\beta} + h^{\alpha\beta}\left(\overline{\nabla}_{\nu}\overline{\nabla}_{\mu}h_{\alpha\beta} + \overline{\nabla}_{\beta}\overline{\nabla}_{\alpha}h_{\mu\nu} - \overline{\nabla}_{\beta}\overline{\nabla}_{\nu}h_{\mu\alpha} - \overline{\nabla}_{\beta}\overline{\nabla}_{\mu}h_{\nu\alpha}\right) \\ &+ \overline{\nabla}^{\beta}h^{\alpha}{}_{\nu}\left(\overline{\nabla}_{\beta}h_{\mu\alpha} - \overline{\nabla}_{\alpha}h_{\mu\beta}\right) - \overline{\nabla}_{\beta}\left(h^{\alpha\beta} - \frac{1}{2}\bar{g}^{\alpha\beta}h\right)\left(\overline{\nabla}_{\{\nu}h_{\mu\}\alpha} - \overline{\nabla}_{\alpha}h_{\mu\nu}\right)\Bigg\}.\end{aligned} \tag{2.6.38}$$

Gauge transformations: Infinitesimal Coordinate Transformations In the above discussion, we regarded the 'background metric' as a tensor. As a consequence, the metric perturbation $h_{\mu\nu}$ was also a tensor. However, since it is the full metric that enters any generally covariant calculation, it really is the combination $\bar{g}_{\mu\nu} + h_{\mu\nu}$ that transforms as a tensor. As we will now explore, when the coordinate transformation

$$x^{\mu} = x'^{\mu} + \xi^{\mu}(x') \tag{2.6.39}$$

[48] I have checked that eq. (2.6.38) is consistent with the output from xAct [20].

is infinitesimal, in that ξ^μ is small in the same sense that $h_{\mu\nu}$ is small, we may instead attribute all the ensuing coordinate transformations to a transformation of $h_{\mu\nu}$ alone. This will allow us to view 'small' coordinate transformations as gauge transformations, and will also be important for the discussion of the linearized Einstein's equations.

In what follows, we shall view the x and x' in eq. (2.6.39) as referring to the same spacetime point, but expressed within infinitesimally different coordinate systems. Now, transforming from x to x',

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu = \left(\bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x)\right)\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= \left(\bar{g}_{\mu\nu}(x'+\xi) + h_{\mu\nu}(x'+\xi)\right)\left(\mathrm{d}x'^\mu + \partial_{\alpha'}\xi^\mu \mathrm{d}x'^\alpha\right)\left(\mathrm{d}x'^\nu + \partial_{\beta'}\xi^\nu \mathrm{d}x'^\beta\right) \\ &= \left(\bar{g}_{\mu\nu}(x') + \xi^\sigma \partial_{\sigma'}\bar{g}_{\mu\nu}(x') + h_{\mu\nu}(x') + \mathcal{O}\left(\xi^2, \xi\partial h\right)\right)\left(\mathrm{d}x'^\mu + \partial_{\alpha'}\xi^\mu \mathrm{d}x'^\alpha\right)\left(\mathrm{d}x'^\nu + \partial_{\beta'}\xi^\nu \mathrm{d}x'^\beta\right) \\ &= \left(\bar{g}_{\mu\nu}(x') + \xi^\sigma(x')\partial_{\sigma'}\bar{g}_{\mu\nu}(x') + \bar{g}_{\sigma\{\mu}(x')\partial_{\nu'\}}\xi^\sigma(x') + h_{\mu\nu}(x') + \mathcal{O}\left(\xi^2, \xi\partial h\right)\right)\mathrm{d}x'^\mu \mathrm{d}x'^\nu \\ &\equiv \left(\bar{g}_{\mu'\nu'}(x') + h_{\mu'\nu'}(x')\right)\mathrm{d}x'^\mu \mathrm{d}x'^\nu . \end{aligned} \tag{2.6.40}$$

This teaches us that, the infinitesimal coordinate transformation of eq. (2.6.39) amounts to keeping the background metric fixed,

$$\bar{g}_{\mu\nu}(x) \to \bar{g}_{\mu\nu}(x), \tag{2.6.41}$$

but shifting

$$h_{\mu\nu}(x) \to h_{\mu\nu}(x) + \xi^\sigma(x)\partial_\sigma \bar{g}_{\mu\nu}(x) + \bar{g}_{\sigma\{\mu}(x)\partial_{\nu\}}\xi^\sigma(x), \tag{2.6.42}$$

followed by replacing

$$x^\mu \to x'^\mu \qquad \text{and} \qquad \partial_\mu \equiv \frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x'^\mu} \equiv \partial_{\mu'}. \tag{2.6.43}$$

However, since x and x' refer to the same point in spacetime,[49] it is customary within the contemporary physics literature to drop the primes and simply phrase the coordinate transformation as replacement rules:

$$x^\mu \to x^\mu + \xi^\mu(x), \tag{2.6.44}$$

$$\bar{g}_{\mu\nu}(x) \to \bar{g}_{\mu\nu}(x), \tag{2.6.45}$$

$$h_{\mu\nu}(x) \to h_{\mu\nu}(x) + \overline{\nabla}_{\{\mu}\xi_{\nu\}}(x); \tag{2.6.46}$$

where we have recognized

$$\xi^\sigma \partial_\sigma \bar{g}_{\mu\nu} + \bar{g}_{\sigma\{\mu}\partial_{\nu\}}\xi^\sigma = \overline{\nabla}_{\{\mu}\xi_{\nu\}} \equiv (\pounds_\xi \bar{g})_{\mu\nu}(x). \tag{2.6.47}$$

Problem 2.55. Lie Derivative of a tensor If x and x' are infinitesimally nearby coordinate systems related via eq. (2.6.39), show that $T^{\mu_1\cdots\mu_N}{}_{\nu_1\ldots\nu_M}(x)$ (the components of a given

[49] We had, earlier, encountered very similar mathematical manipulations while considering the geometric symmetries that left the metric in the same form upon an active coordinate transformation – an actual displacement from one point to another infinitesimally close by. Here, we are doing a passive coordinate transformation, where x and x' describe the same point in spacetime, but using infinitesimally different coordinate systems.

tensor in the x^μ coordinate basis) and $T^{\mu'_1\dots\mu'_N}{}_{\nu'_1\dots\nu'_M}(x')$ (the components of the same tensor but in the x'^μ coordinate basis) are in turn related via

$$T^{\mu'_1\dots\mu'_N}{}_{\nu'_1\dots\nu'_M}(x') = T^{\mu_1\dots\mu_N}{}_{\nu_1\dots\nu_M}(x \to x') + (\pounds_\xi T)^{\mu_1\dots\mu_N}{}_{\nu_1\dots\nu_M}(x \to x'); \tag{2.6.48}$$

where the Lie derivative of the tensor reads

$$\begin{aligned}(\pounds_\xi T)^{\mu_1\dots\mu_N}{}_{\nu_1\dots\nu_M} &= \xi^\sigma \partial_\sigma T^{\mu_1\dots\mu_N}{}_{\nu_1\dots\nu_M} \\ &- T^{\sigma\mu_2\dots\mu_N}{}_{\nu_1\dots\nu_M}\partial_\sigma \xi^{\mu_1} - \dots - T^{\mu_1\dots\mu_{N-1}\sigma}{}_{\nu_1\dots\nu_M}\partial_\sigma\xi^{\mu_N} \\ &+ T^{\mu_1\dots\mu_N}{}_{\sigma\nu_2\dots\nu_M}\partial_{\nu_1}\xi^\sigma + \dots + T^{\mu_1\dots\mu_N}{}_{\nu_1\dots\nu_{M-1}\sigma}\partial_{\nu_M}\xi^\sigma. \end{aligned} \tag{2.6.49}$$

The $x \to x'$ on the right hand side of eq. (2.6.48) means, the tensor $T^{\mu_1\dots\mu_N}{}_{\nu_1\dots\nu_M}$ and its Lie derivative are to be computed in the x^μ-coodinate basis – but x^μ is to be replaced with x'^μ afterwards.

Explain why the partial derivatives on the right hand side of eq. (2.6.49) may be replaced with covariant ones, namely

$$\begin{aligned}(\pounds_\xi T)^{\mu_1\dots\mu_N}{}_{\nu_1\dots\nu_M} &= \xi^\sigma \nabla_\sigma T^{\mu_1\dots\mu_N}{}_{\nu_1\dots\nu_M} \\ &- T^{\sigma\mu_2\dots\mu_N}{}_{\nu_1\dots\nu_M}\nabla_\sigma \xi^{\mu_1} - \dots - T^{\mu_1\dots\mu_{N-1}\sigma}{}_{\nu_1\dots\nu_M}\nabla_\sigma\xi^{\mu_N} \\ &+ T^{\mu_1\dots\mu_N}{}_{\sigma\nu_2\dots\nu_M}\nabla_{\nu_1}\xi^\sigma + \dots + T^{\mu_1\dots\mu_N}{}_{\nu_1\dots\nu_{M-1}\sigma}\nabla_{\nu_M}\xi^\sigma. \end{aligned} \tag{2.6.50}$$

(Hint: First explain why $\partial_\alpha \xi^\beta = \nabla_\alpha \xi^\beta - \Gamma^\beta{}_{\alpha\sigma}\xi^\sigma$.) That the Lie derivative of a tensor can be expressed in terms of covariant derivatives indicates the former is a tensor.

We defined the Lie derivative of the metric $\bar{g}_{\mu\nu}$ with respect to ξ^α in eq. (2.6.47). Is it consistent with equations (2.6.49) and (2.6.50)? □

Lie Derivative of Vector Note that the Lie derivative of some vector field U^μ with respect to ξ^μ is, according to eq. (2.6.50),

$$\pounds_\xi U^\mu = \xi^\sigma \nabla_\sigma U^\mu - U^\sigma \nabla_\sigma \xi^\mu \tag{2.6.51}$$

$$= \xi^\sigma \partial_\sigma U^\mu - U^\sigma \partial_\sigma \xi^\mu = [\xi, U]^\mu. \tag{2.6.52}$$

We have already encountered the Lie bracket/commutator of vector fields, in eq. (**??**). There, we learned that $[\xi, U] = 0$ iff ξ and U may be integrated to form a 2D coordinate system (at least locally). On the other hand, we may view the Lie derivative with respect to ξ as an active coordinate transformation induced by the displacement $x \to x + \xi$. This in fact provides insight into the above mentioned theorem: if $\pounds_\xi U^\mu = 0$ that means U remains unaltered upon a coordinate transformation induced along the direction of ξ; that in turn indicates, it is possible to move along the integral curve of ξ, bringing us from one integral curve of U to the next – while consistently maintaining the same coordinate value along the latter. Similarly, since $[\xi, U] = -[U, \xi] = -\pounds_U \xi = 0$, the vanishing of the Lie bracket also informs us the coordinate value along the integral curve of ξ may be consistently held fixed while moving along the integral curve of U, since the former is invariant under the flow along U. Altogether, this is what makes a set good 2D coordinates; we may vary one while keeping the other fixed, and vice versa.

Problem 2.56. Gauge transformations of a tensor Consider perturbing a spacetime tensor

$$T^{\mu_1\dots\mu_N}{}_{\nu_1\dots\nu_M} \equiv \overline{T}^{\mu_1\dots\mu_N}{}_{\nu_1\dots\nu_M} + \delta T^{\mu_1\dots\mu_N}{}_{\nu_1\dots\nu_M}, \tag{2.6.53}$$

where $\delta T^{\mu_1\dots\mu_N}{}_{\nu_1\dots\nu_M}$ is small in the same sense that ξ^α and $h_{\mu\nu}$ are small. Perform the infinitesimal coordinate transformation in eq. (2.6.39) on the tensor in eq. (2.6.53) and attribute all the transformations to the $\delta T^{\mu_1\dots\mu_N}{}_{\nu_1\dots\nu_M}$. Write down the ensuing gauge transformation, in direct analogy to eq. (2.6.46). Then justify the statement:

> "If the background tensor is zero, the perturbed tensor is gauge-invariant at first order in infinitesimal coordinate transformations."

Hint: You may work this out from scratch, or you may employ the results from Problem (2.55).

□

2.6.1 Perturbed Flat Spacetimes

In this subsection we shall study perturbations about flat spacetimes

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1. \tag{2.6.54}$$

In 4D, this is the context where gravitational waves are usually studied.

Under a Poincaré transformation in eq. (2.1.7), $x^\mu = \Lambda^\mu{}_\nu x'^\nu + a^\mu$, where $\Lambda^\mu{}_\nu$ satisfies (2.1.5), observe that the metric transforms as

$$g_{\alpha'\beta'}(x') = g_{\mu\nu}(x = \Lambda x' + a)\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \tag{2.6.55}$$

$$= \left(\eta_{\mu\nu} + h_{\mu\nu}(x = \Lambda x' + a)\right)\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \equiv \eta_{\alpha\beta} + h_{\alpha'\beta'}(x'). \tag{2.6.56}$$

Hence, as far as Poincaré transformations are concerned, we may attribute all the transformations to those of the perturbations. In other words, $h_{\mu\nu}$ is a tensor under Poincaré transformations:

$$h_{\alpha'\beta'}(x') = h_{\mu\nu}(x(x'))\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta, \tag{2.6.57}$$

$$x^\mu = \Lambda^\mu{}_\nu x'^\nu + a^\mu. \tag{2.6.58}$$

Since the Riemann tensor is zero when $h_{\mu\nu} = 0$, that means the linearized counterpart $\delta_1 R_{\mu\nu\alpha\beta}$ must be gauge-invariant. More specifically, what we have shown thus far is, under the infinitesimal coordinate transformation

$$x^\mu = x'^\mu + \xi^\mu(x'), \tag{2.6.59}$$

the linearized Riemann tensor written in the x versus x' systems are related as

$$\delta_1 R_{\mu\nu\alpha\beta}(x) = \delta_1 R_{\mu'\nu'\alpha'\beta'}(x') + \mathcal{O}(h^2, \xi \cdot h, \xi^2). \tag{2.6.60}$$

Here, the components $\delta_1 R_{\mu\nu\alpha\beta}$ are written in the x coordinate basis whereas $\delta_1 R_{\mu'\nu'\alpha'\beta'}$ are in the x' basis. But, since x and x' differ by an infinitesimal quantity ξ, we may in fact replace $x' \to x$ on the right hand side:

$$\delta_1 R_{\mu\nu\alpha\beta}(x) = \delta_1 R_{\mu'\nu'\alpha'\beta'}(x' \to x) + \mathcal{O}(h^2, \xi \cdot h, \xi^2). \tag{2.6.61}$$

To solve for the $h_{\mu\nu}$ in eq. (2.6.54), one typically has to choose a specific coordinate system. However, eq. (2.6.61) tells us, the tidal forces encoded within the linearized Riemann tensor yield the same expression *for all infinitesimally nearby coordinate systems.*

Two Common Gauges Two commonly used gauges are the synchronous and de Donder gauges. The former refers to the choice of coordinate system such that all perturbations are spatial:

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + h^{(s)}_{ij}\mathrm{d}x^i \mathrm{d}x^j \qquad \text{(Synchronous gauge).} \tag{2.6.62}$$

The latter is defined by the Lorentz-covariant constraint

$$\partial^\mu h_{\mu\nu} = \frac{1}{2}\partial_\nu h, \qquad h \equiv \eta^{\alpha\beta}h_{\alpha\beta}, \qquad \text{(de Donder gauge).} \tag{2.6.63}$$

The de Donder gauge is particularly useful for obtaining explicit perturbative solutions to Einstein's equations. Whereas, the synchronous gauge is useful for describing proper distances between co-moving free-falling test masses.

One may prove that both gauges always exist. According to eq. (2.6.46), the perturbation in a Minkowski background transforms as

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \tag{2.6.64}$$

$$h \to h + 2\partial_\sigma \xi^\sigma. \tag{2.6.65}$$

Hence, if h_{00} were not zero, we may render it so by choosing $\xi_0 = -(1/2)\int^t h_{00}\mathrm{d}t$; since

$$h_{00} \to h_{00} + 2\partial_0 \xi_0 \tag{2.6.66}$$

$$= h_{00} + 2\partial_0 \frac{-1}{2}\int^t h_{00}\mathrm{d}t = 0. \tag{2.6.67}$$

Moreover, if h_{0i} were not zero, an infinitesimal coordinate transformation would yield

$$h_{0i} \to h_{0i} + \partial_i \xi_0 + \partial_0 \xi_i \tag{2.6.68}$$

$$= h_{0i} - \frac{1}{2}\int^t \partial_i h_{00}\mathrm{d}t + \partial_0 \xi_i. \tag{2.6.69}$$

The right hand side is zero if we choose

$$\xi_i = -\int^t \left(h_{0i} - \frac{1}{2}\int^{t'} \partial_i h_{00}\mathrm{d}t'' \right) \mathrm{d}t'. \tag{2.6.70}$$

That is, by choosing ξ_μ appropriately, $h_{0\mu} = h_{\mu 0}$ may always be set to zero.

As for the de Donder gauge condition in eq. (2.6.63), we first re-write it using eq. (2.6.35)

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h. \tag{2.6.71}$$

Namely,

$$\partial^\mu \bar{h}_{\mu\nu} = \partial^\mu \left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \right) = 0. \tag{2.6.72}$$

Utilizing eq. (2.6.64), we may deduce the gauge-transformation of $\bar{h}_{\mu\nu}$ is

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial \cdot \xi, \qquad \partial \cdot \xi \equiv \partial^\sigma \xi_\sigma. \tag{2.6.73}$$

Now, if eq. (2.6.72) were not obeyed, a gauge transformation would produce

$$\partial^\mu \bar{h}_{\mu\nu} \to \partial^\mu \bar{h}_{\mu\nu} + \partial^\mu \left(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu\right) - \eta_{\mu\nu} \partial^\mu \partial \cdot \xi \tag{2.6.74}$$

$$= \partial^\mu \bar{h}_{\mu\nu} + \partial^2 \xi_\nu. \tag{2.6.75}$$

Therefore, by choosing ξ_ν to be the solution to $\partial^2 \xi_\nu = -\partial^\mu \bar{h}_{\mu\nu}$, we may always switch over to the de Donder gauge of eq. (2.6.72). We also note, suppose $\bar{h}_{\mu\nu}$ already obeys the de Donder gauge condition; then notice the transformed $\bar{h}_{\mu\nu}$ actually remains within the de Donder gauge whenever $\partial^2 \xi_\nu = 0$.

Problem 2.57. Are the synchronous and de Donder gauges "infinitesimally nearby" coordinate systems? □

Problem 2.58. Co-moving geodesics in synchronous gauge Prove that

$$Z^\mu(t) = (t, \vec{Z}_0), \tag{2.6.76}$$

where $\vec{Z}_0$ is time-independent, satisfies the geodesic equation in the spacetime

$$g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}t^2 + g_{ij}(t, \vec{x}) \mathrm{d}x^i \mathrm{d}x^j. \tag{2.6.77}$$

This result translates to the following interpretation: each $\vec{x}$ in eq. (2.6.77) may be viewed as the location of a test mass free-falling in the given spacetime. This co-moving test mass remains still, for all time t, in such a synchronous gauge system. Of course, eq. (2.6.62) is a special case of eq. (2.6.77). □

Linearized Synge's World Function In the weak field metric of eq. (2.6.54), according to eq. (2.4.52), half the square of the geodesic distance between x and x' is

$$\bar{\sigma}(x, x') = \frac{1}{2} \int_0^1 \mathrm{d}\lambda \left(\eta_{\mu\nu} + h_{\mu\nu}(Z)\right) \frac{\mathrm{d}Z^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}Z^\nu}{\mathrm{d}\lambda}; \tag{2.6.78}$$

where the trajectories obey geodesic equation (2.4.54)

$$\frac{\mathrm{d}^2 Z^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu{}_{\alpha\beta} \frac{\mathrm{d}Z^\alpha}{\mathrm{d}\lambda} \frac{\mathrm{d}Z^\beta}{\mathrm{d}\lambda} = 0 \tag{2.6.79}$$

subject to the boundary conditions

$$Z^\mu(\lambda = 0) = x'^\mu \qquad \text{and} \qquad Z^\mu(\lambda = 1) = x^\mu. \tag{2.6.80}$$

If the perturbations were not present, $h_{\mu\nu} = 0$, the geodesic equation is

$$\frac{\mathrm{d}^2 \bar{Z}^\mu}{\mathrm{d}\lambda^2} = 0; \tag{2.6.81}$$

whose solution, in turn, is

$$\bar{Z}^\mu(\lambda) = x'^\mu + \lambda(x-x')^\mu, \tag{2.6.82}$$
$$\dot{\bar{Z}}^\mu(\lambda) = (x-x')^\mu. \tag{2.6.83}$$

When the perturbations are non-trivial, $h_{\mu\nu} \neq 0$, the full solution $Z^\mu = \bar{Z}^\mu + \delta Z^\mu$ should deviate from the zeroth order solution $\bar{Z}^\mu$ at linear order in the perturbations: $\delta Z^\mu \sim \mathcal{O}(h_{\mu\nu})$. One may see this from eq. (2.4.56). Hence, if we insert $Z^\mu = \bar{Z}^\mu + \delta Z^\mu$ into Synge's world function in eq. (2.6.78),

$$\sigma(x,x') = \frac{1}{2}\int_0^1 \mathrm{d}\lambda \left(\eta_{\mu\nu} + h_{\mu\nu}(\bar{Z})\right)(x-x')^\mu(x-x')^\nu \\ - \int_0^1 \delta Z^\mu(\lambda)\left(\eta_{\mu\nu} + h_{\mu\nu}(\bar{Z})\right)\frac{D^2\bar{Z}^\nu}{\mathrm{d}\lambda^2}\mathrm{d}\lambda + \mathcal{O}\left((\delta Z)^2\right); \tag{2.6.84}$$

because the zeroth order geodesic equation is satisfied, namely $\mathrm{d}^2\bar{Z}/\mathrm{d}\lambda^2 = 0$, $D^2\bar{Z}^\mu/\mathrm{d}\lambda^2 = \Gamma^\mu{}_{\alpha\beta}\dot{\bar{Z}}^\alpha\dot{\bar{Z}}^\beta \sim \mathcal{O}(h_{\mu\nu})$ and therefore the second line scales as $\mathcal{O}(h_{\mu\nu}^2)$ and higher. At linear order in perturbation theory, half the square of the geodesic distance between $Z(\lambda=0) = x'$ and $Z(\lambda=1) = x$ is therefore Synge's world function evaluated on the zeroth order geodesic solution – namely, the straight line in eq. (2.6.82).[50]

$$\sigma(x,x') = \frac{1}{2}(x-x')^2 + \frac{1}{2}(x-x')^\mu(x-x')^\nu\int_0^1 h_{\mu\nu}\left(\bar{Z}(\lambda)\right)\mathrm{d}\lambda + \mathcal{O}(h^2) \tag{2.6.85}$$

Proper Distance Between Free-Falling Masses: Synchronous Gauge Consider a pair of free-falling test masses at $(t,\vec{y})$ and $(t',\vec{y}')$. Within the synchronous gauge of eq. (2.6.62), where $h_{\mu 0} = h_{0\mu} = 0$, the square of their geodesic spatial separation at a fixed time $t = t'$ is gotten from eq. (2.6.85) through

$$\ell^2 = -2\sigma(t=t';\vec{y},\vec{y}') \tag{2.6.86}$$
$$= (\vec{y}-\vec{y}')^2 - (y-y')^i(y-y')^j\int_0^1 h_{ij}^{(s)}\left(t,\vec{y}'+\lambda(\vec{y}-\vec{y}')\right)\mathrm{d}\lambda + \mathcal{O}(h^2) \tag{2.6.87}$$

Taking the square root on both sides, and using the Taylor expansion result $(1+z)^{1/2} = 1 + z/2 + \mathcal{O}(z^2)$, we surmise that the synchronous gauge form of the metric in eq. (2.6.62) indeed allows us to readily calculate the proper spatial geodesic distance between pairs of free-falling test masses.

$$\ell(t;\vec{y}\leftrightarrow\vec{y}') = |\vec{y}-\vec{y}'|\left(1 - \frac{1}{2}\widehat{R}^i\widehat{R}^j\int_0^1 h_{ij}^{(s)}\left(t,\bar{Z}(\lambda)\right)\mathrm{d}\lambda + \mathcal{O}(h^2)\right), \tag{2.6.88}$$
$$\widehat{R} \equiv \frac{\vec{y}-\vec{y}'}{|\vec{y}-\vec{y}'|}. \tag{2.6.89}$$

(Remember $\bar{Z}$ in eq. (2.6.82).)

[50]This sort of "first-order-variation-vanishes" argument occurs frequently in field theory as well.

Gravitational Wave Polarization & Oscillation Patterns We may re-phrase eq. (2.6.88) as a fractional distortion of space $\delta\ell/\delta_0$ away from the flat space value of $\ell_0 \equiv |\vec{y} - \vec{y}'|$, due to the presence of the perturbation $h_{ij}^{(s)}$,

$$\left(\frac{\delta\ell}{\ell_0}\right)(t; \vec{y} \leftrightarrow \vec{y}') = -\frac{1}{2}\widehat{R}^i\widehat{R}^j \int_0^1 h_{ij}^{(s)}\left(t, \bar{Z}(\lambda)\right) \mathrm{d}\lambda + \mathcal{O}(h^2). \tag{2.6.90}$$

If we define gravitational waves to be simply the finite frequency portion of the tidal signal in eq. (2.6.90), then we see that the fractional distortion of space due to a passing gravitational wave could consist of up to a maximum of $D(D+1)/2$ distinct oscillatory patterns, in a $D+1$ dimensional weakly curved spacetime. In detail, if we decompose

$$h_{ij}^{(s)}\left(t, \bar{Z}(\lambda)\right) = \int_{\mathbb{R}} \widetilde{h}_{ij}^{(s)}\left(\omega, \bar{Z}(\lambda)\right) e^{-i\omega t}\frac{\mathrm{d}\omega}{2\pi}, \tag{2.6.91}$$

then eq. (2.6.90) reads

$$\widetilde{\left(\frac{\delta\ell}{\ell_0}\right)}(\omega; \vec{y} \leftrightarrow \vec{y}') = -\frac{1}{2}\widehat{R}^i\widehat{R}^j \int_0^1 \widetilde{h}_{ij}^{(s)}\Big(\omega, \vec{y} + \lambda(\vec{y}' - \vec{y})\Big)\mathrm{d}\lambda + \mathcal{O}(h^2). \tag{2.6.92}$$

Now, a direct calculation will reveal

$$\delta_1 R_{0i0j}(t, \vec{x}) = -\frac{1}{2}\partial_0^2 h_{ij}^{(s)}(t, \vec{x}), \qquad \text{(Synchronous gauge)}. \tag{2.6.93}$$

To translate this statement to frequency space, we replace $\partial_0 = \partial_t \to -i\omega$,

$$\delta_1 \widetilde{R}_{0i0j}(\omega, \vec{x}) = \frac{\omega^2}{2}\widetilde{h}_{ij}^{(s)}(\omega, \vec{x}), \qquad \text{(Synchronous gauge)}. \tag{2.6.94}$$

Gravitational waves are associated with time dependent radiative processes, capable of performing dissipative work through their oscillatory tidal forces. To this end, eq. (2.6.94) teaches us it is the finite frequency modes – i.e., the $\omega \neq 0$ portion – of the linearized Riemann tensor that is to be associated with such gravitational radiation. By inserting eq. (2.6.94) into eq. (2.6.92), we see that the finite frequency gravitational-wave-driven fractional distortion of space – namely,

$$\widetilde{\left(\frac{\delta\ell}{\ell_0}\right)}(\omega \neq 0; \vec{y} \leftrightarrow \vec{y}') = \frac{\widehat{R}^i\widehat{R}^j}{\omega^2} \int_0^1 \delta_1 \widetilde{R}_{0i0j}\Big(\omega, \vec{y} + \lambda(\vec{y}' - \vec{y})\Big)\mathrm{d}\lambda + \mathcal{O}(h^2) \tag{2.6.95}$$

– is not only gauge-invariant (since the linearized Riemann components are); it has $(D^2-D)/2 + D = D(D+1)/2$ algebraically independent components, since $\delta_1\widetilde{R}_{0i0j}$ is a symmetric rank-2 spatial tensor in the ij indices.

Problem 2.59. Verify eq. (2.6.93). □

Problem 2.60. 4D Gravitational Wave Polarizations In 3+1 dimensional spacetime, choose the unit vector along the 3$-$axis $\widehat{e}_3$ to be the direction of propagation of the finite frequency $\widetilde{h}_{ij}^{(s)}$ in eq. (2.6.92). Then proceed to build upon Problem (**??**) to decompose the fractional distortion of space in eq. (2.6.92) into its irreducible constituents – i.e., the spin−0, spin−1 and spin−2 finite-frequency waves.

In 4D linearized de Donder gauge General Relativity, only null traveling waves are admitted in vacuum. As we will see in the next problem, this implies only the helicity−2 waves are predicted to exist. However, it is conceivable that alternate theories of gravity could allow for the other irreducible modes to carry gravitational radiation. □

Problem 2.61. Synchronous-de Donder Gauge & Null Traveling 'TT' Waves In this problem we shall see how the gauge-invariant linearized Riemann tensor may be used to relate the synchronous gauge metric perturbation to its de Donder counterpart – at least for source-free traveling waves.

Let us begin by performing a Fourier transform in spacetime,

$$h^{(s)}_{ij}(t,\vec{x}) = \int_{\mathbb{R}} \frac{\mathrm{d}\omega}{2\pi} \int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D} \widetilde{h}^{(s)}_{ij}(\omega,\vec{k}) e^{-i\omega t} e^{+i\vec{k}\cdot\vec{x}}; \tag{2.6.96}$$

so that $\partial_\mu \leftrightarrow -i(\omega, k_i)_\mu$. The associated synchronous gauge Riemann tensor components then read

$$\delta_1 \widetilde{R}_{0i0j}(\omega,\vec{k}) = +\frac{\omega^2}{2} \widetilde{h}^{(s)}_{ij}(\omega,\vec{k}), \qquad \text{(Synchronous gauge)}. \tag{2.6.97}$$

Up to this point, we have not assumed a dispersion relation between ω and $\vec{k}$. Suppose we impose the null condition

$$\omega^2 = \vec{k}^2 \tag{2.6.98}$$

on both the synchronous and de Donder gauge perturbations, so they are both superpositions of traveling waves propagating at unit speed –

$$h^{(s)}_{ij}(t,\vec{x}) = \int_{\mathbb{R}^D} \frac{1}{2} \left\{ \widetilde{h}^{(s)}_{ij}(k) e^{-i|\vec{k}|t} + \widetilde{h}^{(s)}_{ij}(k)^* e^{+i|\vec{k}|t} \right\} e^{i\vec{k}\cdot\vec{x}} \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D}, \qquad k^\mu \equiv (|\vec{k}|, \vec{k}) \tag{2.6.99}$$

– now, verify directly that the corresponding Riemann components are

$$\delta_1 \widetilde{R}_{0i0j}(\omega,\vec{k}) = \frac{\omega^2}{2} \left(\widetilde{h}_{ij} + \widehat{k}_{\{i} \widetilde{h}_{j\}l} \widehat{k}^l + \widehat{k}_i \widehat{k}_j \widetilde{h}_{mn} \widehat{k}^m \widehat{k}^n \right), \qquad \text{(de Donder)}; \tag{2.6.100}$$

$$\widehat{k}^i \equiv k^i/|\vec{k}|, \qquad \omega^2 = \vec{k}^2. \tag{2.6.101}$$

Next, verify $\delta_1 \widetilde{R}_{0i0j}$ in eq. (2.6.100) is transverse and traceless:

$$\delta^{ij} \delta_1 \widetilde{R}_{0i0j} = 0 = \widehat{k}^i \delta_1 \widetilde{R}_{0i0j}. \tag{2.6.102}$$

Finally, demonstrate that such a traveling-wave $\delta_1 \widetilde{R}_{0i0j}$ in de Donder gauge is simply the transverse-traceless (TT) portion of the metric perturbation itself:

$$\delta_1 \widetilde{R}_{0i0j}(\omega,\vec{k}) = \frac{\omega^2}{2} \widetilde{P}_{ijab} \widetilde{h}_{ab}(\omega,\vec{k}), \tag{2.6.103}$$

where the TT projector is

$$\widetilde{P}_{ijab} = \frac{1}{2}\widetilde{P}_{i\{a}\widetilde{P}_{b\}j} - \frac{1}{D-1}\widetilde{P}_{ij}\widetilde{P}_{ab}, \tag{2.6.104}$$

$$\widetilde{P}_{ij} = \delta_{ij} - \widehat{k}_i\widehat{k}_j. \tag{2.6.105}$$

It enjoys the following properties:

$$\widetilde{P}_{ijab} = \widetilde{P}_{abij}, \qquad \widetilde{P}_{jiab} = \widetilde{P}_{ijab}, \qquad \delta^{ij}\widetilde{P}_{ijab} = 0 = \widehat{k}^i\widetilde{P}_{ijab}. \tag{2.6.106}$$

Helicity−2 modes Finally, by choosing $\widehat{k} \equiv \widehat{e}_3$, the unit vector along the 3−axis, verify the claim in the previous problem, that the null traveling waves described by these linearized $\delta_1\widetilde{R}_{0i0j}$ are purely helicity−2 modes only.

Hint: Throughout these calculations, you would need to repeatedly employ the de Donder gauge condition (eq. (2.6.63)) in Fourier spacetime: $k^\mu\widetilde{h}_{\mu\nu} = (1/2)k_\nu\widetilde{h}$, with $k^\mu \equiv (\omega, \vec{k})$. □

From our previous discussion, since the linearized Riemann tensor is gauge-invariant, we may immediately equate the $0i0j$ components in the synchronous (eq. (2.6.97)) and de Donder (eq. (2.6.100)) gauges to deduce: for finite frequencies $|\omega| = |\vec{k}| \neq 0$, the synchronous gauge metric perturbation is the TT part of the de Donder gauge one.

$$\widetilde{h}^{(s)}_{ij}[\text{Synchronous}] = \widetilde{P}_{ijab}\widetilde{h}_{ab}[\text{de Donder}] \tag{2.6.107}$$

That this holds only for finite frequencies – the formulas in equations (2.6.97) and (2.6.100) do not contain $\delta(\omega)$ or $\delta'(\omega)$ terms – because $\omega^2\delta(\omega) = 0 = \omega^2\delta'(\omega)$. More specifically, since eq. (2.6.93) involved a second time derivative on $h^{(s)}_{ij}$, by equating it to the (position-spacetime version of) eq. (2.6.100), we may solve the synchronous gauge metric perturbation only up to its initial value and time derivative:

$$\begin{aligned} h^{(s)}_{ij}(t,\vec{x}) = -2\int_{t_0}^{t}\int_{t_0}^{\tau_2} \delta_1 R_{0i0j}(\tau_1,\vec{x})\mathrm{d}\tau_1\mathrm{d}\tau_2 \\ + (t-t_0)\dot{h}^{(s)}_{ij}(t_0,\vec{x}) + h^{(s)}_{ij}(t_0,\vec{x}). \end{aligned} \tag{2.6.108}$$

Note that the initial velocity term $(t-t_0)\dot{h}^{(s)}_{ij}(t_0,\vec{x})$ is proportional to $\delta'(\omega)$ in frequency space; whereas the initial value $h^{(s)}_{ij}(t_0,\vec{x})$ is proportional to $\delta(\omega)$.

Unlike eq. (2.6.107), eq. (2.6.108) does not depend on specializing to traveling waves obeying the null dispersion relation $k^2 \equiv k_\mu k^\mu = 0$.[51] Moreover, eq. (2.6.108) suggests, up to the two initial condiitions, $h^{(s)}_{ij}$ itself is almost gauge-invariant – afterall it measures something geometrical, eq. (2.6.88), the proper distances between free-falling test masses – and we may attempt to further understand this through the following considerations. Since the synchronous gauge perturbation allows us to easily compute proper distances between co-moving test masses, let us ask how much coordinate freedom is available while still remaining with the synchronous gauge itself. For the 00 component to remain 0, we have from eq. (2.6.64)

$$0 = h^{(s)}_{00} \to 2\partial_0\xi_0 = 0. \tag{2.6.109}$$

[51]More specifically, eq. (2.6.107) holds whenever the linearized vacuum Einstein's equations hold; whereas eq. (2.6.108) is true regardless of the underlying dynamics of the metric perturbations.

That is, ξ_0 needs to be time-independent. For the $0i$ component to remain zero,

$$0 = h^{(s)}_{0i} \to \partial_0 \xi_i + \partial_i \xi_0 = 0. \tag{2.6.110}$$

This allows us to assert

$$\xi_i(t, \vec{x}) = -(t - t_0)\partial_i \xi_0(\vec{x}) + \xi_i(t_0, \vec{x}). \tag{2.6.111}$$

Under such a coordinate transformation, $x \to x + \xi$,

$$h^{(s)}_{ij} \to h^{(s)}_{ij} + \partial_i \xi_j + \partial_j \xi_i \tag{2.6.112}$$

$$= h^{(s)}_{ij}(t, \vec{x}) - 2(t - t_0)\partial_i \partial_j \xi_0(\vec{x}) + \partial_{\{i}\xi_{j\}}(t_0, \vec{x}). \tag{2.6.113}$$

Comparison with eq. (2.6.108) tells us $\partial_i \partial_j \xi_0$ may be identified with the freedom to redefine the initial velocity of $h^{(s)}_{ij}$; and $\partial_{\{i}\xi_{j\}}(t_0, \vec{x})$ its initial value.

2.7 Special Topic: Conformal/Weyl Transformations

In this section, we collect for the reader's reference, the conformal transformation properties of various geometric objects. We shall define a conformal transformation on a metric to be a change of the geometry by an overall spacetime dependent scale. That is,

$$g_{\mu\nu}(x) \equiv \Omega^2(x) \bar{g}_{\mu\nu}(x). \tag{2.7.1}$$

The inverse metric is

$$g^{\mu\nu}(x) = \Omega(x)^{-2} \bar{g}^{\mu\nu}(x), \qquad \bar{g}^{\mu\sigma}\bar{g}_{\sigma\nu} \equiv \delta^\mu_\nu. \tag{2.7.2}$$

We shall now enumerate how the geometric objects/operations built out of $g_{\mu\nu}$ is related to that built out of $\bar{g}_{\mu\nu}$. In what follows, all indices on barred tensors are raised and lowered with $\bar{g}^{\mu\nu}$ and $\bar{g}_{\mu\nu}$ while all indices on un-barred tensors are raised/lowered with $g^{\mu\nu}$ and $g_{\mu\nu}$; the covariant derivative ∇ is with respect to $g_{\mu\nu}$ while the $\overline{\nabla}$ is with respect to $\bar{g}_{\mu\nu}$.

Metric Determinant Since

$$\det g_{\mu\nu} = \det\left(\Omega^2 \bar{g}_{\mu\nu}\right) = \Omega^{2d} \det \bar{g}_{\mu\nu}, \tag{2.7.3}$$

we must also have

$$|g|^{1/2} = \Omega^d |\bar{g}|^{1/2}. \tag{2.7.4}$$

Scalar Gradients The scalar gradient with a lower index is just a partial derivative. Therefore

$$\nabla_\mu \varphi = \overline{\nabla}_\mu \varphi = \partial_\mu \varphi. \tag{2.7.5}$$

while $\nabla^\mu \varphi = g^{\mu\nu}\nabla_\nu \varphi = \Omega^{-2}\bar{g}^{\mu\nu}\overline{\nabla}_\nu \varphi$, so

$$\nabla^\mu \varphi = \Omega^{-2}\overline{\nabla}^\mu \varphi. \tag{2.7.6}$$

Scalar Wave Operator The wave operator $\Box$ in the geometry $g_{\mu\nu}$ is defined as

$$\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu = \nabla_\mu\nabla^\mu. \tag{2.7.7}$$

By a direct calculation, the wave operator $\Box$ with respect to $g_{\mu\nu}$ acting on a scalar ψ is

$$\Box\varphi = \frac{1}{\Omega^2}\left(\frac{d-2}{\Omega}\overline{\nabla}_\mu\Omega\cdot\overline{\nabla}^\mu\varphi + \overline{\Box}\varphi\right), \tag{2.7.8}$$

where $\overline{\Box}$ is the wave operator with respect to $\bar{g}_{\mu\nu}$. We also have

$$\begin{aligned}\Box\left(\Omega^s\psi\right) = \frac{1}{\Omega^2}\Big\{ &\left(s\Omega^{s-1}\overline{\Box}\Omega + s\left(d+s-3\right)\Omega^{s-2}\overline{\nabla}_\mu\Omega\overline{\nabla}^\mu\Omega\right)\psi \\ &+ \left(2s+d-2\right)\Omega^{s-1}\overline{\nabla}_\mu\Omega\overline{\nabla}^\mu\psi + \Omega^s\overline{\Box}\psi\Big\}.\end{aligned} \tag{2.7.9}$$

Christoffel Symbols A direct calculation shows:

$$\Gamma^\mu_{\alpha\beta}[g] = \overline{\Gamma}^\mu_{\alpha\beta}[\bar{g}] + \left(\partial_{\{\alpha}\ln\Omega\right)\delta^\mu_{\beta\}} - \bar{g}_{\alpha\beta}\bar{g}^{\mu\nu}\left(\partial_\nu\ln\Omega\right) \tag{2.7.10}$$

$$= \overline{\Gamma}^\mu_{\alpha\beta}[\bar{g}] + \left(\overline{\nabla}_{\{\alpha}\ln\Omega\right)\delta^\mu_{\beta\}} - \bar{g}_{\alpha\beta}\overline{\nabla}^\mu\ln\Omega. \tag{2.7.11}$$

Riemann Tensor By viewing the difference between $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ as a 'perturbation',

$$g_{\mu\nu} - \bar{g}_{\mu\nu} = \left(\Omega^2 - 1\right)\bar{g}_{\mu\nu} \equiv h_{\mu\nu}, \tag{2.7.12}$$

we may employ the results in §(2.6). In particular, eq. (2.6.24) may be used to infer that the Riemann tensor is

$$\begin{aligned}R^\alpha{}_{\beta\mu\nu}[g] = \bar{R}^\alpha{}_{\beta\mu\nu}[\bar{g}] &+ \overline{\nabla}_\beta\overline{\nabla}_{[\mu}\ln\Omega\delta^\alpha_{\nu]} - \bar{g}_{\beta[\nu}\overline{\nabla}_{\mu]}\overline{\nabla}^\alpha\ln\Omega \\ &+ \delta^\alpha_{[\mu}\overline{\nabla}_{\nu]}\ln\Omega\overline{\nabla}_\beta\ln\Omega + \overline{\nabla}^\alpha\ln\Omega\overline{\nabla}_{[\mu}\ln\Omega\bar{g}_{\nu]\beta} + \left(\overline{\nabla}\ln\Omega\right)^2\bar{g}_{\beta[\mu}\delta^\alpha_{\nu]}.\end{aligned} \tag{2.7.13}$$

Ricci Tensor In turn, the Ricci tensor is

$$R_{\beta\nu}[g] = \bar{R}_{\beta\nu}[\bar{g}] + (2-d)\overline{\nabla}_\beta\overline{\nabla}_\nu\ln\Omega - \bar{g}_{\beta\nu}\overline{\Box}\ln\Omega \tag{2.7.14}$$

$$+ (d-2)\left(\overline{\nabla}_\beta\ln\Omega\overline{\nabla}_\nu\ln\Omega - \bar{g}_{\beta\nu}\left(\overline{\nabla}\ln\Omega\right)^2\right). \tag{2.7.15}$$

Ricci Scalar Contracting the Ricci tensor with $g^{\beta\nu} = \Omega^{-2}\bar{g}^{\beta\nu}$, we conclude

$$\mathcal{R}[g] = \Omega^{-2}\left(\overline{\mathcal{R}}[\bar{g}] + 2(1-d)\overline{\Box}\ln\Omega + (d-2)(1-d)\left(\overline{\nabla}\ln\Omega\right)^2\right). \tag{2.7.16}$$

Weyl Tensor The Weyl tensor, for spacetime dimensions greater than two ($d > 2$), is defined to be the completely trace-free portion of the Riemann tensor:

$$C_{\mu\nu\alpha\beta} \equiv R_{\mu\nu\alpha\beta} - \frac{1}{d-2}\left(R_{\alpha[\mu}g_{\nu]\beta} - R_{\beta[\mu}g_{\nu]\alpha}\right) + \frac{g_{\mu[\alpha}g_{\beta]\nu}}{(d-2)(d-1)}\mathcal{R}[g]. \tag{2.7.17}$$

By a direct calculation, one may verify $C_{\mu\nu\alpha\beta}$ has the same index-symmetries as $R_{\mu\nu\alpha\beta}$ and is indeed completely traceless: $g^{\mu\alpha}C_{\mu\nu\alpha\beta} = 0$. Using equations (2.7.1), (2.7.13), (2.7.14), and

(2.7.16), one may then deduce the Weyl tensor with one upper index is *invariant* under conformal transformations:

$$C^{\mu}{}_{\nu\alpha\beta}[g] = C^{\mu}{}_{\nu\alpha\beta}[\bar{g}]. \tag{2.7.18}$$

If we lower the index μ on both sides,

$$C_{\mu\nu\alpha\beta}[g] = \Omega^2 C_{\mu\nu\alpha\beta}[\bar{g}]. \tag{2.7.19}$$

Let us also record that:

> In spacetime dimensions greater than 3, a metric $g_{\mu\nu}$ is locally conformally flat – i.e., it can be put into the form $g_{\mu\nu} = \Omega^2\eta_{\mu\nu}$ – iff its Weyl tensor is zero.[52]

Einstein Tensor From equations (2.7.1), (2.7.14) and (2.7.16), we may also compute the transformation of the Einstein tensor $G_{\beta\nu} \equiv R_{\beta\nu} - (g_{\beta\nu}/2)\mathcal{R}$.

$$\begin{aligned} G_{\beta\nu}[g] = \overline{G}_{\beta\nu}[\bar{g}] &+ (2-d)\left(\overline{\nabla}_\beta\overline{\nabla}_\nu \ln\Omega - \bar{g}_{\beta\nu}\overline{\Box}\ln\Omega\right) \\ &+ (d-2)\left(\overline{\nabla}_\beta \ln\Omega\overline{\nabla}_\nu \ln\Omega - \bar{g}_{\beta\nu}\frac{3-d}{2}\left(\overline{\nabla}\ln\Omega\right)^2\right) \end{aligned} \tag{2.7.20}$$

Notice the Einstein tensor is invariant under constant conformal transformations: $G_{\beta\nu}[g] = \overline{G}_{\beta\nu}[\bar{g}]$ whenever $\partial_\mu\Omega = 0$.

Problem 2.62. 2D Einstein is Zero Explain why the Einstein tensor is zero in 2D. This implies the 2D Ricci tensor is proportional to the Ricci scalar:

$$R_{\alpha\beta} = \frac{1}{2}g_{\alpha\beta}\mathcal{R}. \tag{2.7.21}$$

Hint: Refer to Problem (1.31). □

Scalar Field Action In d dimensional spacetime, the following action involving the scalar φ and Ricci scalar $\mathcal{R}[g]$,

$$S[\varphi] \equiv \int \mathrm{d}^d x\sqrt{|g|}\frac{1}{2}\left(g^{\alpha\beta}\nabla_\alpha\varphi\nabla_\beta\varphi + \frac{d-2}{4(d-1)}\mathcal{R}\varphi^2\right), \tag{2.7.22}$$

is invariant – up to surface terms – under the simultaneous replacements

$$g_{\alpha\beta} \to \Omega^2 g_{\alpha\beta}, \qquad g^{\alpha\beta} \to \Omega^{-2}g^{\alpha\beta}, \qquad \sqrt{|g|} \to \Omega^d\sqrt{|g|}, \tag{2.7.23}$$

$$\varphi \to \Omega^{1-\frac{d}{2}}\varphi. \tag{2.7.24}$$

The jargon here is that φ transforms covariantly under conformal transformations, with weight $s = 1 - (d/2)$. We see in two dimensions, $d = 2$, a minimally coupled massless scalar theory automatically enjoys conformal/Weyl symmetry.

To reiterate: on the right-hand-sides of these expressions for the Riemann tensor, Ricci tensor and scalar, all indices are raised and lowered with $\bar{g}$; for example, $(\overline{\nabla}A)^2 \equiv \bar{g}^{\sigma\tau}\overline{\nabla}_\sigma A\overline{\nabla}_\tau A$ and $\overline{\nabla}^\alpha A \equiv \bar{g}^{\alpha\lambda}\overline{\nabla}_\lambda A$. The $R^\alpha{}_{\beta\mu\nu}[g]$ is built out of the metric $g_{\alpha\beta}$ but the $\bar{R}^\alpha{}_{\beta\mu\nu}[\bar{g}]$ is built entirely out of $\bar{g}_{\mu\nu}$, etc.

[52]In $d = 3$ dimensions, a spacetime is locally conformally flat iff its Cotton tensor vanishes.

3 Local Conservation Laws

Non-relativistic You would be rightly shocked if you had stored a sealed tank of water on your rooftop only to find its contents gradually disappearing over time – the total mass of water ought to be a constant. Assuming a flat space geometry, if you had instead connected the tank to two pipes, one that pumps water into the tank and the other pumping water out of it, the rate of change of the total mass of the water

$$M \equiv \int_{\text{tank}} \rho(t, \vec{x}) \mathrm{d}^3\vec{x} \tag{3.0.1}$$

in the tank – where t is time, $\vec{x}$ are Cartesian coordinates, and $\rho(t, \vec{x})$ is the water's mass density – is

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\text{tank}} \rho \mathrm{d}^3\vec{x} = -\left(\int_{\text{cross section of 'in' pipe}} + \int_{\text{cross section of 'out' pipe}} \right) \mathrm{d}^2\vec{\Sigma} \cdot (\rho\vec{v}). \tag{3.0.2}$$

Note that $\mathrm{d}^2\vec{\Sigma}$ points *outwards* from the tank, so at the 'in' pipe-tank interface, if the water were indeed following into the pipe, $-\mathrm{d}^2\vec{\Sigma} \cdot (\rho\vec{v}) > 0$ and its contribution to the rate of increase is positive. At the 'out' pipe-tank interface, if the water were indeed following out of the pipe, $-\mathrm{d}^2\vec{\Sigma} \cdot (\rho\vec{v}) < 0$. If we apply Gauss' theorem,

$$\int_{\text{tank}} \dot{\rho} \mathrm{d}^3\vec{x} = -\int_{\text{tank}} \mathrm{d}^3\vec{x} \vec{\nabla} \cdot (\rho\vec{v}). \tag{3.0.3}$$

If we applied the same sort of reasoning to any infinitesimal packet of fluid, with some local mass density ρ, we would find the following local conservation law

$$\dot{\rho} = -\partial_i \left(\rho \cdot v^i \right). \tag{3.0.4}$$

This is a "local" conservation law in the sense that mass cannot simply vanish from one location and re-appear a finite distance away, without first flowing to a neighboring location.

Relativistic We have implicitly assumed a non-relativistic system, where $|\vec{v}| \ll 1$. This is an excellent approximation for most hydrodynamics problems. Strictly speaking, however, relativistic effects – length contraction, in particular – imply that mass density is not a Lorentz scalar. If we define $\rho(t, \vec{x})$ to be the mass density at $(t, \vec{x})$ in a frame instantaneously at rest (aka 'co-moving') with the fluid packet, then the mass density current that is a locally conserved Lorentz vector is given by

$$J^\mu(t, \vec{x}) \equiv \rho(t, \vec{x}) v^\mu(t, \vec{x}). \tag{3.0.5}$$

Along its integral curve v^μ should be viewed as the proper velocity $\mathrm{d}(t, \vec{x})^\mu / \mathrm{d}\tau$ of the fluid packet, where τ is the latter's proper time. Moreover, as long as the velocity v^μ is timelike, which is certainly true for fluids, let us recall it is always possible to find a (local) Lorentz transformation $\Lambda^\mu{}_\nu(t, \vec{x})$ such that

$$(1, \vec{0})^\mu \equiv v'^\mu = \Lambda^\mu{}_\nu(t, \vec{x}) v^\nu(t, \vec{x}). \tag{3.0.6}$$

and the mass density-current is now

$$J'^{\mu} = \rho(t', \vec{x}')v'^{\mu} = \rho(x') \cdot \delta^{\mu}_{0}. \tag{3.0.7}$$

The local conservation law obeyed by this relativistically covariant current J^μ is now (in Cartesian coordinates)

$$\partial_\mu J^\mu = 0; \tag{3.0.8}$$

which in turn is a Lorentz invariant statement. Total mass M in a given global inertial frame at a fixed time t is

$$M \equiv \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} J^0. \tag{3.0.9}$$

To show it is a constant, we take the time derivative, and employ eq. (3.0.8):

$$\dot{M} = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\partial_0 J^0 = -\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\partial_i J^i. \tag{3.0.10}$$

The divergence theorem tells us that this is equal to the flux of J^i at spatial infinity. But there is no J^i at spatial infinity for physically realistic – i.e., isolated – systems.

Problem 3.1. Local Conservation In Curved Spacetimes The equivalence principle tells us: in a local free-falling frame, physics should be nearly identical to that in Minkowski spacetime. In a generic curved spacetime, the local conservation laws we are examining in this section becomes

$$\nabla_\mu J^\mu = \frac{\partial_\mu \left(\sqrt{|g|} J^\mu\right)}{\sqrt{|g|}} = 0. \tag{3.0.11}$$

(Why does the first equality hold?) Argue that this reduces to eq. (3.0.8) in a FNC coordinate system described by equations (2.5.6)-(2.5.8). □

Local Conservation does not imply global conservation In a curved spacetime $g_{\mu\nu}(t,\vec{x})$, if space is still infinite, we could define

$$Q(t) \equiv \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}\sqrt{|g(t,\vec{x})|} J^0(t,\vec{x}). \tag{3.0.12}$$

Then, according to eq. (3.0.11),

$$\dot{Q} = \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}\partial_0 \left\{ \sqrt{|g(t,\vec{x})|} J^0(t,\vec{x}) \right\} \tag{3.0.13}$$

$$= -\int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}\partial_i \left\{ \sqrt{|g(t,\vec{x})|} J^i(t,\vec{x}) \right\} = 0 \tag{3.0.14}$$

as long as $\vec{J}^i$ is isolated in space. An example of such a calculation can be found in the problem below.

However, apart from simple situations (such as the cosmological one below), the interpretation of $Q(t)$ in eq. (3.0.12) is not completely clear. In curved spacetimes, tensor components may not have direct physical meaning; one often needs to – at the very least – re-write them as components in an orthonormal frame (for e.g., $J^{\widehat{\mu}}$). Even then, the total charge gotten by integrating $J^{\widehat{0}}$ over all of space is a coordinate dependent statement; in Minkowski spacetime we can show that $Q = \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x} J^{\widehat{0}}$ is constant *regardless* of the inertial frame $(t, \vec{x})$ chosen (see Problem (5.2) below), but this cannot be expected to hold true in a generic curved spacetime, simply because there is no longer a notion of a global inertial frame. At best, we may go to a free-falling system that is co-moving with the system *locally*, and proceed to apply eq. (3.0.10) but only in a small enough region of spacetime:

$$\dot{M} \overset{\text{FNC}}{=} \int_{\text{small region}} \mathrm{d}^{d-1}\vec{x}\partial_0 J^0 = -\int_{\text{small region}} \mathrm{d}^{d-1}\vec{x}\partial_i J^i = -\int_{\text{boundary of small region}} \mathrm{d}^{d-2}\vec{\Sigma}_i J^i. \tag{3.0.15}$$

We have used here the free-falling Fermi-Normal-Coordinate system, where $g_{\mu\nu} = \eta_{\mu\nu}$ plus corrections that go as Riemann contracted into two displacement vectors; hence, in this limit $\sqrt{g} \to 1$ and $\nabla_\mu J^\mu \to \partial_\mu J^\mu$.

Unless the physical interpretation can be made clear, the local conservation law of eq. (3.0.11) for a current J^μ does not necessarily imply a global conservation law.

Problem 3.2. Electric Charge Conservation in a (Spatially Flat) Expanding Universe Let us consider a d−dimensional universe described by the line element

$$\mathrm{d}s^2 = a(\eta)^2 \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{3.0.16}$$

(The $x^0 = \eta$ in $a(\eta)$ is the time coordinate, not to be confused with the flat metric.) When $d = 4$, this appears to describe our universe at the largest length scales.

Let us now examine an electric current J^μ inhabiting such a universe, where $\nabla_\mu J^\mu = 0$. First verify that the orthonormal frame fields describing a family of co-moving observers is given by

$$\varepsilon^{\widehat{\alpha}}{}_\mu = a(\eta)\delta^\alpha_\mu. \tag{3.0.17}$$

According to this family of observers, they measure a local electric charge density of $J^{\widehat{0}}$ and current flow $J^{\widehat{i}}$. On a constant time η hypersurface, the induced metric can be obtained from eq. (3.0.16) by setting $\mathrm{d}x^0 = 0$ and multiplying throughout by a -1 (so as to get positive distances):

$$H_{ij} = a^2 \delta_{ij}. \tag{3.0.18}$$

Define the total charge on a constant η hypersurface as

$$Q(\eta) \equiv \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}\sqrt{\det H_{ab}} J^{\widehat{0}}. \tag{3.0.19}$$

Explain why this may also be written as

$$Q(\eta) \equiv \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}\sqrt{|g(\eta, \vec{x})|} J^0(\eta, \vec{x}). \tag{3.0.20}$$

Then show that Q is actually independent of $x^0 = \eta$. □

4 Scalar Fields in Minkowski Spacetime

Field theory in Minkowski spacetime indicates we wish to construct partial differential equations obeyed by fields such that they take the same form in all inertial frames – i.e., the PDEs are Lorentz covariant. As a warm-up, we shall in this section study the case of scalar fields.

A scalar field $\varphi(x)$ is an object that transforms, under Poincaré transformations

$$x^\mu = \Lambda^\mu{}_\nu x'^\nu + a^\mu \tag{4.0.1}$$

as simply

$$\varphi(x) = \varphi\left(x^\mu = \Lambda^\mu{}_\nu x'^\nu + a^\mu\right). \tag{4.0.2}$$

To ensure that this is the case, we would like the PDE it obeys to take the same form in the two inertial frames $\{x^\mu\}$ and $\{x'^\mu\}$ related by eq. (4.0.1). The simplest example is the wave equation with some external scalar source $J(x)$. Let's first write it in the x^μ coordinate system.

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\varphi(x) = J(x), \qquad \partial_\mu \equiv \partial/\partial x^\mu. \tag{4.0.3}$$

If putting a prime on the index denotes derivative with respect to x'^μ, namely $\partial_{\mu'} \equiv \partial/\partial x'^\mu$, then by the chain rule,

$$\partial_{\mu'} = \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial}{\partial x^\sigma} = \partial_{\mu'}\left(\Lambda^\sigma{}_\rho x'^\rho + a^\sigma\right)\partial_\sigma \tag{4.0.4}$$

$$= \Lambda^\sigma{}_\mu \partial_\sigma. \tag{4.0.5}$$

Therefore the wave operator indeed takes the same form in both coordinate systems:

$$\eta^{\mu\nu}\partial_{\mu'}\partial_{\nu'} = \eta^{\mu\nu}\Lambda^\sigma{}_\mu\Lambda^\rho{}_\nu\partial_\sigma\partial_\rho \tag{4.0.6}$$

$$= \eta^{\sigma\rho}\partial_\sigma\partial_\rho. \tag{4.0.7}$$

because of Lorentz invariance

$$\eta^{\mu\nu}\Lambda^\sigma{}_\mu\Lambda^\rho{}_\nu = \eta^{\sigma\rho}. \tag{4.0.8}$$

A generalization of the wave equation in eq. (4.0.3) is to add a potential $V(\varphi)$:

$$\partial^2\varphi + V'(\varphi) = J, \tag{4.0.9}$$

where $\partial^2 \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$ and the prime is a derivative with respect to the argument.

Problem 4.1. Yukawa potential in $(3+1)$D Let the potential in eq. (4.0.9) be that of a mass term

$$V(\varphi) = \frac{m^2}{2}\varphi^2. \tag{4.0.10}$$

Consider a static point mass resting at $\vec{x} = 0$ in the $\{x^\mu\}$ inertial frame, namely

$$J(\vec{x}) = J_0\delta^{(3)}(\vec{x}), \qquad J_0 \text{ constant}. \tag{4.0.11}$$

Solve φ. Hint: You may assume the time derivatives in eq. (4.0.9) can be neglected. Then go to Fourier $\vec{k}-$space. You should find

$$\widetilde{\varphi}(\vec{k}) = \frac{J_0}{\vec{k}^2 + m^2}. \tag{4.0.12}$$

You should find a short-range force that, when $m \to 0$, recovers the Coulomb/Newtonian $1/r$ potential.

Next, consider an inertial frame $\{x'^\mu\}$ that is moving relative to the $\{x^\mu\}$ frame at velocity v along the positive x^3 axis. What is $\varphi(x')$ in the new frame? □

5 Electromagnetism in Minkowski Spacetime

In this section we will discuss in some detail Minkowski spacetime electromagnetism to illustrate both its Lorentz and gauge symmetries. It will also provide us the opportunity to introduce the action principle, which is key formulating both classical and quantum field theories.

Maxwell & Lorentz We begin with Maxwell's equations in the following Lorentz covariant form, written in Cartesian coordinates $\{x^\mu\}$ so that $g_{\mu\nu} = \eta_{\mu\nu}$:

$$\partial_\mu F^{\mu\nu} = J^\nu, \qquad \partial_{[\mu} F_{\alpha\beta]} = 0, \qquad F_{\mu\nu} = -F_{\nu\mu}. \tag{5.0.1}$$

The $J^\mu \equiv \rho v^\mu$ is the electromagnetic current. Assuming J^μ is timelike, v^μ is its $d-$proper velocity with $v^2 \equiv v^\mu v_\mu = 1$; and $\rho \equiv J^\mu v_\mu$ is the electric charge in the (local) rest frame where $v^\mu = \delta^\mu_0$. Defined this way, ρ is a Lorentz scalar and J^μ is a Lorentz vector since v^μ is a Lorentz vector. It is then reasonable to suppose $F_{\mu\nu}$ is a rank-2 Lorentz tensor. Specifically, let two inertial frames $\{x^\mu\}$ and $\{x'^\mu\}$ be related via the Lorentz transformation

$$x^\mu = \Lambda^\mu{}_\alpha x'^\alpha, \qquad \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta_{\mu\nu} = \eta_{\alpha\beta}. \tag{5.0.2}$$

Then the Faraday tensor transforms as

$$F_{\alpha'\beta'}(x') = F_{\mu\nu}\left(x(x') = \Lambda \cdot x'\right) \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \tag{5.0.3}$$

Its derivatives are also Lorentz covariant, for keeping in mind eq. (5.0.2),

$$\partial_{\lambda'} F_{\alpha'\beta'}(x') = \frac{\partial x^\sigma}{\partial x'^\lambda} \partial_\sigma F_{\mu\nu}\left(x(x') = \Lambda \cdot x'\right) \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \tag{5.0.4}$$

$$= \Lambda^\sigma{}_\lambda \partial_\sigma F_{\mu\nu}\left(x(x')\right) \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta. \tag{5.0.5}$$

This immediately tells us $\partial_\mu F^{\mu\nu} = \eta^{\mu\alpha} \partial_\mu F_{\alpha\beta} \eta^{\beta\nu}$ in eq. (5.0.1) is a Lorentz vector.

Problem 5.1. 4D Maxwell's Equations in term of $(\vec{E}, \vec{B})$ Let us check that eq. (5.0.1) does in fact reproduce Maxwell's equations in terms of electric E^i and magnetic B^i fields in 4D. Given a Lorentzian inertial frame, define

$$F^{i0} \equiv E^i \qquad \text{and} \qquad F^{ij} \equiv \epsilon^{ijk} B^k; \tag{5.0.6}$$

with $\epsilon^{123} \equiv -1$. Show that the $\partial_\mu F^{\mu\nu} = J^\nu$ from eq. (5.0.1) translates to

$$\vec{\nabla} \cdot \vec{E} = J^0 \qquad \text{and} \qquad \vec{\nabla} \times \vec{B} - \partial_t \vec{E} = \vec{J}. \tag{5.0.7}$$

(The over-arrow refers to the spatial components; for instance $\vec{B} = (B^1, B^2, B^3)$.) The $\partial_{[\alpha} F_{\mu\nu]} = 0$ from eq. (5.0.1) translates to

$$\partial_t \vec{B} + \vec{\nabla} \times \vec{E} = 0 \qquad \text{and} \qquad \vec{\nabla} \cdot \vec{B} = 0. \tag{5.0.8}$$

Hint: Note that $(\vec{\nabla} \times \vec{A})^i = -\epsilon^{ijk} \partial_j A^k$, for any Cartesian vector $\vec{A}$. Also, when you compute $\partial_{[i} F_{jk]}$, you simply need to set $\{i, j, k\}$ to be any distinct permutation of $\{1, 2, 3\}$. (Why?)

Next, verify the Lorentz invariant relations, with $\epsilon^{0123} \equiv -1$:

$$F_{\mu\nu} F^{\mu\nu} = -2\left(\vec{E}^2 - \vec{B}^2\right), \qquad \vec{E}^2 \equiv E^i E^i,\ \vec{B}^2 \equiv B^i B^i, \tag{5.0.9}$$

$$\epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} = 4\partial_\mu \left(\epsilon^{\mu\nu\alpha\beta} A_\nu \partial_\alpha A_\beta\right) = 8\vec{E} \cdot \vec{B}. \tag{5.0.10}$$

How does $F_{\mu\nu} F^{\mu\nu}$ transform under time reversal, $t \equiv x^0 \to -t$? How does it transform under parity flips, $x^i \to -x^i$ (for a fixed i)? Answer the same questions for $\widetilde{F}^{\mu\nu} F_{\mu\nu}$, where the dual of $F_{\mu\nu}$ is

$$\widetilde{F}^{\mu\nu} \equiv \frac{1}{2} \widetilde{\epsilon}^{\mu\nu\alpha\beta} F_{\alpha\beta}. \tag{5.0.11}$$

$d \neq 4$ Can you comment what the analog of the magnetic field ought to be in spacetime dimensions different from 4 – is it still a 'vector'? – and what is the lowest dimension that the magnetic field still exists? How many components does the electric field have in 1+1 dimensions? □

Current conservation Taking the divergence of $\partial_\mu F^{\mu\nu} = J^\nu$ yields the conservation of the electric current as a consistency condition. For, by the antisymmetry $F_{\mu\nu} = -F_{\nu\mu}$, $\partial_\nu \partial_\mu F^{\mu\nu} = (1/2)\partial_\nu \partial_\mu F^{\mu\nu} - (1/2)\partial_\mu \partial_\nu F^{\nu\mu} = 0$.

$$\partial_\mu J^\mu = 0. \tag{5.0.12}$$

Problem 5.2. Total charge is constant in all inertial frames Even though we defined ρ in the $J^\mu = \rho v^\mu$ as the charge density in the local rest frame of the electric current itself, we may also define the charge density $\widehat{J^0} \equiv u_\mu J^\mu$ in the rest frame of an arbitrary family of inertial time-like observers whose worldlines' tangent vector is $u^\mu \partial_\mu = \partial_\tau$. (In other words, in their frame, the spacetime metric is $\mathrm{d}s^2 = (\mathrm{d}\tau)^2 - \mathrm{d}\vec{x} \cdot \mathrm{d}\vec{x}$.) Show that total charge is independent of the Lorentz frame by demonstrating that

$$Q \equiv \int_{\mathbb{R}^D} \mathrm{d}^D \Sigma_\mu J^\mu, \qquad \mathrm{d}^D \Sigma_\mu \equiv \mathrm{d}^D \vec{x} u_\mu, \quad D \equiv d - 1, \tag{5.0.13}$$

is a constant. □

Vector Potential & Gauge Symmetry The other Maxwell equation (cf eq. (5.0.1)) leads us to introduce a vector potential A_μ. For $\partial_{[\mu}F_{\alpha\beta]} = 0 \Leftrightarrow \mathrm{d}F = 0$ tells us, by the Poincaré lemma, that

$$F = \mathrm{d}A \qquad \Leftrightarrow \qquad F_{\mu\nu} = \partial_{[\mu}A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{5.0.14}$$

Notice the dynamics in eq. (5.0.1) is not altered if we add to A_μ any object L_μ that obeys $\mathrm{d}L = 0$, because that does not alter the Faraday tensor: $F = \mathrm{d}(A + L) = F + \mathrm{d}L = F$. Now, $\mathrm{d}L = 0$ means, again by the Poincaré lemma, that $L_\mu = \partial_\mu L$, where L on the right hand side is a scalar. *Gauge symmetry*, in the context of electromagnetism, is the statement that the following replacement involving the gauge potential

$$A_\mu(x) \to A_\mu(x) + \partial_\mu L(x) \tag{5.0.15}$$

leaves the dynamics encoded in Maxwell's equations (5.0.1) unchanged.

The use of the gauge potential A_μ makes the $\mathrm{d}F = 0$ portion of the dynamics in eq. (5.0.1) redundant; and what remains is the vector equation

$$\partial_\mu F^{\mu\nu} = \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = J^\nu. \tag{5.0.16}$$

The symmetry under the gauge transformation of eq. (5.0.15) means that solutions to eq. (5.0.16) cannot be unique – in particular, since A_μ and $A_\mu + \partial_\mu L$ are simultaneously solutions, there really is an infinity of solutions parametrized by the arbitrary function L. In this same vein, by going to Fourier space, namely

$$A_\mu(x) \equiv \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \widetilde{A}_\mu(k) e^{-ik_\mu x^\mu} \qquad \text{and} \qquad J_\mu(x) \equiv \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \widetilde{J}_\mu(k) e^{-ik_\mu x^\mu}, \tag{5.0.17}$$

we may see that the differential operator in eq. (5.0.16) cannot be inverted because it has a zero eigenvalue. Firstly, the Fourier version of eq. (5.0.16) reads

$$-K^{\mu\nu}\widetilde{A}_\mu = \widetilde{J}^\nu, \tag{5.0.18}$$

$$K^{\mu\nu} \equiv k_\sigma k^\sigma \eta^{\mu\nu} - k^\nu k^\mu. \tag{5.0.19}$$

If K^{-1} exists, the solution in Fourier space would be (schematically) $\widetilde{A} = -K^{-1}\widetilde{J}$. However, since $K^{\mu\nu} = K^{\nu\mu}$ is a real symmetric matrix, it must be diagonalizable via an orthogonal transformation, with $\det K^{\mu\nu}$ equal to the product of its eigenvalues. That $\det K^{\mu\nu} = 0$ and therefore K^{-1} does not exist can now be seen by observing that k_μ is in fact its null eigenvector:

$$K^{\mu\nu}k_\mu = (k_\sigma k^\sigma)k^\nu - k^\nu k^\mu k_\mu = 0. \tag{5.0.20}$$

Problem 5.3. Can you explain why eq. (5.0.20) amounts to the statement that $F_{\mu\nu}$ is invariant under the gauge transformation of eq. (5.0.15)? Hint: Consider eq. (5.0.15) in Fourier space.

Lorenz gauge To make $K^{\mu\nu}$ invertible, one *fixes a gauge*. A common choice is the Lorenz gauge; in Fourier spacetime:

$$k^\mu \widetilde{A}_\mu = 0. \tag{5.0.21}$$

In 'position'/real spacetime, this reads instead

$$\partial^\mu A_\mu = 0 \qquad \text{(Lorenz gauge)}. \tag{5.0.22}$$

With the constraint in eq. (5.0.21), Maxwell's equations in eq. (5.0.18) becomes

$$-\left(k_\sigma k^\sigma \widetilde{A}^\nu - k^\nu (k^\mu \widetilde{A}_\mu)\right) = -k_\sigma k^\sigma \widetilde{A}^\nu = \widetilde{J}^\nu. \tag{5.0.23}$$

Now, Maxwell's equations have become invertible:

$$\widetilde{A}_\mu(k) = \frac{\widetilde{J}_\mu(k)}{-k^2}, \qquad k^2 \equiv k_\sigma k^\sigma, \qquad \text{(Lorenz gauge)}. \tag{5.0.24}$$

In position/real spacetime, eq. (5.0.23) is equivalent to

$$\partial^2 A^\nu(x) = J^\nu(x) \qquad \partial^2 \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu. \tag{5.0.25}$$

[53]In the Lorenz gauge, we have d Minkowski scalar wave equations, one for each Cartesian component. We may express its position spacetime solution by inverting the Fourier transform in eq. (5.0.24):

$$A_\mu(x) = \int_{\mathbb{R}^{d-1,1}} \mathrm{d}^d x' G_d^+(x-x') J_{\mu'}(x'), \tag{5.0.26}$$

$$G_d^+(x-x') \equiv \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{e^{-ik\cdot(x-x')}}{-k^2}. \tag{5.0.27}$$

Because A_μ is not gauge-invariant, its physical interpretation can be ambiguous. Classically it is the electromagnetic fields $F_{\mu\nu}$ that exert forces on charges/currents, so we need its solution. In fact, we may take the curl of eq. (5.0.25) to see that

$$\partial^2 F_{\mu\nu} = \partial_{[\mu} J_{\nu]}; \tag{5.0.28}$$

this means, using the same Green's function in eq. (5.0.27):

$$F_{\mu\nu}(x) = \int_{\mathbb{R}^{d-1,1}} \mathrm{d}^d x' G_d^+(x-x') \partial_{[\mu'} J_{\nu']}(x'). \tag{5.0.29}$$

We may verify that equations (5.0.26) and (5.0.27) solve eq. (5.0.25) readily:

$$\begin{aligned}
\partial_x^2 G_d^+(x-x') &= \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\partial_\sigma \partial^\sigma e^{-ik\cdot(x-x')}}{-k^2} \\
&= \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\partial_\sigma(-ik^\rho \delta_\rho^\sigma e^{-ik\cdot(x-x')})}{-k^2} \\
&= \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\partial_\sigma(-ik^\sigma e^{-ik\cdot(x-x')})}{-k^2}
\end{aligned}$$

[53]Eq. (5.0.25) is valid in any dimension $d \geq 3$. In 2D, the $\mathrm{d}F = 0$ portion of Maxwell's equations is trivial – i.e., *any* F would satisfy it – because there cannot be three distinct indices in $\partial_{[\mu} F_{\alpha\beta]} = 0$.

$$= \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{(-ik_\sigma)(-ik^\sigma) e^{-ik\cdot(x-x')}}{-k^2}$$

$$= \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik\cdot(x-x')} = \delta^{(d)}(x-x'); \tag{5.0.30}$$

with a similar calculation showing $\partial_{x'}^2 G_d^+(x-x') = \delta^{(d)}(x-x')$. To sum,

$$\partial_x^2 G_d^+(x-x') = \partial_{x'}^2 G_d^+(x-x') = \delta^{(d)}(x-x'); \tag{5.0.31}$$

Moreover, comparing each Cartesian component of the wave equation in eq. (5.0.25) with the one obeyed by the Green's function in eq. (5.0.31), we may identify the source J of the Green's function itself to be a unit strength spacetime point source at some fixed location x'. It is often useful to think of x as the spacetime location of some observer; so $x^0 - x'^0 \equiv t - t'$ is the time elapsed while $|\vec{x} - \vec{x}'|$ is the observer-source spatial distance. Altogether, we may now view the solution in eq. (5.0.26) as the sum of the field generated by all spacetime point sources, weighted by the physical electric current $J_\mu(x')$.

We now may verify directly that eq. (5.0.26) is indeed a solution to eq. (5.0.25).

$$\partial_x^2 A_\mu(x) = \partial_x^2 \left(\int_{\mathbb{R}^{d-1,1}} \mathrm{d}^d x' G_d^+(x-x') J_{\mu'}(x') \right) = \int_{\mathbb{R}^{d-1,1}} \mathrm{d}^d x' \delta^{(d)}(x-x') J_{\mu'}(x')$$
$$= J_\mu(x). \tag{5.0.32}$$

Lorenz gauge: Existence That we have managed to solve Maxwell's equations using the Lorenz gauge, likely convinces the practical physicist that the Lorenz gauge itself surely exists. However, it is certainly possible to provide a general argument. For suppose $\partial^\mu A_\mu$ were not zero, then all one has to show is that we may perform a gauge transformation (cf. (5.0.15)) that would render the new gauge potential $A'_\mu \equiv A_\mu - \partial_\mu L$ satisfy

$$\partial^\mu A'_\mu = \partial^\mu A_\mu - \partial^2 L = 0. \tag{5.0.33}$$

But all that means is, we have to solve $\partial^2 L = \partial^\mu A_\mu$; and since the Green's function $1/\partial^2$ exists, we have proved the assertion.

Lorenz gauge and current conservation You may have noticed, by taking the divergence of both sides of eq. (5.0.25),

$$\partial^2 \left(\partial^\sigma A_\sigma\right) = \partial^\sigma J_\sigma. \tag{5.0.34}$$

This teaches us the consistency of the Lorenz gauge is intimately tied to the conservation of the electric current $\partial^\sigma J_\sigma = 0$. Another way to see this, is to take the time derivative of the divergence of the vector potential, followed by subtracting and adding the spatial Laplacian of A_0 so that $\partial^2 A_0 = J_0$ may be employed:

$$\partial^\sigma \dot{A}_\sigma = \ddot{A}_0 + \partial^i \dot{A}_i = \partial^0 \partial_0 A_0 + \partial^i \partial_i A_0 + \partial^i \partial_0 A_i - \partial^i \partial_i A_0$$
$$= \partial^2 A_0 - \partial^i \left(\partial_i A_0 - \partial_0 A_i\right)$$
$$\partial_0 \left(\partial^\sigma A_\sigma\right) = J_0 - \partial^i F_{i0}. \tag{5.0.35}$$

Notice the right hand side of the last line is zero if the $\nu = 0$ component of $\partial_\mu F^{\mu\nu} = J^\nu$ is obeyed – and if the latter is obeyed the 'left-hand-side' of Lorenz gauge condition $\partial_\mu A^\mu$ is a time independent quantity.

5.1 4 dimensions

4D Maxwell We now focus on the physically most relevant case of $(3+1)D$. In 4D, the wave operator ∂^2 has the following inverse – i.e., retarded Green's function – that obeys causality:

$$G_4^+(x-x') \equiv \frac{\delta(t-t'-|\vec{x}-\vec{x}'|)}{4\pi|\vec{x}-\vec{x}'|}, \qquad x^\mu = (t,\vec{x}),\ x'^\mu = (t',\vec{x}'), \tag{5.1.1}$$

$$\partial_x^2 G_4^+(x-x') = \partial_{x'}^2 G_4^+(x-x') = \delta^{(4)}(x-x'), \tag{5.1.2}$$

$$\partial_x^2 \equiv \eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} \qquad \partial_{x'}^2 \equiv \eta^{\mu\nu}\frac{\partial}{\partial x'^\mu}\frac{\partial}{\partial x'^\nu}. \tag{5.1.3}$$

To see that G_4^+ obeys causality, that it respects the principle that cause precedes effect, one merely needs to focus on the δ-function in eq. (5.1.1). It is non-zero only when the time elapsed $t-t'$ is precisely equal to the observer-source distance $|\vec{x}-\vec{x}'|$. That is, if the source is located at a spatial distance $R=|\vec{x}-\vec{x}'|$ away from the observer, and if the source emitted an instantaneous flash at time t', then the observer would see a signal at time R later (i.e., at $t=t'+R$). In other words, the retarded Green's function propagates signals on the *forward* light cone of the source.[54]

Problem 5.4. Lorentz covariance Suppose $\Lambda^\alpha{}_\mu$ is a Lorentz transformation; let two inertial frames $\{x^\mu\}$ and $\{x'^\mu\}$ be related via

$$x^\mu = \Lambda^\mu{}_\alpha x'^\alpha. \tag{5.1.4}$$

Suppose we solved the Lorenz gauge Maxwell's equations in the $\{x^\mu\}$ frame, namely

$$\frac{\partial A^\mu(x)}{\partial x^\mu} = 0, \qquad \eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}A_\alpha(x) = J_\alpha(x). \tag{5.1.5}$$

Explain how to solve $A_{\alpha'}(x')$, the solution in the $\{x'^\mu\}$ frame. □

Problem 5.5. Analogy: Driven Simple Harmonic Oscillator Suppose we only Fourier-transformed the spatial coordinates in the Lorenz gauge Maxwell eq. (5.0.25). Show that this leads to

$$\ddot{\widetilde{A}}_\mu(t,\vec{k}) + k^2\widetilde{A}_\mu(t,\vec{k}) = \widetilde{J}_\mu(t,\vec{k}), \qquad k \equiv |\vec{k}|. \tag{5.1.6}$$

[55]Compare this to the simple harmonic oscillator (in flat space), with Cartesian coordinate vector $\vec{q}(t)$, mass m, spring constant σ, and driven by an external force $\vec{f}$:

$$m\ddot{\vec{q}} + \sigma\vec{q} = \vec{f}, \tag{5.1.7}$$

where each over-dot corresponds to a time derivative. Identify k^2 and $\widetilde{J}$ in eq. (5.1.6) with the appropriate quantities in eq. (5.1.7). Even though the Lorenz gauge Maxwell equations are fully

[54]The advanced Green's function $G_4^-(x-x') = \delta(t-t'+|\vec{x}-\vec{x}'|)/(4\pi|\vec{x}-\vec{x}'|)$ also solves eq. (5.0.31), but propagates signals on the past light cone: $t=t'-R$.

[55]This equation actually holds in all dimensions $d \geq 3$.

relativistic, notice the analogy with the non-relativistic driven harmonic oscillator! In particular, when the electric current is not present (i.e., $J_\mu = 0$), the 'mixed-space' equations of (5.1.6) are in fact a collection of free simple harmonic oscillators.

Now, how does one solve eq. (5.1.7)? Explain why the inverse of $(\mathrm{d}/\mathrm{d}t)^2 + k^2$ is

$$G_{\text{SHO}}(t-t', k) = -\int_{\mathbb{R}} \frac{\mathrm{d}\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega^2 - k^2}. \tag{5.1.8}$$

That is, verify that this equation satisfies

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + k^2\right) G_{\text{SHO}}(t-t', k) = \left(\frac{\mathrm{d}^2}{\mathrm{d}t'^2} + k^2\right) G_{\text{SHO}}(t-t', k) = \delta(t-t'). \tag{5.1.9}$$

If one tries to integrate ω over the real line in eq. (5.1.8), one runs into trouble – explain the issue. In other words, eq. (5.1.8) is actually ambiguous as it stands.

Now evaluate the Green's function G^+_{SHO} in eq. (5.1.8) using the contour running just slightly above the real line, i.e., $\omega \in (-\infty + i0^+, +\infty + i0^+)$. You should find

$$G^+_{\text{SHO}}(t-t', k) = \Theta(t-t') \frac{\sin\left(k(t-t')\right)}{k}. \tag{5.1.10}$$

Here, Θ is the step function

$$\Theta(x) = 1, \qquad \text{if } x > 0, \tag{5.1.11}$$

$$= 0, \qquad \text{if } x < 0. \tag{5.1.12}$$

Hence, the mixed-space Maxwell's equations have the solution

$$\widetilde{A}_\mu(t, \vec{k}) = \int_{-\infty}^{t} \mathrm{d}t' G^+_{\text{SHO}}(t-t', k) \widetilde{J}_\mu(t', \vec{k}). \tag{5.1.13}$$

By performing an inverse-Fourier transform, namely

$$A_\mu(x) = \int_{\mathbb{R}^{3,1}} \mathrm{d}^4 x' G^+_4(x-x') J_{\mu'}(x'), \tag{5.1.14}$$

arrive at the expression in eq. (5.1.1) □

Vacuum solution & Massless Spin-1 (Helicity-1) Let us examine the simplest situation in 4D flat spacetime, where there are no electric charges nor currents present: $J_\nu = 0$. In Fourier space, setting $\widetilde{J} = 0$ in eq. (5.1.6) leads us to

$$\ddot{\widetilde{A}}_\mu(t, \vec{k}) + k^2 \widetilde{A}_\mu(t, \vec{k}) = 0, \qquad k \equiv |\vec{k}|. \tag{5.1.15}$$

These are the free simple harmonic oscillators alluded to earlier. The solutions are $\widetilde{A}_\mu(t, \vec{k}) = \exp(\pm ikt)$ for $k \equiv |\vec{k}| \geq 0$. Hence, the general solution is the superposition

$$\begin{aligned} A_\mu &= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \left(a_\mu(\vec{k}) \exp(-ikt + i\vec{k}\cdot\vec{x}) + b_\mu(\vec{k}) \exp(ikt + i\vec{k}\cdot\vec{x})\right) \\ &= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \left(a_\mu(\vec{k}) \exp(-ikt + i\vec{k}\cdot\vec{x}) + b_\mu(-\vec{k}) \exp(ikt - i\vec{k}\cdot\vec{x})\right). \end{aligned} \tag{5.1.16}$$

Referring to eq. (5.0.25), since J_μ is real, so is A_μ. Thus it must be that $a_\mu(\vec{k})^* = b_\mu(-\vec{k})$:

$$A_\mu = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \left(a_\mu(\vec{k}) e^{-ik\cdot x} + a_\mu(\vec{k})^* e^{ik\cdot x} \right). \tag{5.1.17}$$

Since a_μ has been arbitrary thus far, we may write a single plane wave solution to eq. (5.1.6) as

$$\begin{aligned} \mathrm{Re}\left\{ \widetilde{A}_\mu(t,\vec{k}) e^{i\vec{k}\cdot\vec{x}} \right\} &= \mathrm{Re}\left\{ \epsilon_\mu(\vec{k}) e^{-ik\cdot t} e^{i\vec{k}\cdot\vec{x}} \right\} = \mathrm{Re}\left\{ \epsilon_\mu(\vec{k}) e^{-ik\cdot x} \right\}, \\ k_\mu \equiv (k, k_i), \qquad & k \equiv |\vec{k}| = \sqrt{\delta^{ab} k_a k_b}. \end{aligned} \tag{5.1.18}$$

The Lorenz gauge says $k^\mu \widetilde{A}_\mu = 0$. Since the $\exp(-ik_\mu x^\mu)$ are basis functions, it must be that the polarization vector ϵ_μ itself is orthogonal to the momentum vector k^μ:

$$k^\mu \epsilon_\mu(\vec{k}) = 0. \tag{5.1.19}$$

Let us suppose k_i points in the positive 3−axis, so that

$$k_\mu = k(1,0,0,-1) \qquad \text{and} \qquad k^\mu = k(1,0,0,1). \tag{5.1.20}$$

This means the plane wave itself becomes

$$\exp(-ik_\mu x^\mu) = \exp(-ik(t - x^3)); \tag{5.1.21}$$

i.e., it indeed describes propagation in the positive 3−direction. The polarization vector may then be decomposed as follows:

$$\epsilon_\mu = \kappa_+ \ell^+{}_\mu + \kappa_- \ell^-{}_\mu + a_+ \epsilon^+{}_\mu + a_- \epsilon^-{}_\mu; \tag{5.1.22}$$

where the κ and a's are (scalar) complex amplitudes; the null basis vectors $\ell^\pm$ are

$$\ell^\pm{}_\mu \equiv \frac{1}{\sqrt{2}} (1,0,0,\pm 1)^{\mathrm{T}}; \tag{5.1.23}$$

wheres the spatial basis vectors $\epsilon^\pm$ are

$$\epsilon^\pm{}_\mu \equiv \frac{1}{\sqrt{2}} (0,\mp 1, i, 0)^{\mathrm{T}}. \tag{5.1.24}$$

Now, under the following rotation on the $(1,2)$-plane orthogonal to $\vec{k}$, namely

$$\widehat{R}(\theta)^\mu{}_\nu \doteq \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{5.1.25}$$

the null polarization vectors in eq. (5.1.23) remain unchanged ($\widehat{R}(\theta)^\mu{}_\nu \ell^{\pm\nu} = \ell^{\pm\mu}$) – they are the spin-0 modes – while the spatial polarizations in eq. (5.1.24) transform as

$$\epsilon^\pm{}_\mu \widehat{R}(\theta)^\mu{}_\nu = e^{-i(\pm 1)\theta} \epsilon^\pm{}_\nu. \tag{5.1.26}$$

These $\epsilon^\pm{}_\nu$ are the helicity-1 modes.

Problem 5.6. Verify eq. (5.1.26).

We now turn to imposing the Lorenz gauge condition $k^\mu \widetilde{A}_\mu = 0$.

$$k\epsilon_0 + k\epsilon_3 = 0 \qquad \Rightarrow \qquad \epsilon_3 = -\epsilon_0. \tag{5.1.27}$$

Since the 0th component has to be negative the 3rd, the ℓ^+ cannot occur in the decomposition of eq. (5.1.22). But since ℓ^- is proportional to k_μ (cf. eq. (5.1.23)) and $k^2 \equiv k_\nu k^\nu = 0$, we see this remaining spin-0 piece of the polarization tensor simultaneously satisfies the Lorenz gauge and is a gradient term – and hence 'pure gauge' (cf. the $\partial_\mu L$ terms of eq. (5.0.15)) – in position spacetime:

$$\kappa_- \ell^-{}_\mu = \frac{\kappa_-}{\sqrt{2}} \frac{k_\mu}{k}. \tag{5.1.28}$$

Since this term will not contribute to the electromagnetic fields $F_{\mu\nu}$, we may perform a Lorenz-gauge-preserving gauge transformation to cancel this term:

$$\text{Re } \widetilde{A}'_\mu(t, \vec{k}) e^{i\vec{k}\cdot\vec{x}} \equiv \text{Re} \left\{ \epsilon_\mu(\vec{k}) e^{-ik\cdot x} - \frac{\kappa_-}{\sqrt{2}} \frac{k_\mu}{k} e^{-ik\cdot x} \right\}. \tag{5.1.29}$$

And now that we have canceled the 0th and 3rd component of the polarization vector in eq. (5.1.22),

$$\text{Re } \widetilde{A}'_\nu(t, \vec{k}) e^{i\vec{k}\cdot\vec{x}} = \text{Re} \left\{ \left(a_+ \epsilon^+{}_\mu + a_- \epsilon^-{}_\mu \right) e^{-ik\cdot x} \right\}. \tag{5.1.30}$$

General Case When k_i is not necessarily (anti)parallel to the 3−axis, we may continue to define $\ell^-{}_\mu$ to be the normalized version of k_μ, i.e.,

$$\ell^-{}_\mu \equiv \frac{k_\mu}{\sqrt{2}k}, \qquad k \equiv k_0. \tag{5.1.31}$$

The ℓ^+, on the other hand, is the solution to the constraints

$$\ell^+ \cdot \ell^- \equiv \eta^{\mu\nu} \ell^+{}_\mu \ell^-{}_\nu = +1, \tag{5.1.32}$$

$$\epsilon^{(1)} \cdot \ell^+ = \epsilon^{(2)} \cdot \ell^+ = 0; \tag{5.1.33}$$

where the $\epsilon^{(1)}$ and $\epsilon^{(2)}$ are themselves mutually orthogonal spatial basis vectors perpendicular to ℓ^+ – namely

$$\epsilon^{(\text{I})} \cdot \epsilon^{(\text{J})} = -\delta^{\text{IJ}}, \qquad \text{I}, \text{J} \in \{1, 2\} \tag{5.1.34}$$

$$\epsilon^{(1)} \cdot \ell^- = \epsilon^{(2)} \cdot \ell^- = 0. \tag{5.1.35}$$

The spin−1 basis vectors can be constructed from the $\epsilon^{(\text{I})}$ via the definitions

$$\epsilon^\pm \equiv \frac{\mp 1}{\sqrt{2}} \epsilon^{(1)} + \frac{i}{\sqrt{2}} \epsilon^{(2)}. \tag{5.1.36}$$

Altogether, the Minkowski metric would obey the following completeness relation

$$\eta_{\mu\nu} = \ell^+{}_{\{\mu} \ell^-{}_{\nu\}} - \epsilon^{(1)}{}_\mu \epsilon^{(1)}{}_\nu - \epsilon^{(2)}{}_\mu \epsilon^{(2)}{}_\nu \tag{5.1.37}$$

$$= \ell^+{}_{\{\mu} \ell^-{}_{\nu\}} + \epsilon^+{}_{\{\mu} \epsilon^-{}_{\nu\}}. \tag{5.1.38}$$

(3+1)D Spin-1 Waves To sum, given an inertial frame, the electromagnetic vector potential A_μ in vacuum is given by the following superposition of spin-1 waves:

$$A_\mu(x) = \text{Re} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \left(a_+ \epsilon^+{}_\mu(\vec{k}) + a_- \epsilon^-{}_\mu(\vec{k}) \right) e^{-ik\cdot x}, \tag{5.1.39}$$

where $\epsilon^\pm{}_\mu$ are purely spatial polarization tensors orthogonal to the k_i; and, under a rotation by an angle θ around the plane perpendicular to k_i transforms as $\epsilon^\pm \to \exp(-i(\pm 1)\theta)\epsilon^\pm$.[56]

Problem 5.7. Circularly Polarized Light from 4D Spin-1 Consider a single spin-1 plane wave (cf. (5.1.24)) propagating along the 3−axis, with $k_\mu = k(1,0,0,-1)$:

$$A_\mu^\pm(t,x,y,z) \equiv \text{Re}\left\{ a_\pm \epsilon^\pm{}_\mu e^{-ik(t-z)} \right\}, \qquad a_\pm \in \mathbb{R}. \tag{5.1.40}$$

Compute the electric field ${}_\pm E^i = F^{i0}$ and show that these plane waves give rise to circularly polarized light, i.e., for either a fixed time t or spatial location z – the electric field direction rotates in a circular fashion:

$${}_\pm E^i = \frac{k a_\pm}{\sqrt{2}} \left(\pm \sin(k(t-z))\widehat{x}^i + \cos(k(t-z))\widehat{y}^i \right), \tag{5.1.41}$$

where $\widehat{x}$ and $\widehat{y}$ are the unit vectors in the 1− and 2−directions:

$$\widehat{x}^i \doteq (1,0,0) \qquad \text{and} \qquad \widehat{y}^i \doteq (0,1,0). \tag{5.1.42}$$

□

Redshift For each Lorenz-gauge plane wave in an inertial frame $\{x^\mu = (t,\vec{x})\}$,

$$\epsilon^\pm{}_\mu(k)\exp(-ik\cdot x) = \epsilon^\pm{}_\mu(k)\exp(-ik_j x^j)\exp(-i\omega t), \qquad \omega \equiv |\vec{k}|, \tag{5.1.43}$$

we may read off its frequency ω as the coefficient of the time coordinate t. Quantum mechanics tells us ω is also the energy of the associated photon. Suppose a different Lorentz inertial frame $\{x'\}$ is related to the previous through the Lorentz transformation $\Lambda^\alpha{}_\mu$: $x^\alpha = \Lambda^\alpha{}_\mu x'^\mu$. Because the phase in the plane wave solution of eq. (5.1.43) is a scalar, in the $\{x'\}$ Lorentz frame

$$-ik_\alpha x^\alpha = -ik_\alpha \Lambda^\alpha{}_\mu x'^\mu = -i(k_\alpha \Lambda^\alpha{}_0)t' - i(k_\alpha \Lambda^\alpha{}_i)x'^i. \tag{5.1.44}$$

The frequency ω' and hence the photon's energy in this $\{x'\}$ frame is therefore

$$\omega' = k_\alpha \Lambda^\alpha{}_0 = \omega \left(\Lambda^0{}_0 + \widehat{k}_i \Lambda^i{}_0 \right) \tag{5.1.45}$$

$$\widehat{k}_i \equiv k_i/|\vec{k}| = k_i/\omega. \tag{5.1.46}$$

[56] For a given inertial frame and within the Lorenz gauge, we have been able to get rid of the 'pure gauge' spin-0 mode by a gauge transformation, leaving only the massless spin-1 (simple-harmonic) waves. Note however, these waves in eq. (5.1.39) would no longer be an admixture of pure spin-1 modes – simply by viewing them in a different reference frame, i.e., upon a Lorentz boost.

There is a slightly different way to express this redshift result that would help us generalize the analysis to curved spacetime, at least in the high frequency 'JWKB' limit. To extract the frequency directly from the phase $S \equiv k \cdot x$, we may take its time derivative using the unit norm vector $u \equiv \partial_t = \partial_0$ that we may associate with the worldlines of observers at rest in the $\{x\}$ frame:

$$u^\mu \partial_\mu S = \partial_0 (k_\alpha x^\alpha) = \omega. \tag{5.1.47}$$

The observers at rest in the $\{x'\}$ frame have $u' \equiv \partial_{t'} = \partial_{0'}$ as their timelike unit norm tangent vector. (Note: $x^\alpha = \Lambda^\alpha{}_\mu x'^\mu \Leftrightarrow \partial_{\mu'} = \Lambda^\alpha{}_\mu \partial_\alpha$.) The energy of the photon is then

$$\begin{aligned} u'^\alpha \partial_{\alpha'} S = \partial_{t'} S &= \Lambda^\alpha{}_0 \partial_\alpha (k \cdot x) \\ &= \Lambda^\alpha{}_0 k_\alpha = \omega \left(\Lambda^0{}_0 + \widehat{k}_i \Lambda^i{}_0 \right). \end{aligned} \tag{5.1.48}$$

Problem 5.8. Consider a single photon with wave vector $k_\mu = \omega(1, \widehat{n}_i)$ (where $\widehat{n}_i \widehat{n}_j \delta^{ij} = 1$) in some inertial frame $\{x^\mu\}$. Let a family of inertial observers be moving with constant velocity $v^\mu \equiv (1, v^i)$ with respect to the frame $\{x^\mu\}$. What is the photon's frequency ω' in their frame? Compute the redshift formula for ω'/ω. Comment on the redshift result when v^i is (anti)parallel to $\widehat{n}_i$ and when v^i is perpendicular to $\widehat{n}_i$.

Problem 5.9. Dispersion relations Consider the *massive* Klein-Gordon equation in Minkowski spacetime:

$$\left(\partial^2 + m^2 \right) \varphi(t, \vec{x}) = 0, \tag{5.1.49}$$

where φ is a real scalar field. Find the general solution for φ in terms of plane waves $\exp(-ik \cdot x)$ and obtain the dispersion relation:

$$k^2 = m^2 \qquad \Leftrightarrow \qquad E^2 = \vec{p}^2 + m^2, \tag{5.1.50}$$

$$E \equiv k^0, \qquad \vec{p} \equiv \vec{k}. \tag{5.1.51}$$

If each plane wave is associated with a particle of $d-$momentum k_μ, this states that it has mass m. The photon, which obeys $k^2 = 0$, has zero mass.

Bonus: Can you restore the factors of $\hbar$ and c in eq. (5.1.49)? □

5.2 Gauge Invariant Variables for Vector Potential

Although the vector potential A_μ itself is not a gauge invariant object, we will now exploit the spatial translation symmetry of Minkowski spacetime to seek a gauge-invariant set of partial differential equations involving a "scalar-vector" decomposition of A_μ. There are at least two reasons for doing so.

- This will be a warm-up to an analogous analysis for gravitation linearized about a Minkowski "background" spacetime.

- We will witness how, for a given inertial frame, the only portion of the vector potential A_μ that obeys a wave equation is its gauge-invariant "transverse" spatial portion. (Even though every component of A_μ in the Lorenz gauge (cf. eq. (5.0.25)) obeys the wave equation, remember such a statement is not gauge-invariant.) We shall also identify a gauge-invariant scalar potential sourced by charge density.

Scalar-Vector Decomposition The scalar-vector decomposition is the statement that the spatial components of the vector potential may be expressed as a gradient of a scalar α plus a transverse vector α_i:

$$A_i = \partial_i \alpha + \alpha_i, \tag{5.2.1}$$

where by "transverse" we mean

$$\partial_i \alpha_i = 0. \tag{5.2.2}$$

To demonstrate the generality of eq. (5.2.1) we shall first write A_i in Fourier space

$$A_i(t,\vec{x}) = \int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D} \widetilde{A}_i(t,\vec{k}) e^{i\vec{k}\cdot\vec{x}}; \tag{5.2.3}$$

where $\vec{k}\cdot\vec{x} \equiv \delta_{ij}k^i x^j = -k_j x^j$. Every spatial derivative ∂_j acting on $A_i(t,\vec{x})$ becomes in Fourier space a $-ik_j$, since

$$\begin{aligned}
\partial_j A_i &= \int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D} \partial_j \left(i\delta_{ab}k^a x^b\right) \widetilde{A}_i(t,\vec{k}) e^{i\vec{k}\cdot\vec{x}} \\
&= \int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D} \left(i\delta_{ab}k^a \delta^b_j\right) \widetilde{A}_i(t,\vec{k}) e^{i\vec{k}\cdot\vec{x}} \\
&= \int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D} ik^j \widetilde{A}_i(t,\vec{k}) e^{i\vec{k}\cdot\vec{x}} \\
&= \int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D} (-ik_j) \widetilde{A}_i(t,\vec{k}) e^{i\vec{k}\cdot\vec{x}}.
\end{aligned} \tag{5.2.4}$$

As such, the transverse property of $\alpha_i(t,\vec{x})$ would in Fourier space become

$$-ik_i\widetilde{\alpha}_i(t,\vec{k}) = 0. \tag{5.2.5}$$

At this point we simply write down

$$\widetilde{A}_i(t,\vec{k}) = \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}\right)\widetilde{A}_j(t,\vec{k}) + \frac{k_i k_j}{\vec{k}^2}\widetilde{A}_j(t,\vec{k}). \tag{5.2.6}$$

This is mere tautology, of course. However, we may now check that the first term on the left hand side of eq. (5.2.6) is transverse:

$$-ik_i\left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}\right)\widetilde{A}_j(t,\vec{k}) = -i\left(k_j - \frac{\vec{k}^2 k_j}{\vec{k}^2}\right)\widetilde{A}_j(t,\vec{k}) = 0. \tag{5.2.7}$$

The second term on the right hand side of eq. (5.2.6) is a gradient because it is

$$-ik_i \left(\frac{ik_j}{\vec{k}^2} \widetilde{A}_j \right). \tag{5.2.8}$$

To sum, we have identified the α and α_i terms of eq. (5.2.1) as

$$\alpha(t, \vec{x}) = \int_{\mathbb{R}^D} \frac{\mathrm{d}^D \vec{k}}{(2\pi)^D} \frac{ik_j}{\vec{k}^2} \widetilde{A}_j(t, \vec{k}) e^{i\vec{k}\cdot\vec{x}}; \tag{5.2.9}$$

and the transverse portion as

$$\begin{aligned} \alpha_i(t, \vec{x}) &= \int_{\mathbb{R}^D} \frac{\mathrm{d}^D \vec{k}}{(2\pi)^D} P_{ij}(\vec{k}) \widetilde{A}_j(t, \vec{k}) e^{i\vec{k}\cdot\vec{x}}, \\ P_{ij}(\vec{k}) &\equiv \delta_{ij} - \frac{k_i k_j}{\vec{k}^2}. \end{aligned} \tag{5.2.10}$$

Notice it is really the projector P_{ij} that is "transverse"; i.e.

$$k_i P_{ij}(\vec{k}) = 0. \tag{5.2.11}$$

Let us also note that this scalar-vector decomposition is unique, in that – if we have the Fourier-space equation

$$-ik_i\widetilde{\alpha} + \widetilde{\alpha}_i = -ik_i\widetilde{\beta} + \widetilde{\beta}_i, \tag{5.2.12}$$

where $k_i\widetilde{\alpha}_i = k_i\widetilde{\beta}_i = 0$, then

$$\widetilde{\alpha} = \widetilde{\beta} \qquad \text{and} \qquad \widetilde{\alpha}_i = \widetilde{\beta}_i. \tag{5.2.13}$$

For, we may first "dot" both sides of eq. (5.2.12) with $\vec{k}$ and see that – for $\vec{k} \neq \vec{0}$,

$$\vec{k}^2\widetilde{\alpha} = \vec{k}^2\widetilde{\beta} \qquad \Leftrightarrow \qquad \widetilde{\alpha} = \widetilde{\beta}. \tag{5.2.14}$$

Plugging this result back into eq. (5.2.12), we also conclude $\widetilde{\alpha}_i = \widetilde{\beta}_i$.

Now, this scalar-vector decomposition is really just a mathematical fact, and may even be performed in a curved space – as long as the latter is infinite – since it depends on the existence of the Fourier transform and not on the metric structure. (A finite space would call for a discrete Fourier-like series of sorts.) However, to determine its usefulness, we would need to insert it into the partial differential equations obeyed by A_i, where the metric structure does matter. As we now turn to examine, because of the spatial translation symmetry of Minkowski spacetime, Maxwell's equations themselves admit a scalar-vector decomposition. This, in turn, would lead to PDEs for the gauge-invariant portions of A_μ.

Gauge transformations We first examine how the gauge transformation of eq. (5.0.15) is implemented on a scalar-vector decomposed A_μ.

$$A_0 \to A_0 + \dot{L} \tag{5.2.15}$$

$$\begin{aligned} A_i = \partial_i\alpha + \alpha_i &\to \partial_i\alpha + \alpha_i + \partial_i L \\ &= \partial_i(\alpha + L) + \alpha_i. \end{aligned} \tag{5.2.16}$$

From the uniqueness discussion above, we may thus identify the gauge-transformed "scalar" portion of A_i

$$\alpha \to \alpha' \equiv \alpha + L \tag{5.2.17}$$

and the "transverse-vector" portion of A_i to be gauge-invariant:

$$\alpha_i \to \alpha_i. \tag{5.2.18}$$

Let us now identify

$$\Phi \equiv A_0 - \dot{\alpha} \tag{5.2.19}$$

because it is gauge-invariant; for, according to equations (5.2.15) and (5.2.17)

$$\Phi \to A_0 + \dot{L} - \partial_0(\alpha + L) = A_0 - \dot{\alpha}. \tag{5.2.20}$$

In terms of Φ and α_i, the components of the gauge-invariant electromagnetic tensor read

$$F_{0i} \equiv \dot{A}_i - \partial_i A_0 = \dot{\alpha}_i + \partial_i \dot{\alpha} - \partial_i A_0 \tag{5.2.21}$$

$$= \dot{\alpha}_i - \partial_i \Phi \tag{5.2.22}$$

$$F_{ij} = \partial_{[i} A_{j]} = \partial_{[i} \alpha_{j]}. \tag{5.2.23}$$

Electric current We also need to perform a scalar-vector decomposition of the electric current

$$J_\mu \equiv (\rho_{\rm E}, \partial_i \mathcal{J} + \mathcal{J}_i). \tag{5.2.24}$$

Its conservation $\partial^\mu J_\mu = 0$ now reads

$$\dot{\rho}_{\rm E} - \partial_i (\partial_i \mathcal{J} + \mathcal{J}_i) = 0 \tag{5.2.25}$$

$$\dot{\rho}_{\rm E} = \vec{\nabla}^2 \mathcal{J}. \tag{5.2.26}$$

Maxwell's Equations At this point, we are ready to write down Maxwell's equations $\partial^\mu F_{\mu\nu} = J_\nu$. From eq. (5.2.22), the $\nu = 0$ component is

$$-\partial_i F_{i0} = \partial_i(\dot{\alpha}_i - \partial_i \Phi) = -\vec{\nabla}^2 \Phi = \rho_{\rm E}. \tag{5.2.27}$$

The $\nu = i$ component of $\partial^\mu F_{\mu\nu} = J_\nu$, according to eq. (5.2.22) and (5.2.23),

$$\partial_0 F_{0i} - \partial_j F_{ji} = \partial_i \mathcal{J} + \mathcal{J}_i \tag{5.2.28}$$

$$\ddot{\alpha}_i - \partial_i \dot{\Phi} - \partial_j (\partial_j \alpha_i - \partial_i \alpha_j) = \partial_i \mathcal{J} + \mathcal{J}_i \tag{5.2.29}$$

$$\partial^2 \alpha_i - \partial_i \dot{\Phi} = \mathcal{J}_i + \partial_i \mathcal{J}. \tag{5.2.30}$$

As already advertised, we see that the spatial components of Maxwell's equations does admit a scalar-vector decomposition. By the uniqueness argument above, we may read off the "transverse-vector" portion to be

$$\partial^2 \alpha_i = \mathcal{J}_i. \tag{5.2.31}$$

and the "scalar" portion to be

$$-\dot{\Phi} = \mathcal{J}. \tag{5.2.32}$$

We have gotten 3 (groups of) equations – (5.2.27), (5.2.31), (5.2.32) – for 2 sets of variables (Φ, α_i). Let us argue that eq. (5.2.32) is actually redundant. Taking into account eq. (5.2.26), we may take a time derivative of both sides of eq. (5.2.27),

$$-\vec{\nabla}^2\dot{\Phi} = \dot{\rho}_{\rm E} = \vec{\nabla}^2\mathcal{J}. \tag{5.2.33}$$

For the physically realistic case of isolated electric currents, where we may assume implies both $\dot{\Phi} \to 0$ and $\mathcal{J} \to 0$ as the observer-J_i distance goes to infinity, the solution to this above Poisson equation is then unique. This hands us eq. (5.2.32).

> **Gauge-Invariant Formalism** To sum: for physically realistic situations in Minkowski spacetime, if we perform a scalar-vector decomposition of the photon vector potential A_μ through eq. (5.2.1) and that of the current J_μ through eq. (5.2.24), we find a gauge-invariant Poisson equation
>
> $$-\vec{\nabla}^2\Phi = \rho_{\rm E}, \qquad \Phi \equiv A_0 - \dot{\alpha}; \tag{5.2.34}$$
>
> as well as a gauge-invariant wave equation
>
> $$\partial^2\alpha_i = \mathcal{J}_i. \tag{5.2.35}$$

These illuminate the theoretical structure of electromagnetism. As you may recall, our explicit discussions in 4D leading up to the spin-1 modes of eq. (5.1.39) led us to conclude that the non-trivial homogeneous wave solutions of Maxwell's equations are in fact of the "transverse-vector" type. The gauge-invariant formalism for this section thus allows us to identify the source of these spin-1 waves – they are the "transverse-vector" portion of the spatial electric current.

Remark: (1+1)D The one constraint $\partial_i\alpha_i = 0$ obeyed by the spin-1 photon α_i means it has really $D - 1 = d - 2$ independent components, since in Fourier space $k_i\widetilde{\alpha}_i = 0$ implies (for $\vec{k} \neq 0$) the $\{\widetilde{\alpha}_i\}$ are linearly dependent. In particular, in $(1+1)$D $k_1\widetilde{\alpha}_1 = 0$ and as long as $k_1 \neq 0$, the spin-1 photon itself is trivial: $\widetilde{\alpha}_1 = 0$.

6 Action Principles & Classical Field Theory

In §(5), we elucidated the Lorentz and gauge symmetries enjoyed by Maxwell's equations. There is in fact an efficient means to define a theory such that it would enjoy the symmetries one desires. This is the action principle. You may encountered it in (non-relativistic) Classical Mechanics, where Newton's second law emerges from demanding the integral

$$S \equiv \int_{t_\text{i}}^{t_\text{f}} L \text{d}t, \tag{6.0.1}$$

$$L \equiv \frac{1}{2} m \dot{\vec{x}}(t)^2 - V(\vec{x}(t)). \tag{6.0.2}$$

Here, L is called the Lagrangian, and in this context is the difference between the particle's kinetic and potential energy. The action of a field theory also plays a central role in its quantum theory when phrased in the path integral formulation; roughly speaking, $\exp(iS)$ is related to the infinitesimal quantum transition amplitude. For these reasons, we shall study the classical field theories – leading up to General Relativity itself – through the principle of stationary action.

General covariance In field theory one defines an object similar to the one in eq. (6.0.1), except the integrand $\mathcal{L}$ is now a Lagrangian *density* (per unit spacetime volume). To obtain generally covariant equations, we now demand that the Lagrangian density is, possibly up to a total divergence, a scalar under coordinate transformations and other symmetry transformations relevant to the problem at hand.

$$S \equiv \int_{t_\text{i}}^{t_\text{f}} \text{d}^d x \sqrt{|g|} \mathcal{L} \tag{6.0.3}$$

One then demands that the action is extremized under the boundary conditions that the field configurations at some initial t_i and final time t_f are fixed. If the spatial boundaries of the spacetime are a finite distance away, one would also have to impose appropriate boundary conditions there; otherwise, if space is infinite, the fields are usually assumed to fall off to zero sufficiently quickly at spatial infinity – below, we will assume the latter for technical simplicity. (In particle mechanics, the action principle also assumes the initial and final positions of the particle are specified.) In curved spacetime, note that the time coordinate x^0 need not correspond to same variable defining the initial t_i and final t_f times; the latter are really shorthand for any appropriately defined spacelike 'constant-time' hyper-surfaces.

6.1 Scalar Fields

Let us begin with a scalar field φ. For concreteness, we shall form its Lagrangian density $\mathcal{L}(\varphi, \nabla_\alpha \varphi)$ out of φ and its first covariant derivatives $\nabla_\alpha \varphi$. Demanding the resulting action be extremized means its first order variation need to vanish. That is, we shall replace $\varphi \to \varphi + \delta\varphi$ (which also means $\nabla_\alpha \varphi \to \nabla_\alpha \varphi + \nabla_\alpha \delta\varphi$) and demand that the portion of the action linear in $\delta\varphi$ be zero.

$$\begin{aligned} \delta_\varphi S &= \int_{t_\text{i}}^{t_\text{f}} \text{d}^d x \sqrt{|g|} \left(\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\nabla_\alpha \varphi)} \nabla_\alpha \delta\varphi \right) \\ &= \left[\int \text{d}^{d-1} \Sigma_\alpha \frac{\partial \mathcal{L}}{\partial(\nabla_\alpha \varphi)} \delta\varphi \right]_{t_\text{i}}^{t_\text{f}} + \int_{t_\text{i}}^{t_\text{f}} \text{d}^d x \sqrt{|g|} \delta\varphi \left(\frac{\partial \mathcal{L}}{\partial \varphi} - \nabla_\alpha \frac{\partial \mathcal{L}}{\partial(\nabla_\alpha \varphi)} \right) \end{aligned} \tag{6.1.1}$$

Because the initial and final field configurations $\varphi(t_\mathrm{i})$ and $\varphi(t_\mathrm{f})$ are assumed fixed, their respective variations are zero by definition: $\delta\varphi(t_\mathrm{i}) = \delta\varphi(t_\mathrm{f}) = 0$. This sets to zero the first term on the second equality. At this point, the requirement that the action be stationary means $\delta_\varphi S$ be zero for any small but arbitrary $\delta\varphi$, which in turn implies the coefficient of $\delta\varphi$ must be zero. That leaves us with the Euler-Lagrangian equations

$$\frac{\partial\mathcal{L}}{\partial\varphi} = \nabla_\alpha \frac{\partial\mathcal{L}}{\partial(\nabla_\alpha\varphi)}. \tag{6.1.2}$$

We may now consider a coordinate transformation $x(x')$. Assuming $\mathcal{L}$ is a coordinate scalar, this means the only ingredient that is not a scalar is the derivative with respect to $\nabla_\alpha\varphi$. Since

$$\frac{\partial x^\alpha}{\partial x'^\mu}\nabla_\alpha\varphi(x) = \nabla_{\mu'}\varphi(x') \equiv \nabla_{\mu'}\varphi\left(x(x')\right), \tag{6.1.3}$$

we have

$$\frac{\partial\mathcal{L}}{\partial(\nabla_\alpha\varphi(x))} = \frac{\partial(\nabla_{\mu'}\varphi(x'))}{\partial(\nabla_\alpha\varphi(x))}\frac{\partial\mathcal{L}}{\partial(\nabla_{\mu'}\varphi(x'))} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial\mathcal{L}}{\partial(\nabla_{\mu'}\varphi(x'))}. \tag{6.1.4}$$

That is, $\partial\mathcal{L}/\partial(\nabla_\alpha\varphi(x))$ transforms as a rank-1 vector; and $\nabla_\alpha\{\partial\mathcal{L}/\partial(\nabla_\alpha\varphi(x))\}$ is its divergence, i.e., a scalar. Altogether, we have thus demonstrated that the Euler-Lagrange equations in eq. (6.1.2), for a scalar field φ, is itself a scalar. This is a direct consequence of the fact that $\mathcal{L}$ is a coordinate scalar by construction. A common example of such a scalar action is

$$S[\varphi] \equiv \int \mathrm{d}^d x\sqrt{|g|}\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - V(\varphi)\right), \tag{6.1.5}$$

where V is its scalar potential.

Internal Global O_N Symmetry To provide an example of a symmetry other than the invariance under coordinate transformations, let us consider the following action involving $N > 1$ scalar fields $\{\varphi^\mathrm{I}|\mathrm{I} = 1, 2, 3, \dots, N\}$:

$$S \equiv \int \mathrm{d}^d x\sqrt{|g|}\mathcal{L}\left(g^{\mu\nu}\nabla_\mu\varphi^\mathrm{I}\nabla_\nu\varphi^\mathrm{I}, \varphi^\mathrm{I}\varphi^\mathrm{I}\right). \tag{6.1.6}$$

With summation covention in force, we see that the sum over the scalar field label 'I' is simply a dot product in 'field space'. This in turn leads us to observe that the action is invariant under a global rotation:

$$\varphi^\mathrm{I} \equiv \widehat{R}^\mathrm{I}{}_\mathrm{J}\varphi'^\mathrm{J}, \tag{6.1.7}$$

where $\widehat{R}^\mathrm{I}{}_\mathrm{A}\widehat{R}^\mathrm{J}{}_\mathrm{B}\delta_\mathrm{IJ} = \delta_\mathrm{AB}$. (By 'global' rotation, we mean the rotation matrices $\{\widehat{R}^\mathrm{I}{}_\mathrm{J}\}$ do not depend on spacetime.) Explicitly,

$$\int \mathrm{d}^d x\sqrt{|g|}\mathcal{L}\left(g^{\mu\nu}\nabla_\mu\varphi^\mathrm{I}\nabla_\nu\varphi^\mathrm{I}, \varphi^\mathrm{I}\varphi^\mathrm{I}\right) = \int \mathrm{d}^d x\sqrt{|g|}\mathcal{L}\left(g^{\mu\nu}\nabla_\mu\varphi'^\mathrm{I}\nabla_\nu\varphi'^\mathrm{I}, \varphi'^\mathrm{I}\varphi'^\mathrm{I}\right). \tag{6.1.8}$$

Let us now witness, because we have constructed a Lagrangian density that is invariant under such an internal O_N symmetry, the resulting equations of motion transform covariantly under rotations. Firstly, the I-th Euler-Lagrange equation, gotten by varying eq. (6.1.6) with respect to φ^{I}, reads

$$\frac{\partial \mathcal{L}}{\partial \varphi^{\mathrm{I}}} = \nabla_\alpha \frac{\partial \mathcal{L}}{\partial(\nabla_\alpha \varphi^{\mathrm{I}})}. \tag{6.1.9}$$

Under rotation, eq. (6.1.7) is equivalent to

$$\left(\widehat{R}^{-1}\right)^{\mathrm{J}}{}_{\mathrm{I}} \varphi^{\mathrm{I}} = \varphi'^{\mathrm{J}}, \tag{6.1.10}$$

which in turn tells us

$$\left(\widehat{R}^{-1}\right)^{\mathrm{J}}{}_{\mathrm{I}} \nabla_\alpha \varphi^{\mathrm{I}} = \nabla_\alpha \varphi'^{\mathrm{J}}. \tag{6.1.11}$$

Therefore eq. (6.1.9) becomes

$$\frac{\partial \varphi'^{\mathrm{J}}}{\partial \varphi^{\mathrm{I}}} \frac{\partial \mathcal{L}}{\partial \varphi'^{\mathrm{J}}} = \frac{\partial \nabla_\alpha \varphi'^{\mathrm{J}}}{\partial \nabla_\alpha \varphi^{\mathrm{I}}} \nabla_\alpha \frac{\partial \mathcal{L}}{\partial(\nabla_\alpha \varphi^{\mathrm{J}})}, \tag{6.1.12}$$

$$\left(\widehat{R}^{-1}\right)^{\mathrm{J}}{}_{\mathrm{I}} \frac{\partial \mathcal{L}}{\partial \varphi'^{\mathrm{J}}} = \left(\widehat{R}^{-1}\right)^{\mathrm{J}}{}_{\mathrm{I}} \nabla_\alpha \frac{\partial \mathcal{L}}{\partial(\nabla_\alpha \varphi^{\mathrm{J}})}. \tag{6.1.13}$$

The PDEs for our O_N-invariant scalar field theory transforms covariantly as a vector under global rotation of the fields $\{\varphi^{\mathrm{I}}\}$.

6.2 Maxwell's Electromagnetism

We have seen how Maxwell's equations are both gauge-invariant and Lorentz invariant. (In fact, the former meant we had to gauge fix the equations before we could solve them.) In curved spacetime, the gauge-invariant electromagnetic tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu \tag{6.2.1}$$

does not actually contain any metric because the Christoffel symbols cancel out. We may form a coordinate scalar as follows:

$$\mathcal{L}_{\text{Maxwell}} \equiv -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{6.2.2}$$

We now claim that, given an externally prescribed electric current J^μ, Maxwell's equations arise from the following action:

$$S_{\text{Maxwell}} \equiv \int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} \mathrm{d}^d x \sqrt{|g|} \left(\mathcal{L}_{\text{Maxwell}} - A_\mu J^\mu\right). \tag{6.2.3}$$

The $A_\mu J^\mu$ term, under gauge transformation $A_\mu \to A_\mu + \partial_\mu L$, is altered as

$$\begin{aligned} A_\mu J^\mu &\to A_\mu J^\mu + \nabla_\mu L \cdot J^\mu \\ &= A_\mu J^\mu + \nabla_\mu (L \cdot J^\mu) - L \nabla_\mu J^\mu \end{aligned} \tag{6.2.4}$$

If we require that $L(t_\mathrm{i}) = L(t_\mathrm{f}) = 0$, then we see that such a gauge transformation changes the Maxwell action in eq. (6.2.3) as

$$S_{\text{Maxwell}} \to S_{\text{Maxwell}} - \left[\int \mathrm{d}^{d-1}\Sigma_\mu (L \cdot J^\mu)\right]_{t_\mathrm{i}}^{t_\mathrm{f}} + \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}^d x\sqrt{|g|} L \nabla_\mu J^\mu \tag{6.2.5}$$

$$\to S_{\text{Maxwell}} + \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}^d x\sqrt{|g|} L \nabla_\mu J^\mu. \tag{6.2.6}$$

We have already witnessed in §(5) how gauge-invariance is intimately tied to current conservation: here we see that the Maxwell action would not be invariant under gauge transformations unless J^μ is conserved.

Let us now proceed to vary the Maxwell action, and see how Maxwell's equations emerge. Consider

$$A_\mu \to A_\mu + \delta A_\mu \tag{6.2.7}$$

and read off the first order in δA_μ terms in the resulting action:

$$\begin{aligned}
\delta_A S_{\text{Maxwell}} &= \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}^d x\sqrt{|g|}\left(-\frac{1}{4}\partial_{[\mu}\delta A_{\nu]}F^{\mu\nu} - \frac{1}{4}F^{\mu\nu}\partial_{[\mu}\delta A_{\nu]} - \delta A_\mu J^\mu\right) \\
&= \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}^d x\sqrt{|g|}\left(-\frac{1}{2}\nabla_\mu \delta A_\nu F^{[\mu\nu]} - \delta A_\mu J^\mu\right) \\
&= \left[-\int \mathrm{d}^{d-1}\Sigma_\mu \delta A_\nu F^{\mu\nu}\right]_{t_\mathrm{i}}^{t_\mathrm{f}} + \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}^d x\sqrt{|g|}\delta A_\nu \left(\nabla_\mu F^{\mu\nu} - J^\nu\right).
\end{aligned} \tag{6.2.8}$$

Let us notice not all the components of the vector potential A_μ need to be fixed at t_i and t_f for the first term of the last equality to vanish. As a simple example, suppose we focus on the Minkowski case and let t_i and t_f correspond to constant x^0-surfaces, then we have $\int \mathrm{d}^{d-1}\Sigma_\mu \delta A_\nu F^{\mu\nu} = \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}\delta A_i F^{0i}$ because, by the antisymmetry of $F^{\mu\nu}$, $F^{00} = 0$. In any case, once the boundary field configurations are fixed, $\delta A_i(t_\mathrm{i}) = \delta A_i(t_\mathrm{f}) = 0$, we have to demand the remaining coefficient of δA_ν be zero.

$$\nabla_\mu F^{\mu\nu} = J^\nu \tag{6.2.9}$$

This is Maxwell's equations in curved spacetime. Here, we are using A_μ as our fundamental field variable; but if we were instead working with $F_{\mu\nu}$, we need to impose $F_{\mu\nu} = -F_{\nu\mu}$ and $\mathrm{d}F = 0$:

$$\nabla_{[\alpha}F_{\mu\nu]} = \partial_{[\alpha}F_{\mu\nu]} = 0. \tag{6.2.10}$$

Of course, by the Poincaré lemma, this does imply $F = \mathrm{d}A$, i.e., $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Problem 6.1. Conservation of electric current Explain why

$$\nabla_\nu \nabla_\mu F^{\mu\nu} = 0 \tag{6.2.11}$$

is an identity. Therefore, by taking the divergence on both sides of eq. (6.2.9), we see that Maxwell's equations in curved spacetime continue to require the conservation of its electric current. □

Electromagnetism of point charges As a non-trivial application of the action principle for electromagnetism, we will now demonstrate why the following action describes the electromagnetism of N point electric charges.

$$S' \equiv -\frac{1}{4}\int_{t_\text{i}}^{t_\text{f}} \text{d}^d x\sqrt{|g|}F_{\mu\nu}F^{\mu\nu} - \sum_{a=1}^{N}\left(m_a \int_{t_\text{i}}^{t_\text{f}} \text{d}\lambda_a \sqrt{g_{\mu\nu}\dot{z}_a^\mu \dot{z}_a^\nu} + q_a \int_{t_\text{i}}^{t_\text{f}} A_\mu(z_a)\text{d}x^\mu\right), \qquad (6.2.12)$$

where m_a and q_a are respectively the mass and electric charge of the ath point particle; and $\dot{z}_a^\mu \equiv \text{d}z_a^\mu/\text{d}\lambda_a$. In particular, this action leads to Maxwell's equations sourced by point charges

$$\nabla_\mu F^{\mu\nu} = J^\nu \qquad (6.2.13)$$

where the electric current here is

$$J^\nu = \sum_a q_a \int \text{d}\lambda_a \frac{\text{d}z_a^\nu}{\text{d}\lambda_a}\frac{\delta^{(d)}\left(x - z(\lambda_a)\right)}{\sqrt[4]{g(x)g(z)}}; \qquad (6.2.14)$$

as well as the covariant Lorentz force law

$$m_a \frac{D^2 z_a^\mu}{\text{d}\tau_a^2} = q_a F^\mu{}_\nu \frac{\text{d}z_a^\nu}{\text{d}\tau_a}, \qquad (6.2.15)$$

where τ_a is the proper time of the ath point charge and covariant acceleration on the left-hand-side is

$$\frac{D^2 z_a^\mu}{\text{d}\tau_a^2} \equiv \frac{\text{d}^2 z_a^\mu}{\text{d}\tau_a^2} + \Gamma^\mu{}_{\alpha\beta}\frac{\text{d}z_a^\alpha}{\text{d}\tau_a}\frac{\text{d}z_a^\beta}{\text{d}\tau_a}. \qquad (6.2.16)$$

Gauge symmetry Let us observe S' in eq. (6.2.12) is gauge invariant. We already know the $F_{\mu\nu}F^{\mu\nu}$ is gauge invariant, so we only need to check the $A_\mu \text{d}x^\mu$ term. Upon the replacement $A_\mu \text{d}x^\mu \to A_\mu \text{d}x^\mu + \partial_\mu L \text{d}x^\mu = A_\mu \text{d}x^\mu + \text{d}L$, and as long as L is chosen to vanish at the initial t_i and final t_f times of the trajectories

$$\begin{aligned}\sum_a \int A_\mu(z_a)\text{d}x^\mu \to \sum_a &\left(\int A_\mu(z_a)\text{d}x^\mu + \int \text{d}L(z_a)\right) \\ &= \sum_a \left(\int A_\mu(z_a)\text{d}x^\mu + L(t_\text{f}, \vec{z}_a(t_\text{f})) - L(t_\text{i}, \vec{z}_a(t_\text{i}))\right) \\ &= \sum_a \int A_\mu(z_a)\text{d}x^\mu. \end{aligned} \qquad (6.2.17)$$

Variational calculation We demand the action be stationary under the variation of the gauge field as well as the individual trajectories $\{z_a^\mu\}$. By re-writing the $A_\mu \text{d}x^\mu$ terms as

$$-\sum_a q_a \int A_\mu \text{d}x^\mu = -\int_{t_\text{i}}^{t_\text{f}} \text{d}^d x\sqrt{|g(x)|}A_\mu(x)\sum_a q_a \int \text{d}\lambda_a \frac{\text{d}z_a^\mu}{\text{d}\lambda_a}\frac{\delta^{(d)}\left(x - z_a\right)}{\sqrt[4]{|g(x)g(z_a)|}}, \qquad (6.2.18)$$

the δ-functions tell us we are dealing with point charges. If we compare this expression against the $A_\mu J^\mu$ term in eq. (6.2.3) – namely, by reading off the coefficient of A_μ – this allows us to identify the electric current in eq. (6.2.14). We have thus arrived at eq. (6.2.13).

Next, we vary the action with respect to the ath trajectory z_a, namely

$$z_a^\mu \to z_a^\mu + \delta z_a^\mu; \tag{6.2.19}$$

assuming $\delta z_a(t_\mathrm{i}) = \delta z_a(t_\mathrm{f}) = 0$. The terms linear in the small perturbation δz_a^μ are

$$\begin{aligned}
\delta_{z_a} S' &= \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}\tau_a \delta z_a^\alpha g_{\alpha\beta} m_a \frac{D^2 z_a^\beta}{\mathrm{d}\tau_a^2} - q_a \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}\tau_a \left(\delta z_a^\alpha \partial_\alpha A_\beta \dot{z}_a^\beta + A_\beta \delta \dot{z}_a^\beta \right) \\
&= \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}\tau_a \delta z_a^\alpha \left\{ g_{\alpha\beta} \cdot m_a \frac{D^2 z_a^\beta}{\mathrm{d}\tau_a^2} - q_a \left(\partial_\alpha A_\beta - \partial_\beta A_\alpha \right) \dot{z}_a^\beta \right\} \\
&= \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}\tau_a \delta z_a^\alpha g_{\alpha\beta} \left\{ m_a \frac{D^2 z_a^\beta}{\mathrm{d}\tau_a^2} - q_a F^\beta{}_\gamma \dot{z}_a^\gamma \right\}.
\end{aligned} \tag{6.2.20}$$

We have thus recovered eq. (6.2.15). Note that, in the first equality above, we have changed variables from λ_a to proper time τ_a *after variation*.

Problem 6.2. Lorentz force law in 4D flat spacetime Express the 4D Minkowski spacetime version of the Lorentz force law in eq. (6.2.15) in terms of electric $E^i \equiv F^{i0}$ and magnetic fields $F^{ij} = \epsilon^{0ijk} B^k$. (The ϵ^{0ijk} is the Levi-Civita tensor in flat Minkowski spacetime, with $\epsilon_{0123} \equiv 1$.) Express your time derivatives with respect to coordinate time. You should find the zeroth component of eq. (6.2.15) to be redundant; to arrive at this conclusion more rapidly you may want to start with the action in eq. (6.2.12) but written in coordinate time t:

$$S'_\mathrm{pp} \equiv -\sum_{a=1}^N \left(m_a \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t \sqrt{\eta_{\mu\nu} \dot{z}_a^\mu \dot{z}_a^\nu} + q_a \int_{t_\mathrm{i}}^{t_\mathrm{f}} A_\mu(z_a) \dot{z}_a^\mu \mathrm{d}t \right), \tag{6.2.21}$$

where $\dot{z}_a^\mu \equiv \mathrm{d}z_a^\mu / \mathrm{d}t$. □

Problem 6.3. Lorenz gauge Let us impose the Lorenz gauge condition in curved spacetime,

$$\nabla^\mu A_\mu = 0. \tag{6.2.22}$$

Show that Maxwell's equation in eq. (6.2.9) then reads

$$\Box A_\nu - R_\nu{}^\sigma A_\sigma = J_\nu, \qquad \Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu = \nabla_\mu \nabla^\mu. \tag{6.2.23}$$

Hint: You may need to 'commute' the covariant derivatives in the term $\nabla^\mu \nabla_\nu A_\mu$. □

Problem 6.4. Weyl invariance in (3+1)D Consider replacing the metric $g_{\mu\nu}$ by multiplying it with an overall scalar function $\Omega(x)^2$, i.e.,

$$g_{\mu\nu}(x) \to \Omega(x)^2 g_{\mu\nu}(x). \tag{6.2.24}$$

Show that the Maxwell action in eq. (6.2.3) is invariant in 4 spacetime dimensions if we simultaneously make the replacement in eq. (6.2.24) and

$$J^\mu(x) \to \Omega(x)^p J^\mu(x) \tag{6.2.25}$$

for an appropriate p – what is p? – as well as

$$A_\mu(x) \to A_\mu(x). \tag{6.2.26}$$

□

Problem 6.5. Null Geodesics & Weyl Transformations Suppose two geometries $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ are related via a Weyl transformation

$$g_{\mu\nu}(x) = \Omega(x)^2 \bar{g}_{\mu\nu}(x). \tag{6.2.27}$$

Consider the null geodesic equation in the geometry $g_{\mu\nu}(x)$,

$$k'^\sigma \nabla_\sigma k'^\mu = 0, \qquad g_{\mu\nu} k'^\mu k'^\nu = 0 \tag{6.2.28}$$

where ∇ is the covariant derivative with respect to $g_{\mu\nu}$; as well as the null geodesic equation in $\bar{g}_{\mu\nu}(x)$,

$$k^\sigma \overline{\nabla}_\sigma k^\mu = 0, \qquad \bar{g}_{\mu\nu} k^\mu k^\nu = 0; \tag{6.2.29}$$

where $\overline{\nabla}$ is the covariant derivative with respect to $\bar{g}_{\mu\nu}$. Show that

$$k^\mu = \Omega^2 \cdot k'^\mu. \tag{6.2.30}$$

Hint: First show that the Christoffel symbol $\overline{\Gamma}^\mu{}_{\alpha\beta}[\bar{g}]$ built solely out of $\bar{g}_{\mu\nu}$ is related to $\Gamma^\mu{}_{\alpha\beta}[g]$ built out of $g_{\mu\nu}$ through the relation

$$\Gamma^\mu{}_{\alpha\beta}[g] = \bar{\Gamma}^\mu{}_{\alpha\beta}[\bar{g}] + \delta^\mu_{\{\beta} \overline{\nabla}_{\alpha\}} \ln \Omega - \bar{g}_{\alpha\beta} \overline{\nabla}^\mu \ln \Omega. \tag{6.2.31}$$

Remember to use the constraint $g_{\mu\nu} k'^\mu k'^\nu = 0 = \bar{g}_{\mu\nu} k^\mu k^\nu$.

A spacetime is said to be conformally flat if it takes the form

$$g_{\mu\nu}(x) = \Omega(x)^2 \eta_{\mu\nu}. \tag{6.2.32}$$

Solve the null geodesic equation explicitly in such a spacetime. □

Problem 6.6. Non-Minimal Electromagnetic-Gravitational Interactions in 4D Consider adding the following action to the Maxwell one of eq. (6.2.3):

$$S_{\text{EM-Gravity I}} \equiv \int_{t_i}^{t_f} \mathrm{d}^4 x \sqrt{|g|} C_{6,1} R^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}. \tag{6.2.33}$$

By ensuring the dimension of the Maxwell action in eq. (6.2.3) is the same as that of $S_{\text{EM-Gravity I}}$, determine the dimension of $C_{6,1}$. That is, $[C_{6,1}] = \text{Mass}^p$ – what is p? Write down as many such actions as you can with coefficients that share the same mass dimension. (Hint: Evaluate the $\sqrt{|g|}$ and Lagrangian density in a Fermi Normal Coordinate System.)

Note that quantum corrections to electromagnetism in 4D curved spacetimes does in fact generate such terms – and infinitely many more! – in addition to the Maxwell action of eq. (6.2.3). For low energy processes, photons interact with gravity in increasingly complicated ways through the exchange of virtual electron-positrons propagating in spacetime, and the coefficient $C_{6,1}$ and its analogs would scale as some power of $1/m_e$, where m_e is the electron mass. To see such interactions are indeed quantum in nature, put back the factors of $\hbar$; i.e., write down $S_{\text{EM-Gravity I}}$ as

$$S_{\text{EM-Gravity I}} = \int_{t_\text{i}}^{t_\text{f}} \mathrm{d}^4x \sqrt{|g|} C'_{6,1} \hbar^q m_e^p R^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}, \tag{6.2.34}$$

with the appropriate powers of q and p and some dimensionless $C'_{6,1}$. Finally, by comparing the length scales involved, i.e., $F_{\mu\nu}F^{\mu\nu}$ versus $C_{6,1}R^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$, describe qualitatively the relative importance of such quantum effects encoded with this $C_{6,1}$ term and its cousins. □

Problem 6.7. Hodge dual formulation of Maxwell's equations Define the dual of the Faraday tensor as

$$\widetilde{F}^{\mu_1 \dots \mu_{d-2}} \equiv \frac{1}{2} \widetilde{\epsilon}^{\mu_1 \dots \mu_{d-2} \alpha\beta} F_{\alpha\beta}. \tag{6.2.35}$$

Verify the Hodge dual formulation of Maxwell's equations (6.2.9) and (6.2.10):

$$\chi \cdot \widetilde{\epsilon}^{\mu_1 \dots \mu_{d-1} \nu} \nabla_{\mu_1} \widetilde{F}_{\mu_2 \dots \mu_{d-1}} = J^\nu, \tag{6.2.36}$$

$$\nabla_\sigma \widetilde{F}^{\sigma \mu_3 \dots \mu_{d-1}} = 0; \tag{6.2.37}$$

and work out the numerical constant χ. □

Problem 6.8. Second order form of Maxwell's Equations By taking the divergence of the Bianchi portion of Maxwell's equations, $\mathrm{d}F = 0$, followed by using the geometric Bianchi identity $R_{\mu[\nu\alpha\beta]} = 0$, show that

$$\Box F_{\alpha\beta} + R_{\mu\nu\alpha\beta} F^{\mu\nu} + R^\sigma{}_{[\alpha} F_{\beta]\sigma} = -\nabla_{[\alpha} \nabla^\sigma F_{\beta]\sigma}, \qquad \Box \equiv \nabla_\sigma \nabla^\sigma. \tag{6.2.38}$$

Next, use the other Maxwell's equation, $\mathrm{div} F = J$, to obtain the second order form of Maxwell's equations:

$$\Box F_{\alpha\beta} + R_{\mu\nu\alpha\beta} F^{\mu\nu} + R^\sigma{}_{[\alpha} F_{\beta]\sigma} = \nabla_{[\alpha} J_{\beta]}. \tag{6.2.39}$$

This indicates electromagnetic fields $F_{\mu\nu}$ obey a wave equation in curved spacetimes. □

6.3 JWKB Approximation and Gravitational Redshift

In this section we will apply the JWKB (more commonly dubbed WKB) approximation to study the vacuum (i.e., $J_\mu = 0$ limit of) Maxwell's equations in eq. (6.2.23). At leading orders in perturbation theory, we will argue – in the limit where the wavelength of the photons are much shorter than that of the background geometric curvature – that photons propagate on the light cone and their polarization tensors are largely parallel transported along their null geodesics. We will also see that the photon's phase S would allow us to define its frequency as the number density of constant-S surfaces piercing the timelike worldline of the observer. This also leads us to recognize that, not only is $k^\mu \equiv \nabla^\mu S$ null it obeys the geodesic equation $k^\sigma \nabla_\sigma k^\mu = 0$.

Eikonal/Geometric Optics/JWKB Ansatz We will begin by postulating that the vector potential can be modeled as the (real part of) a slowly varying amplitude a_μ multiplied by a rapidly oscillating phase $\exp(iS)$:

$$A_\mu = \text{Re}\left\{a_\mu \exp(iS/\epsilon)\right\}. \tag{6.3.1}$$

[57]The $\{a_\mu\}$ can be complex but S is real. We shall also allow the amplitude itself to be a power series in ϵ:

$$a_\mu = \sum_{\ell=0}^{\infty} \epsilon^\ell \, {}_\ell a_\mu. \tag{6.3.2}$$

The $0 < \epsilon \ll 1$ is a fictitious parameter that reminds us of the hierarchy of length scales in the problem – specifically, ϵ should be viewed as the ratio between the short wavelength of the photon to the long wavelength of the background geometric curvature. To this end, we shall re-write the vacuum version of the Lorenz-gauge Maxwell's equation (6.2.23) with ϵ^2 multiplying the wave operator $\Box$:

$$\Box A_\mu - \epsilon^2 R_\mu{}^\sigma A_\sigma = 0. \tag{6.3.3}$$

In a locally freely-falling frame (i.e., flat coordinate system), this equation takes the schematic form

$$\partial^2 A - \epsilon^2 (\partial^2 g) A = 0. \tag{6.3.4}$$

The first term from the left goes as $A/(\text{wavelength of } A)^2$ while the second as $A/(\text{wavelength of } g)^2$, and as thus already advertised ϵ^2 is a power counting parameter reminding us of the relative strength of the two terms.

Wave Equation Plugging the ansatz of eq. (6.3.1) into eq. (6.3.3):

$$\begin{aligned} 0 &= (\Box a_\mu - \epsilon^2 R_\mu{}^\sigma a_\sigma) e^{iS/\epsilon} + 2\nabla_\sigma a_\mu \frac{i}{\epsilon} \nabla^\sigma S \cdot e^{iS/\epsilon} + a_\mu \nabla_\sigma \left(i(\nabla^\sigma S/\epsilon) e^{iS/\epsilon}\right) \\ &= (\Box a_\mu - \epsilon^2 R_\mu{}^\sigma a_\sigma) e^{iS/\epsilon} + 2\nabla_\sigma a_\mu \frac{i}{\epsilon} (\nabla^\sigma S) \cdot e^{iS/\epsilon} + a_\mu \left(i(\Box S/\epsilon) e^{iS/\epsilon} + (i\nabla S/\epsilon)^2 e^{iS/\epsilon}\right). \end{aligned} \tag{6.3.5}$$

Employing the power series of eq. (6.3.2),

$$\begin{aligned} 0 = \; &\Box \, {}_0 a_\mu + \epsilon \Box \, {}_1 a_\mu + \epsilon^2 \Box \, {}_2 a_\mu + \dots \\ &- R_\mu{}^\sigma \left(\epsilon^2 \, {}_0 a_\sigma + \epsilon^3 \, {}_1 a_\sigma + \dots\right) \\ &+ 2i\epsilon^{-1} (\nabla \, {}_0 a_\mu \cdot \nabla S) + 2i\epsilon^0 (\nabla \, {}_1 a_\mu \cdot \nabla S) + 2i\epsilon (\nabla \, {}_2 a_\mu \cdot \nabla S) + \dots \end{aligned}$$

[57]Recall that this ansatz becomes an exact solution in Minkowski spacetime, where $S = \pm k_\mu x^\mu$ and both k_μ and a_μ are constant.

$$
\begin{aligned}
&+ i\epsilon^{-1}\,{}_0a_\mu\Box S + i\epsilon^0\,{}_1a_\mu\Box S + i\epsilon\,{}_2a_\mu\Box S + \dots \\
&- (\nabla S)^2\left(\epsilon^{-2}\,{}_0a_\mu + \epsilon^{-1}\,{}_1a_\mu + \epsilon^0\,{}_2a_\mu + \epsilon\,{}_3a_\mu + \dots\right).
\end{aligned} \tag{6.3.6}
$$

Negative Two Setting the coefficient of ϵ^{-2} to zero

$$
k_\mu k^\mu = 0, \qquad k_\mu \equiv \nabla_\mu S. \tag{6.3.7}
$$

Because S is a scalar, $\nabla_\nu k_\mu = \nabla_\nu\nabla_\mu S = \nabla_\mu\nabla_\nu S = \nabla_\mu k_\nu$ and hence

$$
0 = \nabla_\nu(k^2) = 2k^\mu\nabla_\nu k_\mu = 2k^\mu\nabla_\mu k_\nu. \tag{6.3.8}
$$

That is, the gradient of the phase S sweeps out null geodesics in spacetime:

$$
(k\cdot\nabla)k^\mu = 0. \tag{6.3.9}
$$

Negative One Setting the coefficient of ϵ^{-1} to zero:

$$
0 = {}_0a_\mu\Box S + 2\nabla^\sigma S\nabla_\sigma\,{}_0a_\mu, \tag{6.3.10}
$$

$$
0 = \overline{{}_0a_\mu}\Box S + 2\nabla^\sigma S\nabla_\sigma\,\overline{{}_0a_\mu}, \tag{6.3.11}
$$

where the second line is simply the complex conjugate of the first. Note that $\nabla|a|^2 = (\nabla a)\bar{a} + a(\nabla\bar{a})$. Guided by this, we may multiply the first equation by $\overline{{}_0a^\mu}$ and the second equation by ${}_0a^\mu$, followed by adding them.

$$
0 = |\,{}_0a|^2\Box S + 2\,\overline{{}_0a^\mu}\nabla^\sigma S\nabla_\sigma\,{}_0a_\mu \tag{6.3.12}
$$

$$
0 = |\,{}_0a|^2\Box S + 2\,{}_0a^\mu\nabla^\sigma S\nabla_\sigma\,\overline{{}_0a_\mu} \tag{6.3.13}
$$

$$
0 = 2|\,{}_0a|^2\Box S + 2\nabla^\sigma S\nabla_\sigma|\,{}_0a|^2, \qquad |\,{}_0a|^2 \equiv {}_0a_\mu\,\overline{{}_0a^\mu}. \tag{6.3.14}
$$

The right hand side of the final equation can be expressed as a divergence.

$$
0 = \nabla_\sigma\left(|\,{}_0a|^2\nabla^\sigma S\right) = \nabla_\sigma\left(|\,{}_0a|^2 k^\sigma\right) \tag{6.3.15}
$$

Up to an overall normalization constant, we may interpret $n^\sigma \equiv |\,{}_0a|^2k^\sigma$ as a photon number current, and this equation as its conservation law.

We turn to examining the derivative along $k \equiv \nabla S$ the normalized leading order photon amplitude ${}_0a_\mu/\sqrt{|\,{}_0a|^2}$:

$$
\nabla^\sigma S\nabla_\sigma\left(\frac{{}_0a_\mu}{\sqrt{|\,{}_0a|^2}}\right) = \frac{\nabla^\sigma S\nabla_\sigma\,{}_0a_\mu}{\sqrt{|\,{}_0a|^2}} - \frac{{}_0a_\mu}{2(|\,{}_0a|^2)^{3/2}}\nabla^\sigma S\nabla_\sigma|\,{}_0a|^2. \tag{6.3.16}
$$

Eq. (6.3.15) says $\nabla S\cdot\nabla|\,{}_0a|^2 = -|\,{}_0a|^2\Box S$, while eq. (6.3.10), in turn, states ${}_0a_\mu\Box S = -2\nabla^\sigma S\nabla_\sigma\,{}_0a_\mu$.

$$
\begin{aligned}
\nabla^\sigma S\nabla_\sigma\left(\frac{{}_0a_\mu}{\sqrt{|\,{}_0a|^2}}\right) &= \frac{\nabla^\sigma S\nabla_\sigma\,{}_0a_\mu}{\sqrt{|\,{}_0a|^2}} + \frac{|\,{}_0a|^2}{2(|\,{}_0a|^2)^{3/2}}\,{}_0a_\mu\Box S \\
&= \frac{\nabla^\sigma S\nabla_\sigma\,{}_0a_\mu}{\sqrt{|\,{}_0a|^2}} - \frac{\nabla^\sigma S\nabla_\sigma\,{}_0a_\mu}{\sqrt{|\,{}_0a|^2}} = 0.
\end{aligned} \tag{6.3.17}
$$

Lorenz gauge Let us not forget the Lorenz gauge condition: $0 = \nabla^\mu A_\mu = ((\nabla^\mu a_\mu) + (i/\epsilon)\nabla^\mu S a_\mu)e^{iS/\epsilon}$.

$$\begin{aligned} 0 &= \nabla^\mu {}_0a_\mu + \epsilon\nabla^\mu {}_1a_\mu + \epsilon^2\nabla^\mu {}_2a_\mu + \dots \\ &\quad + i\epsilon^{-1}\nabla S \cdot {}_0a + i\epsilon^0\nabla S \cdot {}_1a + i\epsilon\nabla S \cdot {}_2a + \dots \end{aligned} \tag{6.3.18}$$

Negative One Setting the coefficient of ϵ^{-1} to zero, we find the leading order polarization vector must be orthogonal to the wave vector:

$$k^\mu {}_0a_\mu = 0. \tag{6.3.19}$$

Zero Setting the coefficient of ϵ^0 to zero,

$$k^\mu {}_1a_\mu = i\nabla^\mu {}_0a_\mu. \tag{6.3.20}$$

This is telling us that the polarization vector does not remain perpendicular to k^μ at the next order.

To summarize, we have worked out the first two orders of the Lorenz gauge vacuum Maxwell's equations in the JWKB/eikonal/geometric optics limit. Up to this level of accuracy, perturbation theory teaches us:

- The gradient of the phase of the photon field $k^\mu \equiv \nabla^\mu S$ – which we may interpret as its dominant direction of propagation – follows null geodesics in the curved spacetime.
- The photon number current is covariantly conserved.
- The normalized polarization vector is parallel transported along k^μ.
- This same wave vector is orthogonal to the polarization of the photon at leading order; and the first deviation to non-orthogonality occurring at the next order is proportional to the divergence of the polarization vector itself.

Problem 6.9. Electromagnetic Fields In classical theory, it is the electromagnetic fields $F_{\mu\nu}$ that are directly observable, as opposed to the gauge-dependent vector potential A_μ. In the leading order of the JWKB approximation, argue that

$$F_{\mu\nu} \approx \text{Re}\left\{\frac{i}{\epsilon}k_{[\mu}a_{\nu]}\exp\left(iS/\epsilon\right)\right\}, \qquad k_\mu \equiv \nabla_\mu S. \tag{6.3.21}$$

Why is $k^\mu F_{\mu\nu} \approx 0$? This might appear at first sight to depend on the Lorenz gauge condition in eq. (6.3.19), but argue that the Lorenz gauge condition continues to hold – at the leading JWKB approximation – upon any gauge transformation of the form

$$A_\mu \to A_\mu + \text{Re}\left\{\nabla_\mu\left(\ell \cdot \exp(iS/\epsilon)\right)\right\}, \tag{6.3.22}$$

where ℓ is a slowly varying function of spacetime compared to the phase $\exp(iS/\epsilon)$. □

Gravitational Redshift As alluded to at the beginning of this section, the frequency of light according to a timelike observer may be defined as the number density of constant phase surfaces piercing its worldline. This, in turn, may be formalized using the unit normal vector $u^\mu \partial_\mu$ tangent to the said worldline:

$$\omega \equiv |u \cdot \nabla S| = \left| \frac{\mathrm{d}S}{\mathrm{d}\tau} \right| = k^{\widehat{0}} = k_{\widehat{0}}, \tag{6.3.23}$$

where τ is the observer's proper time. In other words, the frequency *is* the zeroth component of the wave vector ($\equiv$ momentum) in an orthonormal basis in the observer's frame.

Static Spherically Symmetric Metrics Near the surface of the Earth, we may model its geometry – at least as a first pass! – as a static spherically symmetric one, given by

$$\mathrm{d}s^2 = (A(r)\mathrm{d}t)^2 - (B(r)\mathrm{d}r)^2 - r^2\mathrm{d}\Omega^2, \tag{6.3.24}$$

$$A(r) = \sqrt{1 - \frac{r_{s,\mathrm{E}}}{r}}, \qquad B(r) = \frac{1}{\sqrt{1 - r_{s,\mathrm{E}}/r}}, \qquad r_{s,\mathrm{E}} \equiv 2G_\mathrm{N} M_\mathrm{E}. \tag{6.3.25}$$

The associated Lagrangian for the geodesic equation is

$$L = \frac{1}{2}\left((Ak^0)^2 - (Bk^r)^2 - (rk^\theta)^2 - (r\sin(\theta)k^\phi)^2 \right); \tag{6.3.26}$$

where $k^\mu \equiv \mathrm{d}(t, r, \theta, \phi)^\mu/\mathrm{d}\lambda$. (Remember this Lagrangian yields $k^\mu \nabla_\mu k^\nu = 0$.) The static assumption allows us to immediately identify 'energy' E as the conserved quantity

$$E = \frac{\partial L}{\partial k^0} = A^2 k^0. \tag{6.3.27}$$

An observer at a fixed position ($\mathrm{d}r = \mathrm{d}\theta = 0$) has proper time

$$\mathrm{d}\tau = A(r)\mathrm{d}t. \tag{6.3.28}$$

This allows us to identify the 0th vierbein $\varepsilon^{\widehat{0}}{}_\mu \mathrm{d}x^\mu = A\mathrm{d}t$ as the observer worldline's unit timelike tangent vector. That, in turn, inform us eq. (6.3.27) is now

$$E = A(r)k^{\widehat{0}} = A(r)\omega(r) \qquad \Rightarrow \qquad \omega(r) = \frac{E}{A(r)}. \tag{6.3.29}$$

Here, we have recalled from our JWKB discussion above that $k^{\widehat{0}}$ *is* the frequency of the photon measured by our observer. We may now send electromagnetic waves between observers at different radii r_1 and r_2 (say, the bottom and top ends of the Pound-Rebka experiment) near the surface of the Earth:

$$\frac{\omega(r_2)}{\omega(r_1)} = \frac{A(r_1)}{A(r_2)}. \tag{6.3.30}$$

Compare this result to the time dilation result we worked out in Problem (2.29). See also the Wikipedia article on the Pound-Rebka experiment, the first verification of the gravitational time dilation effect. □

Problem 6.10. Co-Moving Redshift in Cosmology At large scales, we live in a universe well described by a spatially flat Friedmann-Lemaître–Robertson–Walker (FLRW) universe:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a(t)^2 \mathrm{d}\vec{x} \cdot \mathrm{d}\vec{x}, \qquad a(t) > 0. \tag{6.3.31}$$

The observers at rest in this geometry – the ones that witness a perfectly isotropic Cosmic Microwave Background sky (i.e., with a zero dipole) – have trajectories described by

$$Z^\mu = (t, \vec{Z}_0), \qquad \vec{Z}_0 \text{ constant}. \tag{6.3.32}$$

Exploiting the spatial translation symmetry of this geometry, we may postulate the following ansatz for the Lorenz gauge vector potential:

$$A_\mu = \mathrm{Re}\left\{ a_\mu(kt) e^{i\Sigma(t)} e^{i\vec{k}\cdot\vec{x}} \right\}, \qquad k \equiv |\vec{k}|; \tag{6.3.33}$$

where the slowly-varying amplitude a_μ does not depend on the spatial coordinates $\{x^i\}$.

Show that, to leading order, the frequency of the photon according to a co-moving observer redshifts as $1/a$. Specifically, demonstrate that

$$\frac{\omega(t_{\text{observer}})}{\omega(t_{\text{emission}})} = \frac{a(t_{\text{emission}})}{a(t_{\text{observer}})} \equiv (1+z)^{-1}; \tag{6.3.34}$$

where z is dubbed the redshift parameter. Since z is observable[58] – oftentimes inferred using atomic spectral lines – we may employ it to determine the epoch of emission of a given signal. □

[58]Example: the Cosmic Microwave Background radiation is roughly 2.7 Kelvins, emitted from 'the last scattering surface' roughly 380,000 years after the Big Bang and as observed from our vantage point has $z \approx 1,100$.

7 Dynamics for Spacetime Geometry: General Relativity

7.1 Einstein-Hilbert Action & Einstein's Field Equations

We now turn to the action principle for gravitation itself. Up till this point, we have been discussing "kinematical" aspects of curved spacetimes – geodesic equations obeyed by isolated bodies, geometric curvature exerting tidal forces on macroscopic systems, etc. We now turn to its dynamics: what generates a given spacetime in the first place; what is its 'source'? This will lead us to Einstein's equations for General Relativity. In all, we will gather that the study of the dynamics of spacetime cannot, in general, be divorced from the study of the dynamics of other fields and the motion of material bodies residing within it.

> "Spacetime tells matter how to move; matter tells spacetime how to curve."
> John A. Wheeler, in *Geons, Black Holes, and Quantum Foam* (2000), p. 235.

One reason why we spent the previous Chapter (§(6)) discussing action principles is because we will now obtain the equations of General Relativity itself by first seeking its corresponding 'Einstein-Hilbert' action. The first guiding principle to an action-based gravitational dynamics is that we need to find a Lagrangian density that is a coordinate scalar, since we would definitely want the ensuing equations for the metric to coordinate-transform covariantly as tensors. The available scalars intrinsic to the geometry itself must be built out of the Riemann tensor; Ricci tensor and scalar; and the Levi-Civita pseudo-tensor. The second guiding principle is that the resulting equations for the metric should not contain terms with more than 2 time derivatives. This is closely related to why Newton's second law is the way it is: why does particle mechanics involve the second time derivative of position, but no higher derivatives? A theorem due to Ostrogradsky[59] tells us higher derivative systems with non-degenerate Lagrangians that also interact with other systems are generically unstable in that their energy can become arbitrarily negative. This means they can impart an infinite amount of positive energy to other systems while respecting energy conservation, by simply lowering their own energy to negative infinity.

It turns out, in (3+1)D, these two guiding principles lead us to the following Einstein-Hilbert action:[60]

$$S_{\mathrm{GR}} \equiv -\frac{1}{16\pi G_{\mathrm{N}}} \int \mathrm{d}^4x \sqrt{|g|} \left(\mathcal{R} + 2\Lambda\right). \tag{7.1.1}$$

The coefficient $-(16\pi G_{\mathrm{N}})^{-1}$ is determined by ensuring that Newtonian gravity is recovered in the weak gravity limit; whereas Λ is known as the cosmological constant. Since the discovery of the accelerating expansion of the universe[61] the case for a non-zero Λ in our universe has strengthened considerably. We now turn to understanding how to obtain from eq. (7.1.1) Einstein's equations for General Relativity. To this end, we will in fact require that the Einstein-Hilbert action in eq. (7.1.1) plus an arbitrary matter action be stationary under the variation of spacetime geometry

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}. \tag{7.1.2}$$

[59]See Woodard [11] arXiv: 1506.02210 for a detailed exposition

[60]In higher dimensions, Lovelock has worked out other possibilities consistent with the above two principles.

[61]See the resulting Nobel prize citation here.

That is, if

$$S_{\text{total}} \equiv S_{\text{GR}} + \int \mathrm{d}^d x \sqrt{|g|} \mathcal{L}_{\text{matter}}, \tag{7.1.3}$$

we would like to examine the consequence of $\delta_g S_{\text{total}} = 0$.

We will relegate the technical details to a later section, but collect here the relevant results. The first order variation of the volume factor $\sqrt{|g|}$ and the Ricci scalar, upon the variation of the metric is

$$\delta_g \sqrt{|g|} = -\sqrt{|g|} \cdot \frac{1}{2} \delta g^{\alpha\beta} g_{\alpha\beta}, \tag{7.1.4}$$

$$= \sqrt{|g|} \cdot \frac{1}{2} g^{\alpha\beta} \delta g_{\alpha\beta}; \tag{7.1.5}$$

and

$$\delta_g \mathcal{R} = \delta g^{\alpha\beta} R_{\alpha\beta} - \nabla_\alpha \nabla_\beta \delta g^{\alpha\beta} + g_{\alpha\beta} \Box \delta g^{\alpha\beta} \tag{7.1.6}$$

$$= -\delta g_{\alpha\beta} R^{\alpha\beta} + \nabla^\alpha \nabla^\beta \delta g_{\alpha\beta} - g^{\alpha\beta} \Box \delta g_{\alpha\beta}. \tag{7.1.7}$$

A note of caution: $\delta g^{\alpha\beta} \neq g^{\alpha\mu} g^{\beta\nu} \delta g_{\mu\nu}$ and $\delta g^{\alpha\beta} g_{\alpha\mu} g_{\beta\nu} \neq \delta g_{\mu\nu}$. Rather, because

$$g^{\alpha\sigma} g_{\sigma\nu} = \delta^\alpha_\nu \qquad \Rightarrow \qquad \delta g^{\alpha\sigma} g_{\sigma\nu} = -g^{\alpha\sigma} \delta g_{\sigma\nu}. \tag{7.1.8}$$

Contracting both sides with $g_{\alpha\mu}$ and with $g^{\nu\beta}$,

$$\delta g^{\alpha\beta} g_{\alpha\mu} g_{\beta\nu} = -\delta g_{\mu\nu}, \tag{7.1.9}$$

$$\delta g^{\alpha\beta} = -\delta g_{\mu\nu} g^{\alpha\mu} g^{\beta\nu}. \tag{7.1.10}$$

That is, respectively raising and lowering both indices of $\delta g_{\alpha\beta}$ and $\delta g^{\alpha\beta}$ cost a minus sign. In the same vein,

$$\frac{\partial}{\partial g^{\alpha\beta}} g^{\mu\nu} = \frac{1}{2} \delta^{\{\mu}_\alpha \delta^{\nu\}}_\beta, \tag{7.1.11}$$

$$\frac{\partial}{\partial g^{\alpha\beta}} g_{\mu\nu} = -\frac{1}{2} g_{\alpha\{\mu} g_{\nu\}\beta}, \tag{7.1.12}$$

$$\frac{\partial}{\partial g_{\alpha\beta}} g_{\mu\nu} = \frac{1}{2} \delta^\alpha_{\{\mu} \delta^\beta_{\nu\}}, \tag{7.1.13}$$

$$\frac{\partial}{\partial g_{\alpha\beta}} g^{\mu\nu} = -\frac{1}{2} g^{\alpha\{\mu} g^{\nu\}\beta}. \tag{7.1.14}$$

With these results, and performing the variation in $d-$dimensions to emphasize the results are for the most past really dimension independent,

$$\delta_g S_{\text{total}} = -\frac{1}{16\pi G_{\text{N}}} \int \mathrm{d}^d x \Bigg(\delta_g \sqrt{|g|} \left(\mathcal{R} + 2\Lambda - 16\pi G_{\text{N}} \mathcal{L}_{\text{matter}} \right) + \sqrt{|g|} \delta_g \mathcal{R} - 16\pi G_{\text{N}} \sqrt{|g|} \delta g^{\alpha\beta} \frac{\partial \mathcal{L}_{\text{matter}}}{\partial g^{\alpha\beta}} \Bigg) \tag{7.1.15}$$

$$= -\frac{1}{16\pi G_{\rm N}} \int {\rm d}^d x \sqrt{|g|} \delta g^{\alpha\beta} \Bigg\{ R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \mathcal{R} - \Lambda g_{\alpha\beta}$$

$$- 8\pi G_{\rm N} \left(2\frac{\partial \mathcal{L}_{\rm matter}}{\partial g^{\alpha\beta}} - g_{\alpha\beta} \mathcal{L}_{\rm matter} \right) \Bigg\} \tag{7.1.16}$$

$$+ (16\pi G_{\rm N})^{-1} \int {\rm d}^{d-1} \Sigma_\alpha \left(\nabla_\beta \delta g^{\alpha\beta} - g_{\mu\nu} \nabla^\alpha \delta g^{\mu\nu} \right). \tag{7.1.17}$$

Equivalently,

$$\delta_g S_{\rm total} = +\frac{1}{16\pi G_{\rm N}} \int {\rm d}^d x \sqrt{|g|} \delta g_{\alpha\beta} \Bigg\{ R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} \mathcal{R} - \Lambda g^{\alpha\beta}$$

$$- 8\pi G_{\rm N} \left(-2\frac{\partial \mathcal{L}_{\rm matter}}{\partial g_{\alpha\beta}} - g^{\alpha\beta} \mathcal{L}_{\rm matter} \right) \Bigg\} \tag{7.1.18}$$

$$- (16\pi G_{\rm N})^{-1} \int {\rm d}^{d-1} \Sigma_\alpha \left(\nabla^\beta \delta g_{\alpha\beta} - g^{\mu\nu} \nabla^\alpha \delta g_{\mu\nu} \right). \tag{7.1.19}$$

Note: while raising and lowering both indices of $\delta g_{\alpha\beta}$ and $\delta g^{\alpha\beta}$ cost a minus sign, $\delta_g \mathcal{L}_{\rm matter} = (\partial \mathcal{L}_{\rm matter}/\partial g_{\mu\nu}) \delta g_{\mu\nu} = (\partial \mathcal{L}_{\rm matter}/\partial g^{\mu\nu}) \delta g^{\mu\nu}$.

Setting to zero the coefficient of $\delta g^{\alpha\beta}$ or $\delta g_{\alpha\beta}$ in these variational results,[62] we have arrived at the celebrated Einstein's equations for General Relativity, describing how spacetime curvature is sourced by matter in a given physical system:

$$\boxed{G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G_{\rm N} T_{\mu\nu},} \tag{7.1.20}$$

[63]where the Einstein tensor is

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R}; \tag{7.1.21}$$

and the energy-momentum-shear-stress tensor (often dubbed stress-energy tensor or stress tensor for short) is computed from the matter Lagrangian density as

$$T_{\mu\nu} \equiv 2\frac{\partial \mathcal{L}_{\rm matter}}{\partial g^{\mu\nu}} - g_{\mu\nu} \mathcal{L}_{\rm matter} \tag{7.1.22}$$

$$= \frac{2}{\sqrt{|g|}} \frac{\partial}{\partial g^{\mu\nu}} \left(\sqrt{|g|} \mathcal{L}_{\rm matter} \right),$$

$$T^{\mu\nu} \equiv -2\frac{\partial \mathcal{L}_{\rm matter}}{\partial g_{\mu\nu}} - g^{\mu\nu} \mathcal{L}_{\rm matter} \tag{7.1.23}$$

[62]A remark is in order regarding the surface terms in equations (7.1.17) and (7.1.19). Variational principles are usually defined by requiring the spatial boundary conditions of the fields – and not their derivatives – be fixed; and similarly, initial and final field configurations – and not their derivatives – be specified. Strictly speaking, therefore, in addition to the Einstein-Hilbert action, a York-Gibbons-Hawking action [12, 13] involving the extrinsic curvature is needed on the boundary of the spacetime region we are applying the action principle.

[63]Eq. (7.1.20) is valid in any dimension greater than 2. In 1D, space(time) is always flat; while in 2D the Einstein tensor is identically zero, and eq. (7.1.20) becomes $g_{\mu\nu} = -(8\pi G_{\rm N}/\Lambda) T_{\mu\nu}$.

$$= -\frac{2}{\sqrt{|g|}}\frac{\partial}{\partial g_{\mu\nu}}\left(\sqrt{|g|}\mathcal{L}_{\text{matter}}\right).$$

I personally find these formulas confusing to use because of the minus signs incurred in equations (7.1.12) and (7.1.14), which can in turn be traced back to the signs incurred from raising/lowering the indices of $\delta g_{\alpha\beta}$ and $\delta g^{\alpha\beta}$. On the other hand, from equations (7.1.16) and (7.1.18), we see that one may instead vary the matter action any manner we please and simply read off the stress tensor $T^{\alpha\beta}$ as the coefficient of $-(1/2)\sqrt{|g|}\delta g_{\alpha\beta}$,

$$\delta_g S_{\text{matter}} = -\frac{1}{2}\int \mathrm{d}^d x\sqrt{|g|}\delta g_{\alpha\beta}T^{\alpha\beta}; \tag{7.1.24}$$

or the stress tensor $T_{\alpha\beta}$ as the coefficient of $+(1/2)\sqrt{|g|}\delta g^{\alpha\beta}$,

$$\delta_g S_{\text{matter}} = +\frac{1}{2}\int \mathrm{d}^d x\sqrt{|g|}\delta g^{\alpha\beta}T_{\alpha\beta}. \tag{7.1.25}$$

Problem 7.1. Einstein's Equation for Ricci Tensor In $d-$spacetime dimensions, show that Einstein's equations in eq. (7.1.20) can be re-written as

$$R_{\mu\nu} = -\frac{2\Lambda}{d-2}g_{\mu\nu} + 8\pi G_{\text{N}}\left(T_{\mu\nu} - \frac{g_{\mu\nu}}{d-2}T\right), \qquad T \equiv g^{\sigma\rho}T_{\sigma\rho}. \tag{7.1.26}$$

Hint: Start by taking the "trace" of eq. (7.1.20) – i.e., contract both sides with the inverse metric. This will allow you to solve the Ricci scalar in terms of Λ and T. □

Problem 7.2. Second order form of Einstein's Equations Starting from the Bianchi identity in eq. (2.4.47), first show that

$$\Box R^{\mu\nu}{}_{\alpha\beta} = -\nabla^\gamma\nabla_{[\alpha}R^{\mu\nu}{}_{\beta]\gamma}, \tag{7.1.27}$$

$$\nabla_\gamma R^{\gamma\nu}{}_{\alpha\beta} = \nabla_{[\alpha}R^{\nu}{}_{\beta]}. \tag{7.1.28}$$

Use these results and General Relativity in the form of eq. (7.1.26) to gather that

$$\Box R^{\mu\nu}{}_{\alpha\beta} + [\nabla^\gamma, \nabla_{[\alpha}]R^{\mu\nu}{}_{\beta]\gamma} = 8\pi G_{\text{N}}\nabla_{[\alpha}\nabla^{[\mu}\left(T^{\nu]}{}_{\beta]} - \delta^{\nu]}{}_{\beta]}\frac{T}{d-2}\right). \tag{7.1.29}$$

Note that the $[\nabla^\gamma, \nabla_{[\alpha}]R^{\mu\nu}{}_{\beta]\gamma}$ is really an expression quadratic in the Riemann tensor. Therefore eq. (7.1.29) may be viewed as a nonlinear wave equation for the Riemann tensor sourced by matter. □

Stress-Tensor Example As an example, let us work out the stress tensor of the scalar field in eq. (6.1.5). With

$$\mathcal{L}[\varphi] \equiv \frac{1}{2}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - V(\varphi), \tag{7.1.30}$$

we have the variation

$$\delta_g \left\{ \int \mathrm{d}^d x \sqrt{|g|} \mathcal{L}[\varphi] \right\} = \int \mathrm{d}^d x \sqrt{|g|} \left(-\frac{1}{2} \delta g^{\mu\nu} g_{\mu\nu} \mathcal{L}[\varphi] + \frac{1}{2} \delta g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \right)$$
$$= \frac{1}{2} \int \mathrm{d}^d x \sqrt{|g|} \delta g^{\mu\nu} \left(\partial_\mu \varphi \partial_\nu \varphi - g_{\mu\nu} \mathcal{L}[\varphi] \right). \qquad (7.1.31)$$

Therefore, the stress tensor of the scalar field of eq. (6.1.5) is

$$T_{\mu\nu}[\varphi] = \nabla_\mu \varphi \nabla_\nu \varphi - g_{\mu\nu} \left(\frac{1}{2} \nabla_\sigma \varphi \nabla^\sigma \varphi - V[\varphi] \right). \qquad (7.1.32)$$

Problem 7.3. Electromagnetic Stress Tensor Starting from the electromagnetic action

$$S_{\text{Maxwell}} = -\frac{1}{4} \int \mathrm{d}^d x \sqrt{|g|} F_{\mu\nu} F^{\mu\nu}, \qquad F_{\mu\nu} = \partial_{[\mu} A_{\nu]}, \qquad (7.1.33)$$

show that the stress-energy tensor of electromagnetic fields is

$$T_{\alpha\beta} = -F_{\alpha\sigma} F_\beta{}^\sigma + \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu}. \qquad (7.1.34)$$

Comment on the sign convention dependence of this expression – i.e., what happens to it if you use the 'mostly plus' metric?

4D θ-term In $(3+1)$D consider instead the following action

$$S'_{\text{EM}} \equiv \int \mathrm{d}^4 x \sqrt{|g|} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \theta \widetilde{F}^{\mu\nu} F_{\mu\nu} \right), \qquad (7.1.35)$$
$$\widetilde{F}^{\mu\nu} \equiv \frac{1}{2} \widetilde{\epsilon}^{\mu\nu\alpha\beta} F_{\alpha\beta}; \qquad (7.1.36)$$

where θ is just a numerical constant. What is the stress tensor of this theory? What are the equations-of-motion for the gauge field A_μ? Hint: For the last question, show that

$$\widetilde{\epsilon}^{\mu\nu\alpha\beta} (\nabla_\mu A_\nu)(\nabla_\alpha A_\beta) = \nabla_\mu \left(\widetilde{\epsilon}^{\mu\nu\alpha\beta} A_\nu (\nabla_\alpha A_\beta) \right). \qquad (7.1.37)$$

4D Cosmology In (3+1)D spatially flat cosmologies, where in Cartesian coordinates

$$\mathrm{d}s^2 = a(\eta)^2 \eta_{\mu\nu}, \qquad (7.1.38)$$

show that the electromagnetic stress tensor $T^{(\text{FLRW})}_{\widehat{\alpha}\widehat{\beta}}$ in an orthonormal basis $\{\varepsilon^{\widehat{\alpha}}{}_\mu = a(\eta) \delta^\alpha_\mu\}$ is related to its flat spacetime counterpart $\overline{T}_{\widehat{\alpha}\widehat{\beta}}$ as

$$T^{(\text{FLRW})}_{\widehat{\alpha}\widehat{\beta}} = a(\eta)^{-4} \overline{T}_{\widehat{\alpha}\widehat{\beta}}. \qquad (7.1.39)$$

This $1/a^4$ describes how the energy-momentum of photons redshifts in cosmology. □

Problem 7.4. Point Particle Stress Tensor Starting from the action of a point particle of mass m,

$$S_{\rm pp} \equiv -m \int \mathrm{d}\lambda \sqrt{g_{\mu\nu}\dot{z}^\mu \dot{z}^\nu}, \qquad \dot{z}^\mu \equiv \frac{\mathrm{d}z^\mu}{\mathrm{d}\lambda}; \tag{7.1.40}$$

show that its stress-energy tensor written using its proper time τ is

$$_{\rm (pp)}T^{\mu\nu} = m \int \mathrm{d}\tau \dot{z}^\mu \dot{z}^\nu \frac{\delta^{(d)}(x - z(\tau))}{\sqrt[4]{g(x)g(z)}}, \qquad \dot{z}^\mu \equiv \frac{\mathrm{d}z^\nu}{\mathrm{d}\tau}. \tag{7.1.41}$$

□

Geodesics from General Covariance In these notes, we have emphasized the role of symmetries as an important guiding principle to understanding dynamics of both matter and gravitation. In this spirit, we may see that the geodesic equation for the collective motion of a material body of total mass m may also be argued from symmetry. Specifically, let us attempt to follow the motion of its center-of-mass $z^\mu(\lambda)$, where λ is some appropriate parametrization. Since $z^\mu(\lambda)$ sweeps out a worldline in spacetime, we may postulate that its dynamics may be derived from an action principle $S_{\rm pp}$ that is generally-covariant; namely, $S_{\rm pp}$ needs to be a coordinate scalar and $\delta_{z^\mu} S_{\rm pp} = 0$ would yield its dynamics. Just as we used the d-dimensional volume measure $\mathrm{d}^d x\sqrt{|g|}$ to integrate a Lagrangian density to form a coordinate scalar encoding the dynamics of some spacetime-filling field theory, to associate a coordinate scalar with a 1D worldline, we shall first figure out the appropriate 1D volume measure. To this end, the induced 1D metric on the particle's worldline is

$$H_{\lambda\lambda} = g_{\mu\nu}\frac{\mathrm{d}z^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}z^\nu}{\mathrm{d}\lambda}. \tag{7.1.42}$$

From this, we see the infinitesimal 1D volume is simply the proper time

$$\mathrm{d}\tau = \mathrm{d}\lambda\sqrt{|\det H_{\lambda\lambda}|} = \mathrm{d}\lambda\sqrt{g_{\mu\nu}\frac{\mathrm{d}z^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}z^\nu}{\mathrm{d}\lambda}}. \tag{7.1.43}$$

And the simplest action for the dynamics for $z^\mu(\lambda)$ is therefore its integral

$$S_{\rm pp} = -m\int \mathrm{d}\tau = -m\int \mathrm{d}\lambda\sqrt{g_{\mu\nu}\frac{\mathrm{d}z^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}z^\nu}{\mathrm{d}\lambda}}. \tag{7.1.44}$$

The coefficient in front can be fixed by dimensional analysis if we demand the action itself to be dimensionless in natural units, a fact that follows from the path integral formulation of quantum mechanics. Since $[\mathrm{d}\tau]$ = [Time], the coefficient must have dimensions of [Mass], whereas the $-$ sign is gotten by ensuring that in flat spacetime and in the non-relativistic limit, the usual $\int \mathrm{d}t L_{\text{free particle}} = +\int \mathrm{d}t(M/2)\vec{v}^2$ is recovered (recall, too, Problem (2.42)). As advertised, demanding that eq. (7.1.44) be stationary yields the geodesic equation for $z^\mu(\lambda)$.[64]

[64]Strictly speaking, however, because the stress tensor that follows from eq. (7.1.44) contains $\delta^{(d-1)}$-functions – see eq. (7.1.41) – the associated Einstein's equations (7.1.20) run into mathematical difficulties [16].

From this viewpoint, the geodesic motion and, hence, the (approximate) Weak Equivalence Principle discussed in the section enveloping eq. (2.5.1) are no longer separate postulates but really consequences of the symmetry and action-principle based considerations emphasized in the later sections of these notes. Furthermore, one may readily demonstrate that the Weak Equivalence Principle is not an exact statement for realistic macroscopic bodies, since there are infinite number of 'non-minimal' worldline actions; for instance, the addition of

$$\int \mathrm{d}\tau C_{\mathrm{T}} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} \tag{7.1.45}$$

to eq. (7.1.44) would replace the right hand side of the geodesic equation $D^2 z^\mu / \mathrm{d}\tau^2 = 0$ with a term describing non-zero tidal forces.

Problem 7.5. $(N+1)-$dimensional object in $d-$spacetime Write down the action for a structureless $(N+1) \leq d$ dimensional object residing in a $d-$dimensional curved spacetime. What is the physical dimension of the coefficient, i.e., the analog of m and μ for, respectively, the point particle and relativistic string? Hint: Start by giving coordinates ξ^A to the hypersurface swept out by this object. How many components are there? What is the induced metric? □

7.2 Meaning of $T^{\widehat{\mu}\widehat{\nu}}$ & Energy-Momentum Conservation

Components of the Energy-Momentum Tensor[65] We now enumerate the physical meaning of each component of the energy-momentum-shear-stress tensor. For simplicity let us focus on the $(3+1)$D case.

Given a curved geometry, we may always choose to express it in an orthonormal frame,

$$g_{\mu\nu} = \varepsilon^{\widehat{\alpha}}{}_{\mu} \varepsilon^{\widehat{\beta}}{}_{\nu} \eta_{\alpha\beta}, \tag{7.2.1}$$

so that

$$u^\mu \equiv \varepsilon_{\widehat{0}}{}^{\mu} \tag{7.2.2}$$

may be viewed as the timelike vector tangent to a family of observers and the

$$\{\varepsilon_{\widehat{a}}{}^{\nu} \partial_\nu | a = 1, 2, 3\} \tag{7.2.3}$$

are their spatial 'standard rulers'. We will now define the components of the energy-momentum-shear-stress tensor of some physical substance – fluids, electromagnetic fields, neutrinos, scalar fields etc. – residing in this curved spacetime.

> In relativistic language, the (μ, ν) component of the stress tensor in an orthonormal frame – i.e., $T^{\widehat{\mu}\widehat{\nu}}$ – is the ν-th component of the substance's momentum per unit spacetime volume orthogonal to the μ-th direction.

Let us break this down into a time + space decomposition.

[65] A more detailed treatment may be found in Chapter 4 of Schutz [3]; and Chapter 5 of Misner, Thorne, Wheeler [4].

- $T^{\widehat{0}\widehat{\nu}} = T^{\widehat{\nu}\widehat{0}}$ is the 4-momentum per unit spatial volume in the rest frame of the observers. In particular, $T^{\widehat{0}\widehat{0}}$ is the energy density and $T^{\widehat{0}\widehat{i}} = T^{\widehat{i}\widehat{0}}$ is the 3-momentum density.
- $T^{\widehat{i}\widehat{i}}$, with no sum over i, is the *pressure* – force per unit proper area – in the i-th direction.
- $T^{\widehat{i}\widehat{j}} = T^{\widehat{j}\widehat{i}}$, for $i \neq j$, is the *shear*. For a fixed j, it is the i-th component of the 3-momentum per unit time – i.e., force – exerted upon the 2D area perpendicular to the j-th direction. If the momentum is flowing strictly along the j-th direction, since $i \neq j$ by assumption, that means $T^{\widehat{i}\widehat{j}} = T^{\widehat{j}\widehat{i}} = 0$ – this is precisely why $T^{\widehat{i}\widehat{j}} = T^{\widehat{j}\widehat{i}}$ is shear.

Symmetry If a physical system in curved spacetime can be described by an action principle, we have seen how its stress tensor may be obtained from the variation of its action with respect to the metric. And since the metric is symmetric, the resulting $T^{\mu\nu}$ would automatically be symmetric; i.e., $\delta g_{\alpha\beta} T^{\alpha\beta}$ would not retain any antisymmetric portion of $T^{\alpha\beta}$. More generally, however, it is possible to argue on physical grounds that $T^{\mu\nu} = T^{\nu\mu}$. For instance, that $T^{\widehat{0}\widehat{i}} = T^{\widehat{i}\widehat{0}}$ holds essentially from the equivalence between energy and mass. Whereas, if it were not the case that $T^{\widehat{i}\widehat{j}} = T^{\widehat{j}\widehat{i}}$ (for $i \neq j$), it would be possible to pathologically exert a finite amount of torque on an infinitesimal volume of a material body.[66]

Why? The physical interpretation delineated here for the components of $T^{\widehat{\mu}\widehat{\nu}}$ is really an assertion. Let us attempt to justify it partially, by appealing to the flat spacetime limit, where the momentum of a classical field theory may be viewed as the conserved Noether current of spacetime translation symmetry. Specifically, let us analyze the canonical scalar field theory of eq. (6.1.5) but with $g_{\mu\nu} = \eta_{\mu\nu}$.

$$\mathcal{L}(x) = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi(x)\partial_\nu\varphi(x) - V\left(\varphi(x)\right). \tag{7.2.4}$$

Since $\mathcal{L}$ is Lorentz invariant, we may consider an infinitesimal spacetime displacement,

$$x^\mu = x'^\mu + a^\mu, \tag{7.2.5}$$

for constant but 'small' a^μ.

$$\mathcal{L}(x) = \mathcal{L}(x') + a^\mu \partial_{\mu'}\mathcal{L}(x') + \mathcal{O}\left(a^2\right). \tag{7.2.6}$$

On the other hand, $\partial/\partial x^\mu = \partial_\mu = \partial_{\mu'} = \partial/\partial x'^\mu$ and

$$\mathcal{L}(x'+a) = \mathcal{L}(x') + \frac{\partial \mathcal{L}}{\partial\varphi(x')} a^\nu \partial_{\nu'}\varphi(x') + \frac{\partial \mathcal{L}}{\partial\partial_{\mu'}\varphi(x')} a^\nu \partial_{\nu'}\partial_{\mu'}\varphi(x') + \mathcal{O}\left(a^2\right) \tag{7.2.7}$$

$$= \mathcal{L}(x') + a^\nu \partial_{\nu'}\varphi(x') \left\{ \frac{\partial \mathcal{L}}{\partial\varphi(x')} - \partial_{\mu'} \frac{\partial \mathcal{L}}{\partial\partial_{\mu'}\varphi(x')} \right\} + a^\nu \partial_{\mu'} \left(\partial_{\nu'}\varphi(x') \frac{\partial \mathcal{L}}{\partial\partial_{\mu'}\varphi(x')} \right) + \mathcal{O}\left(a^2\right). \tag{7.2.8}$$

Using the equations-of-motion for the scalar field

$$\frac{\partial \mathcal{L}}{\partial\varphi(x')} - \partial_{\mu'} \frac{\partial \mathcal{L}}{\partial\partial_{\mu'}\varphi(x')} = 0, \tag{7.2.9}$$

[66] See §4.5 of Schutz [3] and §5.7 of Misner, Thorne, and Wheeler [4] for a pedagogical discussion.

eq. (7.2.8) becomes

$$\mathcal{L}(x'+a) = \mathcal{L}(x') + a^\nu \partial_{\mu'} \left(\partial_{\nu'}\varphi(x') \frac{\partial \mathcal{L}}{\partial \partial_{\mu'}\varphi(x')} \right). \tag{7.2.10}$$

We may now equate the linear-in-a^ν terms on the right hand sides of equations (7.2.6) and (7.2.10), and find the following conservation law:

$$\partial_{\mu'} \left\{ a^\gamma \left(\partial_{\gamma'}\varphi(x') \frac{\partial \mathcal{L}}{\partial \partial_{\mu'}\varphi(x')} - \delta^\mu_\gamma \mathcal{L}(x') \right) \right\} = 0. \tag{7.2.11}$$

By setting $a^\gamma = \delta^\gamma_\nu$, for a fixed ν, we may identify the conserved quantity inside the $\{\dots\}$ as the Noether momentum p_ν due to translation symmetry along the ν-th direction.[67] Doing so now allows us to identify the conserved stress tensor

$$T^\mu{}_\nu = \partial_{\nu'}\varphi(x') \frac{\partial \mathcal{L}}{\partial \partial_{\mu'}\varphi(x')} - \delta^\mu_\nu \mathcal{L}(x'). \tag{7.2.12}$$

Applying this to eq. (7.2.4), we obtain

$$T^\mu{}_\nu = \partial_\nu\varphi \partial^\mu \varphi - \delta^\mu_\nu \left(\frac{1}{2}(\partial\varphi)^2 - V(\varphi) \right). \tag{7.2.13}$$

This is simply the flat spacetime limit of the stress tensor in eq. (7.1.32). Unfortunately, this same Noether procedure fails to reproduce even the electromagnetic stress tensor in eq. (7.1.34); to do so, it turns out this Noether current needs to be augmented by additional terms to form the Belinfante-Rosenfeld tensor [14, 15].

Problem 7.6. Electromagnetic $T^{\widehat{\mu}\widehat{\nu}}$ in 4D Minkowski In (3+1)D flat spacetime, recalling the relationship between $F^{\mu\nu}$ and the electric/magnetic fields in eq. (5.0.6), verify that the electromagnetic energy density is

$$T^{\widehat{0}\widehat{0}} = T^{00} = \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2 \right), \qquad \vec{E}^2 \equiv E^i E^i,\ \vec{B}^2 \equiv B^i B^i; \tag{7.2.14}$$

whereas the Poynting vector is

$$T^{\widehat{0}\widehat{i}} = T^{0i} = \left(\vec{E} \times \vec{B} \right)^i. \tag{7.2.15}$$

□

Problem 7.7. Electromagnetic Stress Tensor in JWKB Limit Compute the stress tensor of a single electromagnetic wave

$$A_\mu = \mathrm{Re}\left\{ a_\mu \exp\left(iS/\epsilon \right) \right\} \tag{7.2.16}$$

[67] As a simple parallel to the situation here: in classical mechanics, because the free Lagrangian $L = (1/2)\dot{\vec{x}}^2$ is space-translation invariant, $\partial L/\partial x^i = 0$, we may identify the momentum $p_i \equiv \partial L/\partial \dot{x}^i$ as the corresponding Noether charge.

obeying the Lorenz gauge condition $\nabla^\mu A_\mu = 0$ in the high frequency limit – i.e., apply the discussion in §(6.3) – and show it is null:

$$T_{\mu\nu} = -k_\mu k_\nu \mathcal{A}_\sigma \mathcal{A}^\sigma, \qquad\qquad k_\mu \equiv \nabla_\mu S. \tag{7.2.17}$$

where

$$\mathcal{A}_\mu \equiv \mathrm{Re}\left\{a_\mu \exp\left((i/\epsilon)S + i\pi/2\right)\right\}. \tag{7.2.18}$$

□

Problem 7.8. Perfect Fluids Stress Tensor A *perfect* fluid is defined as a fluid that appears isotropic in its co-moving frame. Equivalently, one may also define a perfect fluid to be one where, in a co-moving frame, there is no heat conduction ($T^{\widehat{0}\widehat{i}} = 0$) nor viscosity ($T^{\widehat{i}\widehat{j}} = 0$ for $i \neq j$).[68]

Let U^μ correspond to the $d-$velocity of the fluid, and for concreteness assume it is timelike, so

$$U^2 \equiv g_{\mu\nu} U^\mu U^\nu = +1. \tag{7.2.19}$$

In an orthonormal frame $\{U^\mu, \varepsilon_{\widehat{i}}^{\ \mu}\}$, where

$$g^{\mu\nu} = U^\mu U^\nu - \delta^{ij} \varepsilon_{\widehat{i}}^{\ \mu} \varepsilon_{\widehat{j}}^{\ \nu}, \tag{7.2.20}$$

the definition of the perfect fluid amounts to the statement that the only non-zero components of its stress tensor are

$$T^{\widehat{0}\widehat{0}} = \rho \qquad \text{and} \qquad T^{\widehat{i}\widehat{i}} = p \quad (\text{No sum over } i). \tag{7.2.21}$$

(That is, $T^{\widehat{0}\widehat{i}} = 0 = T^{\widehat{i}\widehat{j}}$, for $i \neq j$.) Argue that the curved spacetime perfect fluid stress tensor can be expressed in terms of its $d-$velocity U^μ; rest frame energy density ρ and pressure density p; and the spacetime metric tensor $g^{\mu\nu}$ as follows:

$$T^{\mu\nu} = (\rho + p)\, U^\mu U^\nu - p g^{\mu\nu}. \tag{7.2.22}$$

Specifically, since this is a tensor, simply re-write it in an orthonormal basis and show that eq. (7.2.21) is recovered. Finally, explain how eq. (7.2.22) would be modified if you were using the 'mostly plus' sign convention for the metric. □

Positive Cosmological Constant As Negative Pressure One may identify the cosmological constant Λ term in Einstein's equations as a 'perfect fluid' with negative pressure $p = -\rho$. First re-write eq. (7.1.20) as

$$G^{\mu\nu} = 8\pi G_{\mathrm{N}} \left(T^{\mu\nu}[\text{matter}] + T^{\mu\nu}[\text{CC}]\right), \tag{7.2.23}$$

[68]As Chapter 4 of Schutz [3] explains, even if there is no momentum flow due to transport of the substance in question, there could be heat transfer – a flow of energy described by $T^{\widehat{0}\widehat{i}}$.

where

$$T^{\mu\nu}[\mathrm{CC}] \equiv \frac{\Lambda}{8\pi G_{\mathrm{N}}} g^{\mu\nu}. \tag{7.2.24}$$

Comparing this $T_{\mu\nu}[\mathrm{CC}]$ to the perfect fluid stress tensor in eq. (7.2.22), we immediately infer

$$p[\mathrm{CC}] = -\rho[\mathrm{CC}] = -\frac{\Lambda}{8\pi G_{\mathrm{N}}}, \tag{7.2.25}$$

which is negative for $\Lambda > 0$. Physically, it is this negative pressure that is responsible for the repulsive nature of gravity on scales larger than $\sim 1/\sqrt{\Lambda}$.

Conservation Because of the Bianchi identity in eq. (2.4.50) and the covariant constancy of the metric, we see that demanding the consistency of Einstein's equations leads to the conservation of energy-momentum of the total energy-momentum $T_{\mu\nu}$ of all the matter in the system encoded within $\mathcal{L}_{\text{matter}}$:

$$\nabla^\mu \left(G_{\mu\nu} - \Lambda g_{\mu\nu}\right) = 0 = \nabla^\mu T_{\mu\nu}. \tag{7.2.26}$$

Let us now elucidate why $\nabla^\mu T_{\mu\nu} = 0$ is non-trivial. Specifically, it is not a trivial identity, but holds when the $T_{\mu\nu}$ of all matter is evaluated on the solutions of its/their relevant equations of motion – this is why we quoted John Wheeler above.

We shall assume that the matter Lagrangian $\mathcal{L}_{\text{matter}}$ is a coordinate scalar. That means the matter action itself may be evaluated in any coordinate system we wish, namely

$$\int \mathrm{d}^d x \sqrt{|g(x)|} \mathcal{L}_{\text{matter}}(x) = \int \mathrm{d}^d x' \sqrt{|g(x')|} \mathcal{L}_{\text{matter}}(x'); \tag{7.2.27}$$

where $g(x)$ is the determinant of the metric in the x coordinate system while $g(x')$ is that in the x' system. By considering an infinitesimal coordinate transformation

$$x^\alpha = x'^\alpha + \xi^{\alpha'}(x'), \tag{7.2.28}$$

where $\xi^{\alpha'}$ is to be considered 'small' in the same sense that the field variations $\delta g_{\mu\nu}$, $\delta\varphi$, δA_μ, etc. are small; we may now exploit the coordinate invariance of the matter action to show that $\nabla_\mu T^{\mu\nu} = 0$ whenever the associated matter equations-of-motion are obeyed. Starting from eq. (7.2.28),

$$\frac{\partial x^\alpha}{\partial x'^\mu} = \delta^\alpha_\mu + \partial_{\mu'} \xi^{\alpha'}(x'), \qquad \partial_{\mu'} \equiv \frac{\partial}{\partial x'^\mu} \tag{7.2.29}$$

$$\frac{\partial x'^\alpha}{\partial x^\mu} = \delta^\alpha_\mu - \partial_{\mu'} \xi^{\alpha'}(x') + \mathcal{O}\left(\xi^2\right); \tag{7.2.30}$$

where the second line follows from the first because $\partial x/\partial x'$ is the inverse of $\partial x'/\partial x$ (and vice versa). These in turn tell us

$$\begin{aligned} g_{\mu'\nu'}(x') &= g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \\ &= g_{\alpha\beta}(x' + \xi) \left(\delta^\alpha_\mu + \partial_{\mu'} \xi^{\alpha'}\right) \left(\delta^\beta_\nu + \partial_{\nu'} \xi^{\beta'}\right) \end{aligned}$$

$$
\begin{aligned}
&= g_{\mu\nu}(x') + \xi^{\sigma'}(x')\partial_{\sigma'} g_{\mu\nu}(x') + g_{\sigma\{\mu}(x')\partial_{\nu'\}}\xi^{\sigma'}(x') && (7.2.31)\\
&= g_{\mu\nu}(x') + \pounds_{\xi(x')} g_{\mu\nu}, && (7.2.32)\\
\pounds_{\xi(x')} g_{\mu\nu} &\equiv \xi^{\sigma'}(x')\partial_{\sigma'} g_{\mu\nu}(x') + g_{\sigma\{\mu}(x')\partial_{\nu'\}}\xi^{\sigma'}(x') && (7.2.33)\\
&= \nabla_{\mu'}\xi_{\nu'}(x') + \nabla_{\nu'}\xi_{\mu'}(x'). && (7.2.34)
\end{aligned}
$$

Above, in the third equality onwards, the $g_{\mu\nu}(x')$ are the metric components in the x coordinate system, but with x^α replaced with x'^α; moreover, the covariant derivative in the Lie derivative of the metric in eq. (7.2.34) is with respect to $g_{\mu\nu}(x')$ (as opposed to $g_{\mu'\nu'}(x')$). We also have

$$
g^{\mu'\nu'}(x') = g^{\mu\nu}(x') - \nabla^{\mu'}\xi^{\nu'}(x') - \nabla^{\nu'}\xi^{\mu'}(x'). \tag{7.2.35}
$$

Problem 7.9. Using $g^{\mu'\sigma'}(x')g_{\sigma'\nu'}(x') = \delta^\mu_\nu$, can you derive eq. (7.2.35) from eq. (7.2.31) without performing the transformation of eq. (7.2.28)? □

These results imply, using the matrix algebra results $\det(X\cdot Y) = \det X \cdot \det Y$ and $\det \exp(\ln X) = \exp(\mathrm{Tr}\,[\ln X])$ for matrices X and Y,

$$
\begin{aligned}
\det g_{\mu'\nu'}(x') &= \det\left(g_{\mu\nu}(x') + \nabla_{\mu'}\xi_{\nu'}(x') + \nabla_{\nu'}\xi_{\mu'}(x')\right)\\
&= (\det g_{\mu\nu}(x'))\cdot\det\left(\delta^\alpha_\beta + \nabla^{\alpha'}\xi_{\beta'}(x') + \nabla_{\beta'}\xi^{\alpha'}(x')\right)\\
&= (\det g_{\mu\nu}(x'))\left(1 + 2\nabla_{\sigma'}\xi^{\sigma'} + \mathcal{O}\left(\xi^2\right)\right). && (7.2.36)
\end{aligned}
$$

Hence, the volume form itself is

$$
\mathrm{d}^d x' \sqrt{|\det g_{\mu'\nu'}(x')|} = \mathrm{d}^d x' \sqrt{|\det g_{\mu\nu}(x')|}\left(1 + \nabla_{\sigma'}\xi^{\sigma'} + \mathcal{O}\left(\xi^2\right)\right). \tag{7.2.37}
$$

If we are dealing with scalar fields as our 'matter', then

$$
\begin{aligned}
\varphi'(x') &\equiv \varphi(x(x')) = \varphi(x' + \xi)\\
&= \varphi(x') + \xi^{\sigma'}\nabla_{\sigma'}\varphi(x') + \mathcal{O}\left(\xi^2\right) && (7.2.38)\\
&\equiv \varphi(x') + \pounds_{\xi(x')}\varphi + \mathcal{O}\left(\xi^2\right).
\end{aligned}
$$

Problem 7.10. Lie Derivatives of Covariant Derivatives For a rank-1 tensor V_β and rank-2 tensor $\Sigma_{\alpha\beta}$, under the infinitesimal coordinate transformation of eq. (7.2.28), show that

$$
\begin{aligned}
V_{\mu'}(x') \equiv V_\alpha(x(x'))\frac{\partial x^\alpha}{\partial x'^\mu} &= V_\mu(x') + \xi^{\sigma'}\nabla_{\sigma'}V_\mu(x') + \nabla_{\mu'}\xi^{\sigma'}V_\sigma(x') && (7.2.39)\\
&\equiv V_\mu(x') + \pounds_{\xi(x')}V_\mu
\end{aligned}
$$

and

$$
\begin{aligned}
\Sigma_{\mu'\nu'}(x') &\equiv \Sigma_{\alpha\beta}(x(x'))\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\\
&= \Sigma_{\mu\nu}(x') + \xi^{\sigma'}\nabla_{\sigma'}\Sigma_{\mu\nu}(x') + \nabla_{\mu'}\xi^{\sigma'}\Sigma_{\sigma\nu}(x') + \nabla_{\nu'}\xi^{\sigma'}\Sigma_{\mu\sigma}(x') && (7.2.40)\\
&\equiv \Sigma_{\mu\nu}(x') + \pounds_{\xi(x')}\Sigma_{\mu\nu};
\end{aligned}
$$

where all the tensor components on the left hand sides are in the x' coordinate basis whereas those on the right hand sides are in the x coordinate basis, but with all the x^α replaced with x'^α. Next, consider the follow derivative in the x' coordinate basis

$$\Sigma_{\alpha'\beta'}(x') \equiv \nabla_{\alpha'} V_{\beta'}. \tag{7.2.41}$$

Verify that, if x and x' are related through eq. (7.2.28), then the infinitesimal coordinate transformation rules of eq. (7.2.40) can be obtained by simultaneously implementing the metric transformation rules of eq. (7.2.31) and rank-1 tensor rules of eq. (7.2.39). Hint: Start from

$$\nabla_{\alpha'} V_{\beta'} = \partial_{\alpha'} V_{\beta'} - \Gamma^{\sigma'}{}_{\alpha'\beta'} V_{\sigma'}, \tag{7.2.42}$$

where $\Gamma^{\sigma'}{}_{\alpha'\beta'}$ is built out of the metric components in the x' coordinate basis, namely $g_{\mu'\nu'}(x')$. Next, employ the result that, under the variation

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu} \tag{7.2.43}$$

the Christoffel symbols transform as

$$\Gamma^\mu{}_{\alpha\beta} \to \Gamma^\mu{}_{\alpha\beta} + \frac{1}{2} g^{\mu\sigma} \left(\nabla_{\{\alpha} h_{\beta\}\sigma} - \nabla_\sigma h_{\alpha\beta} \right); \tag{7.2.44}$$

where the ∇ is with respect to $g_{\mu\nu}(x)$ (as opposed to $g_{\mu\nu} + h_{\mu\nu}$). For a derivation of this Christoffel transformation rule, see Problem (2.52) below. □

The above problem provides a non-trivial example of the following rule. To switch from writing a tensorial expression in the x-coordinate system to an infinitesimally different x'-coordinate system, where $x^\mu \equiv x'^\mu + \xi^\mu(x')$ (for 'small' ξ), one merely needs to replace within in *all fields* – including that of the metric itself – with themselves plus their Lie derivatives, followed by replacing all x with x'.

Returning to the situation at hand, since the action itself is a coordinate scalar, we may switch from writing it in the x coordinate system to the x' coordinate system. Up to linear order in $\xi^{\alpha'}(x')$, we may employ eq. (7.2.28) to re-express all the fields within it – both metric and matter fields – in the x coordinate basis but with x^α replaced with x'^α plus their Lie derivatives with respect to $\xi^{\alpha'}(x')$. But since the $\{x'^\alpha\}$ are merely integration variables, the whole process may be summarized as keeping all the $\mathrm{d}^d x$ and partial derivatives $\{\partial_\mu \equiv \partial/\partial x^\mu\}$ fixed,

$$\mathrm{d}^d x \to \mathrm{d}^d x \qquad \text{and} \qquad \frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x^\mu}; \tag{7.2.45}$$

and replacing the fields as

$$g_{\mu\nu} \to g_{\mu\nu} + \nabla_{\{\mu}\xi_{\nu\}} \equiv g_{\mu\nu} + \pounds_\xi g_{\mu\nu} \tag{7.2.46}$$

$$\psi \to \psi + \pounds_\xi \psi \tag{7.2.47}$$

$$\nabla_\alpha \psi \to \nabla_\alpha \psi + \pounds_\xi g_{\mu\nu} \frac{\partial(\nabla_\alpha \psi)}{\partial g_{\mu\nu}} + \nabla_\alpha \pounds_\xi \psi \tag{7.2.48}$$

$$= \nabla_\alpha \psi + 2\nabla_\mu \xi_\nu \frac{\partial(\nabla_\alpha \psi)}{\partial g_{\mu\nu}} + \nabla_\alpha \pounds_\xi \psi \tag{7.2.49}$$

where we have used $\pounds_\xi g_{\mu\nu} = \nabla_{\{\mu}\xi_{\nu\}}$ in the last line; and, for notational convenience, we are now collectively denoting the matter fields using a single symbol ψ.[69] At this juncture, we may thus write our matter action in the x' coordinate system as

$$S_{\text{matter}} = \int \mathrm{d}^d x \sqrt{|g(x)|} \mathcal{L}_{\text{matter}}(x) = \int \mathrm{d}^d x' \sqrt{|g(x')|} \mathcal{L}_{\text{matter}}(x') \tag{7.2.50}$$

$$\begin{aligned}
&= \int \mathrm{d}^d x \sqrt{|g(x)|} \left(1 + \nabla_\sigma \xi^\sigma + \mathcal{O}\left(\xi^2\right)\right) \\
&\times \left(\mathcal{L}_{\text{matter}} + \frac{\partial \mathcal{L}_{\text{matter}}}{\partial g_{\alpha\beta}} \nabla_{\{\alpha}\xi_{\beta\}} + \frac{\partial \mathcal{L}_{\text{matter}}}{\partial \psi} \pounds_\xi \psi + \frac{\partial \mathcal{L}_{\text{matter}}}{\partial \nabla_\alpha \psi} \nabla_\alpha \pounds_\xi \psi + \mathcal{O}\left(\xi^2\right)\right) \\
&= S_{\text{matter}} \\
&+ \int \mathrm{d}^d x \sqrt{|g(x)|} \xi_\beta \nabla_\alpha \left\{-2 \frac{\partial \mathcal{L}_{\text{matter}}}{\partial g_{\alpha\beta}} - g^{\alpha\beta} \mathcal{L}_{\text{matter}}\right\} \\
&+ \int \mathrm{d}^d x \sqrt{|g(x)|} \pounds_\xi \psi \left(\frac{\partial \mathcal{L}_{\text{matter}}}{\partial \psi} - \nabla_\alpha \frac{\partial \mathcal{L}_{\text{matter}}}{\partial \nabla_\alpha \psi}\right) + \text{surface terms.}
\end{aligned} \tag{7.2.51}$$

Provided we arrange ξ to vanish sufficiently quickly at spatial infinity, and for $\xi(t_{\mathrm{i}}) = \xi(t_{\mathrm{f}}) = 0$, all relevant surface terms ought to vanish. Furthermore, *if* the equations-of-motion are satisfied

$$\frac{\partial \mathcal{L}_{\text{matter}}}{\partial \psi} = \nabla_\alpha \frac{\partial \mathcal{L}_{\text{matter}}}{\partial \nabla_\alpha \psi}, \tag{7.2.52}$$

then what remains from this variational calculation is – recall eq. (7.1.23) –

$$S_{\text{matter}} = S_{\text{matter}} + \int \mathrm{d}^d x \sqrt{|g(x)|} \xi_\beta \nabla_\alpha T^{\alpha\beta}. \tag{7.2.53}$$

Remember all we have done is to switch from writing the matter action in the x coordinate system to the infinitesimally nearby x' one; and because we have assumed the matter action is a coordinate scalar, these linear-in-ξ terms must vanish:

$$\nabla_\alpha T^{\alpha\beta}[\text{total}] = 0. \tag{7.2.54}$$

It is worth reiterating, it is the total stress-energy tensor of all the matter in the system that is conserved.

On the other hand, suppose we demand $\nabla_\alpha T^{\alpha\beta} = 0$; then

$$S_{\text{matter}} = S_{\text{matter}} + \int \mathrm{d}^d x \sqrt{|g(x)|} \pounds_\xi \psi \left(\frac{\partial \mathcal{L}_{\text{matter}}}{\partial \psi} - \nabla_\alpha \frac{\partial \mathcal{L}_{\text{matter}}}{\partial \nabla_\alpha \psi}\right). \tag{7.2.55}$$

Since we have assumed the matter action is a coordinate scalar – i.e., all linear-in-ξ terms must vanish – that means either eq. (7.2.52) must be satisfied or $\pounds_\xi \psi = 0$ for arbitrary but infinitesimal ξ.

To summarize our findings – provided the dynamics of matter fields are encoded within a coordinate scalar action principle:

[69] The factor 2 in front of $\nabla_\mu \xi_\nu$ on the last line arises because $\partial/\partial g_{\mu\nu}$ acting on a tensorial object should return a tensor symmetric in $\mu \leftrightarrow \nu$.

- If the equations-of-motion of all matter in a system are obeyed, their total energy-momentum-shear-stress tensor is conserved.

- If the total energy-momentum-shear-stress tensor of all matter is required to be divergence-free, i.e., conserved, and if the matter fields involved are not invariant under arbitrary active coordinate transformations, then eq. (7.2.52) is satisfied. In general, since eq. (7.2.52) could involve multiple matter fields, this does not imply their individual equations-of-motion are satisfied – the latter is true, however, if there is a single matter field in the system.

Example 1 Let us see explicitly that the stress energy tensor (eq. (7.1.32)) of the canonical scalar field in eq. (6.1.5) is conserved. First we should work out its equations-of-motion. Demanding the action be stationary,

$$\begin{aligned}
&\delta_\varphi \left\{ \int \mathrm{d}^d x \sqrt{|g|} \left(\frac{1}{2} \nabla_\mu \varphi \nabla^\mu \varphi - V(\varphi) \right) \right\} \\
&= \int \mathrm{d}^d x \sqrt{|g|} \left(\nabla_\mu \delta\varphi \nabla^\mu \varphi - V'(\varphi) \delta\varphi \right) \\
&= \int \mathrm{d}^d x \sqrt{|g|} \delta\varphi \left(-\Box\varphi - V'(\varphi) \right) + \text{surface terms.}
\end{aligned} \tag{7.2.56}$$

Setting to zero the coefficient of $\delta\varphi$,

$$\Box\varphi + V'(\varphi) = 0. \tag{7.2.57}$$

Let us now proceed to take the divergence of eq. (7.1.32),

$$\begin{aligned}
\nabla^\mu T_{\mu\nu}[\varphi] &= \Box\varphi \nabla_\nu \varphi + \nabla^\mu \varphi \nabla_\mu \nabla_\nu \varphi - \nabla_\nu \left(\frac{1}{2} \nabla_\sigma \varphi \nabla^\sigma \varphi - V(\varphi) \right) \\
&= \Box\varphi \nabla_\nu \varphi + \nabla^\mu \varphi \nabla_\mu \nabla_\nu \varphi - \nabla_\sigma \nabla_\nu \varphi \nabla^\sigma \varphi + V'(\varphi) \nabla_\nu \varphi \\
&= (\Box\varphi + V'(\varphi)) \nabla_\nu \varphi.
\end{aligned} \tag{7.2.58}$$

We see that, if eq. (7.2.57) is satisfied, then $\nabla^\mu T_{\mu\nu}[\varphi] = 0$. On the other hand, if we demand $\nabla^\mu T_{\mu\nu}[\varphi] = 0$, then either eq. (7.2.57) is satisfied or $\nabla_\nu \varphi = 0$. Notice the latter is precisely the condition that φ be invariant under arbitrary active coordinate transformations – $\xi^\sigma \nabla_\sigma \varphi = 0$ for any 'small' ξ^σ.

Example 2 This sort of action-based variational argument can also be used to derive the Bianchi identity obeyed by Einstein's tensor in eq. (2.4.50). Specifically, since $\int \mathrm{d}^d x \sqrt{|g(x)|} \mathcal{R}(x)$ is a coordinate scalar, we have – taking into account equations (7.1.6), (7.2.28), (7.2.35) and (7.2.37) –

$$\begin{aligned}
&\int \mathrm{d}^d x \sqrt{|g(x)|} \mathcal{R}(x) = \int \mathrm{d}^d x' \sqrt{|g(x')|} \mathcal{R}(x') \qquad (7.2.59) \\
&= \int \mathrm{d}^d x' \sqrt{|\det g_{\mu\nu}(x')|} \left(1 + \nabla_{\sigma'} \xi^{\sigma'} + \mathcal{O}\left(\xi^2\right) \right) \left(\mathcal{R}(x') - 2\nabla^{\alpha'} \xi^{\beta'} R_{\alpha\beta}(x') + \mathcal{O}\left(\xi^2\right) \right) \\
&= \int \mathrm{d}^d x \sqrt{|g(x)|} \mathcal{R}(x)
\end{aligned}$$

$$
\begin{aligned}
&+ 2\int \mathrm{d}^d x' \sqrt{|\det g_{\mu\nu}(x')|} \left\{ \xi^{\beta'} \nabla^{\alpha'} \left(R_{\alpha\beta}(x') - \frac{1}{2} g_{\alpha\beta}(x') \mathcal{R}(x') \right) + \mathcal{O}\left(\xi^2\right) \right\} + \text{surface terms} \\
&= \int \mathrm{d}^d x \sqrt{|g(x)|} \mathcal{R}(x) \\
&+ 2\int \mathrm{d}^d x' \sqrt{|\det g_{\mu\nu}(x')|} \left\{ \xi^{\beta'} \nabla^{\alpha'} G_{\alpha\beta}(x') + \mathcal{O}\left(\xi^2\right) \right\} + \text{surface terms.}
\end{aligned}
$$

It is not surprising that the coefficient of the linear-in-ξ terms yield the Einstein tensor, since essentially the same variational calculation was performed leading up to Einstein's General Relativity in eq. (7.1.20). Furthermore, since ξ was small but arbitrary, we may always arrange for it to be such that the surface terms vanish. And by general covariance – because $\int \mathrm{d}^d x \sqrt{|g(x)|} \mathcal{R}(x)$ is a coordinate scalar – the remaining linear-in-ξ terms must vanish. At this juncture, setting the coefficient of $\xi^{\beta'}$ to zero in the last line in fact hands us $\nabla^\alpha G_{\alpha\beta} = 0$.

Global Conservation of Energy-Momentum in Time-Translation Invariant Geometries In flat spacetime, we may directly interpret T^{00} as the energy/mass density and $T^{0i} = T^{i0}$ as the energy/mass per unit time per volume perpendicular to the ith direction. Defining the energy-momentum vector $P^\mu \equiv T^{0\mu}$,

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^{d-1}} P^\mu(t,\vec{x}) \mathrm{d}^{d-1}\vec{x} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^{d-1}} T^{0\mu}(t,\vec{x}) \mathrm{d}^{d-1}\vec{x} \tag{7.2.60}
$$

$$
= -\int_{\mathbb{R}^{d-1}} \partial_i T^{i\mu}(t,\vec{x}) \mathrm{d}^{d-1}\vec{x} = 0. \tag{7.2.61}
$$

Therefore, *total* energy/mass-momentum

$$
\int_{\mathbb{R}^{d-1}} P^\mu(t,\vec{x}) \mathrm{d}^{d-1}\vec{x} \tag{7.2.62}
$$

is conserved in any inertial frame.

Problem 7.11. Conserved Current from Isometries Even though a generic curved spacetime does not, in general, enjoy the same number of symmetries as flat Minkowski spacetime – suppose a given spacetime *does* admit a Killing vector ξ_μ, show that

$$
J^\alpha \equiv T^{\alpha\beta} \xi_\beta \tag{7.2.63}
$$

is a conserved current; i.e., show that $\nabla_\alpha J^\alpha = 0$. In such a case, as long as $T^{\alpha\beta}$ describes a physically isolated system, explain why

$$
Q \equiv \int \mathrm{d}^{d-1}\vec{y} \sqrt{|H(\vec{y})|} n_\alpha J^\alpha \tag{7.2.64}
$$

is the conserved charge corresponding to the symmetry generated by ξ_ν. Here, $\mathrm{d}^{d-1}\vec{y}\sqrt{H[\vec{y}]} n_\alpha$ is the area element of the surface perpendicular to the unit timelike vector n_α. For simplicity, assume it is possible to choose coordinates such that $x^\alpha = (t, \vec{y})$ and $n_\alpha \mathrm{d}x^\alpha \propto \mathrm{d}t$.

Remark When the geometry is Minkowski spacetime, the J^α in this problem of course then reduces to the usual energy-momentum current. □

Problem 7.12. Show that the electromagnetic stress tensor in eq. (7.1.34) is conserved in vacuum ($J_\nu = 0$) after imposing $\nabla_\mu F^{\mu\nu} = 0$ and $\mathrm{d}F = 0$. Hint: $\mathrm{d}F = 0$ is actually an identity here, because eq. (7.1.33) does assume $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$. Use this to argue that $-\nabla^{[\mu} F^{\nu]}{}_\alpha = \nabla_\alpha F^{\mu\nu}$ and hence $\nabla^\mu T_{\mu\nu}[\mathrm{EM}] = -(\nabla^\mu F_{\mu\sigma}) F_\nu{}^\sigma$. □

Problem 7.13. Geodesic Equation from Stress Tensor Conservation Can you show that demanding the conservation of the point particle stress tensor in eq. (7.1.41), $\nabla_{\mu\ (\mathrm{pp})} T^{\mu\nu} = 0$, leads to the geodesic equation $D^2 z^\mu / \mathrm{d}\tau^2 = 0$? Hint: You may want to first show

$$\nabla_\mu T^{\mu\nu} = \frac{\partial_\mu \left(\sqrt{|g|} T^{\mu\nu} \right)}{\sqrt{|g|}} + \Gamma^\nu{}_{\alpha\beta} T^{\alpha\beta} \tag{7.2.65}$$

before applying it to the point particle stress tensor. □

Problem 7.14. Perfect Fluid Equations from Stress Tensor Conservation Suppose n is the number density of the particles in the co-moving frame of the fluid, and suppose further that particle number is conserved. That means

$$\nabla_\mu N^\mu = 0, \qquad N^\mu \equiv n U^\mu. \tag{7.2.66}$$

Now, demand that the stress tensor of the perfect fluid in eq. (7.2.22) be conserved: $\nabla_\mu T^{\mu\nu} = 0$. Start by writing

$$\nabla_\mu \left\{ (\rho + p) U^\mu U^\nu - p g^{\mu\nu} \right\} = \nabla_\mu \left\{ \left(\frac{\rho + p}{n} \right) N^\mu U^\nu - p g^{\mu\nu} \right\}; \tag{7.2.67}$$

and by taking into account the number conservation of eq. (7.2.66) – first show that the time component, i.e., $U_\nu \nabla_\mu T^{\mu\nu} = 0$, yields

$$N^\mu \nabla_\mu \left(\frac{\rho + p}{n} \right) = U^\mu \nabla_\mu p \equiv \frac{\mathrm{d}p}{\mathrm{d}\tau}, \tag{7.2.68}$$

where τ is the proper time of each infinitesimal packet of the fluid element. (Hint: You may need to use $U^\mu \nabla_\nu U_\mu = 0$; explain why this is true.) Carry out the differentiation to further deduce that eq. (7.2.68) is equivalent to

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = \frac{\rho + p}{n} \frac{\mathrm{d}n}{\mathrm{d}\tau}. \tag{7.2.69}$$

By using eq. (7.2.68), $\nabla_\mu T^{\mu\nu} = 0$ now translates to a generally covariant Newton's 2nd law:

$$(\rho + p)\, a^\mu = (g^{\mu\nu} - U^\mu U^\nu)\, \nabla_\nu p, \tag{7.2.70}$$

where the acceleration is

$$a^\nu \equiv U^\mu \nabla_\mu U^\nu. \tag{7.2.71}$$

Notice, in this relativistic context, that "inertial mass" is no longer ρ but actually energy plus pressure $\rho + p$. Can you explain why the gradient of pressure is force, $f^\nu \sim \nabla^\nu p$? Notice the $g^{\mu\nu} - U^\mu U^\nu$ is purely spatial. To this end: focus on the spatial components; go to a FNC system (cf. (2.5.6)-(2.5.8)); and consider an infinitesimal slab of fluid between x^i and $x^i + \mathrm{d}x^i$. □

Problem 7.15. Perfect Fluids: A Simple Model In this problem, we will consider a perfect fluid whose Lagrangian density $\mathcal{L}(n^2/2)$ depends on spacetime solely through the number density n – recall eq. (7.2.66). The particle number current is defined as

$$N^\mu \equiv \widetilde{\epsilon}^{\mu\alpha_1\dots\alpha_{d-1}} \nabla_{\alpha_1}\Phi^1 \nabla_{\alpha_2}\Phi^2 \dots \nabla_{\alpha_{d-1}}\Phi^{d-1}, \tag{7.2.72}$$

$$n \equiv |N_\sigma N^\sigma|^{1/2}; \tag{7.2.73}$$

and the $\{\Phi^{\rm I}|{\rm I} = 1, 2, \dots, d-1\}$ are scalar fields.

Calculate the equations-of-motion of these $d-1$ scalar fields and the associated stress-energy tensor of this theory. The latter will verify that this is, indeed, a simple model for a perfect fluid. Make sure you identify the pressure and energy densities – explain your reasoning! – in terms of the Lagrangian density.

Finally, in the spatially flat FLRW cosmology of eq. (3.0.16), verify that

$$\Phi^{\rm I} = x^{\rm I}, \qquad {\rm I} \in \{1, 2, \dots, d-1\}, \tag{7.2.74}$$

are exact solutions of this simple perfect fluid model. Since the $\{x^{\rm I}\}$ are the spatial Cartesian coordinates of the universe we may identify eq. (7.2.74) as describing the equilibrium configuration of the fluid itself. □

Problem 7.16. Alternate Point Particle Action Show that the following action

$$S_{\rm pp} \equiv -\frac{1}{2}\int {\rm d}s \left(e g_{\mu\nu}\dot{z}^\mu \dot{z}^\nu + \frac{m^2}{e} \right), \qquad \dot{z}^\mu \equiv \frac{{\rm d}z^\mu}{{\rm d}s}, \tag{7.2.75}$$

describes the dynamics of a point particle. Here, $e(s)$ acts like a Lagrange multiplier. Vary this action with respect to both $z^\mu(s)$ and $e(s)$ and argue this is equivalent to first solving for $e(s)$ in terms of the trajectory $z^\mu(s)$ and inserting it back into the action to obtain, for $m \neq 0$,

$$S_{\rm pp} = -m \int {\rm d}s \sqrt{g_{\mu\nu}\dot{z}^\mu \dot{z}^\nu}. \tag{7.2.76}$$

The action in eq. (7.2.75) is more general than the one in eq. (7.2.76) because the former is able to describe massless particles, where $m = 0$. □

Problem 7.17. Weyl Invariance Suppose a matter action

$$S_m \equiv \int {\rm d}^d x \sqrt{|g|} \mathcal{L}_m \tag{7.2.77}$$

is invariant under the replacement

$$g_{\mu\nu}(x) \to f(x) g_{\mu\nu}(x), \tag{7.2.78}$$

where $f(x)$ is an arbitrary smooth function. Argue that this implies the stress tensor of this theory $T^{\alpha\beta}$ is traceless, i.e.,

$$g_{\alpha\beta} T^{\alpha\beta} = 0. \tag{7.2.79}$$

(Hint: Step through the above metric-variation arguments, but let $g_{\mu\nu}(x) + \delta g_{\mu\nu}(x) \equiv (1 + \varepsilon(x)) g_{\mu\nu}(x)$ for a 'small' but arbitrary function $\varepsilon(x)$.)

In 4D, explain why the electromagnetic stress tensor in eq. (7.1.34) is traceless and also verify it through a direct calculation. □

Problem 7.18. Sign conventions Comment on the sign convention dependence of the Christoffel symbols, Riemann tensor, Ricci tensor, Ricci scalar, and the Einstein tensor. Hint: Count powers of the metric. □

Problem 7.19. Weyl Tensor The Weyl tensor $C_{\mu\nu\alpha\beta}$ is defined as the traceless part of the Riemann tensor. Because $R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta} = -R_{\mu\nu\beta\alpha}$, the Riemann is already traceless when its first two or last two indices are contracted. Check that the decomposition

$$R_{\mu\nu\alpha\beta} = C_{\mu\nu\alpha\beta} - \frac{g_{\mu[\alpha}g_{\beta]\nu}}{(d-2)(d-1)}\mathcal{R} + \frac{1}{d-2}\left(R_{\alpha[\mu}g_{\nu]\beta} - R_{\beta[\mu}g_{\nu]\alpha}\right). \tag{7.2.80}$$

does in fact yield a $C_{\mu\nu\alpha\beta}$ that is completely traceless: $g^{\mu\alpha}C_{\mu\nu\alpha\beta} = 0$. Make sure you also check that Weyl has the same index symmetries as Riemann (cf. eq. (2.4.38)).

Using the form of Einstein's General Relativity in eq. (7.1.26), verify that the Riemann tensor now reads

$$\begin{aligned} R_{\mu\nu\alpha\beta} = C_{\mu\nu\alpha\beta} &- \frac{2\Lambda}{(d-1)(d-2)}g_{\mu[\alpha}g_{\beta]\nu} \\ &+ 8\pi G_{\mathrm{N}}\left\{\frac{1}{d-2}\left(g_{\mu[\alpha}T_{\beta]\nu} - g_{\nu[\alpha}T_{\beta]\mu}\right) - \frac{2T}{(d-1)(d-2)}g_{\mu[\alpha}g_{\beta]\nu}\right\}. \end{aligned} \tag{7.2.81}$$

This implies, in situations where the cosmological constant can be neglected ($\Lambda = 0$) and when a geodesic observer is located in an empty region of spacetime – according to the geodesic deviation eq. (2.5.19), tidal forces are exerted by the Weyl tensor

$$\nabla_U\nabla_U\xi^\mu = -C^\mu{}_{\nu\alpha\beta}U^\nu\xi^\alpha U^\beta. \tag{7.2.82}$$

□

Problem 7.20. Inflationary cosmology Work out the equations-of-motion for a cosmology driven by a single scalar field ϕ, described by the action:

$$\int \mathrm{d}^4x\sqrt{|g|}\left(-\frac{\mathcal{R}}{16\pi G_{\mathrm{N}}} + \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right), \tag{7.2.83}$$

where V is an arbitrary potential. Assume that the geometry of the universe is spatially flat, namely either

$$\mathrm{d}s^2 = a(\eta)^2\left(\mathrm{d}\eta^2 - \mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}\right) \tag{7.2.84}$$

or

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a(t)^2\mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}; \tag{7.2.85}$$

and further assume the scalar field ϕ only depends on conformal or observer time. Hint: You should find that the scalar field obeys a damped and driven simple harmonic equation-of-motion. □

7.3 Symmetries and Spacetimes

Einstein's equations are nonlinear, and it is therefore very difficult to solve them in general. To make progress, we shall therefore impose physically well motivated symmetries on the metric, and attempt to solve Einstein's equations.

7.3.1 Spherical Bodies

No astrophysical body is perfectly spherical, of course, but to solve for the metric generated by the Sum, so that we may study motion around it, for example – we shall make this spherical symmetry assumption as a 'zeroth order' approximation. To this end, recall that the most general spherically symmetric metric is

$$\mathrm{d}s^2 = A(t,r)\mathrm{d}t^2 - B(t,r)\mathrm{d}r^2 - r^2\mathrm{d}\Omega. \tag{7.3.1}$$

Since A and B are arbitrary, it is technically easier to first re-write them as[70]

$$\mathrm{d}s^2 = e^{2\psi(t,r)}\left(1 - \frac{M(t,r)}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{M(t,r)}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega. \tag{7.3.2}$$

(All we have done is swap the two free functions A and B for the free functions ψ and M.) At this point, we may compute

$$r^2 G^t{}_t = \partial_r M = 8\pi G_{\mathrm{N}} r^2 T^0{}_0, \tag{7.3.3}$$

$$r^2 G_t{}^r = -\partial_t M = 8\pi G_{\mathrm{N}} r^2 T_0{}^r, \tag{7.3.4}$$

$$r(G^0{}_0 - G^r{}_r) = 2\left(1 - \frac{M}{r}\right)\partial_r\psi = 8\pi G_{\mathrm{N}} r(T^0{}_0 - T^r{}_r). \tag{7.3.5}$$

Outside the body, $T_{\mu\nu} = 0$ and therefore M must be constant and ψ independent of r. But then $e^{2\psi(t)}(\mathrm{d}t)^2$ may be redefined as the new time displacement

$$e^{\psi(t)}\mathrm{d}t \to \mathrm{d}t \tag{7.3.6}$$

while leaving the rest of the metric un-altered. This yields the Schwarzschild metric

$$\mathrm{d}s^2 = \left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{r_s}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \tag{7.3.7}$$

Notice we did not assume anything else about the matter $T_{\mu\nu}$.

> The geometry outside a spherically symmetric body, whose $T_{\mu\nu}$ can even be $t-$dependent, is that of Schwarzschild.

Inside the body, we will presumably have to integrate the equations for M and ψ in terms of the stress tensor components in the (t,r) subspace. **YZ: To be continued**

Problem 7.21. Verify equations (7.3.3)–(7.3.5). □

[70] Such a parametrization can be found in Poisson's *A Relativist's Toolkit*.

7.3.2 Cosmology and Maximal Symmetry

To study our universe on the largest length scales, we make the simplifying assumption that it is homogeneous and isotropic everywhere in space. This amounts to, assuming at a given time t, that the spatial geometry is maximally symmetric. This, in turn, yields the following metrics

$$\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}t^2 - a(t)^2 \mathrm{d}\vec{x}^2, && \text{(Spatially flat)}, && (7.3.8)\\
&= \mathrm{d}t^2 - a(t)^2\left(\mathrm{d}\vec{x}^2 + \frac{(R^{-1}\vec{x}\cdot\mathrm{d}\vec{x})^2}{1-\vec{x}^2/R^2}\right), && \text{(Spatial Curvature } 6/(a\cdot R)^2\text{)}, && (7.3.9)\\
&= \mathrm{d}t^2 - a(t)^2\left(\mathrm{d}\vec{x}^2 - \frac{(R^{-1}\vec{x}\cdot\mathrm{d}\vec{x})^2}{1+\vec{x}^2/R^2}\right), && \text{(Spatial Curvature } -6/(a\cdot R)^2\text{)}. && (7.3.10)
\end{aligned}$$

de Sitter Spacetime versus Minkowski When $\Lambda = 0$ in eq. (7.1.20) and spacetime is empty (i.e., $T_{\mu\nu} = 0$) we may readily verify that flat Minkowski spacetime $g_{\mu\nu} = \eta_{\mu\nu}$ is a solution to Einstein's equations – since the Riemann and therefore the Einstein tensor are both zero. When $\Lambda \neq 0$ and when spacetime is empty $T_{\mu\nu} = 0$, notice that flat/Minkowski spacetime is no longer a solution to Einstein's General Relativity in eq. (7.1.20): $G_{\mu\nu}[\eta] - \Lambda\eta_{\mu\nu} = -\Lambda\eta_{\mu\nu} \neq 0$. Over the past decades, cosmological probes have provided evidence that $\Lambda > 0$. For instance, systematic observations of distant Type Ia supernovae tell us, the universe appears to be expanding more and more quickly. In the next 3 problems, we shall explore a cosmological solution that satisfies Einstein's equations with $\Lambda > 0$ that approximates such a situation. The two key properties that make de Sitter the analog of Minkowski spacetime are: the former is a vacuum solution and has the same maximally allowed number of Killing vectors (i.e., $d(d+1)/2$ of them) – just like the latter.

Problem 7.22. de Sitter as Empty Spacetime with $\Lambda > 0$ Use the results of §(2.7) to verify that de Sitter spacetime, written here in a 'conformally flat' form

$$\mathrm{d}s^2 = \Omega^2\left(\mathrm{d}\eta^2 - \mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}\right), \qquad (7.3.11)$$

$$\Omega \equiv -\frac{1}{H\eta}, \qquad \eta \in (-\infty, 0),\ H > 0, \qquad (7.3.12)$$

is the solution to General Relativity in eq. (7.1.20) with $\Lambda > 0$ but without any matter: $T_{\mu\nu} = 0$. In particular, you need to find the relationship between the Hubble constant H and the cosmological constant Λ in arbitrary dimension d.

By finding the appropriate coordinate transformation $\eta = \eta(t)$, show that eq. (7.3.11) can be re-written as

$$\mathrm{d}s^2 = \mathrm{d}t^2 - e^{2Ht}\mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}, \qquad t \in \mathbb{R}. \qquad (7.3.13)$$

Why is t the 'observer' time? (Hint: Can you find the normalized tangent vector to the observer that remains at rest at some arbitrary spatial location $\vec{x}$?)

We see that the proper length between two arbitrary spatial locations $\vec{x}$ and $\vec{x}'$ on a constant-t hyper-surface grows exponentially with increasing time. de Sitter spacetime, as already alluded to, therefore describes a universe with an exponential rate of expansion. As our universe expands and matter within it gets more diluted – and if the 'Dark Energy' that is driving our 4D universe can be entirely attributed to the cosmological constant term of Einstein's General Relativity in eq. (7.1.20) – then eq. (7.3.11) will become a better and better approximation of the geometry of our universe on the largest distance scales. □

Problem 7.23. de Sitter as a Maximally Symmetric Spacetime Using the results of §(**??**), verify that the de Sitter spacetime of eq. (7.3.11) has the following Riemann tensor:

$$R_{\mu\nu\alpha\beta} = \frac{\mathcal{R}}{d(d-1)} g_{\mu[\alpha} g_{\beta]\nu}. \tag{7.3.14}$$

Also verify that the Ricci tensor and scalar are

$$R_{\mu\nu} = \frac{\mathcal{R}}{d} g_{\mu\nu} \qquad \text{and} \qquad \mathcal{R} = -\frac{2d\Lambda}{d-2}. \tag{7.3.15}$$

de Sitter spacetime is a maximally symmetric spacetime, with $d(d+1)/2$ Killing vectors.[71] Verify that the following are Killing vector of eq. (7.3.11):

$$T^\mu \partial_\mu \equiv -H x^\mu \partial_\mu, \qquad\qquad x^\mu \equiv (\eta, x^i). \tag{7.3.16}$$

and

$$K^\mu_{(i)} \partial_\mu \equiv x^i T^\mu \partial_\mu - H \bar{\sigma} \partial_{x^i}, \tag{7.3.17}$$

$$\bar{\sigma} \equiv \frac{1}{2}\left(\eta^2 - \vec{x}^2\right) = \frac{1}{2}\eta_{\mu\nu} x^\mu x^\nu. \tag{7.3.18}$$

(Hint: It is easier to use the right hand side of eq. (2.5.35) in eq. (2.5.36).) Can you write down the remaining Killing vectors? (Hint: Think about the symmetries on a constant-η surface.) Using (some of) these $d(d+1)/2$ Killing vectors and eq. (2.5.64), explain why the Ricci scalar of the de Sitter geometry is a spacetime constant.

Note that, since $-Hx^0 = -H\eta = 1/\Omega(\eta)$ (cf. (7.3.12)), the Killing vector in eq. (7.3.16) may also be expressed as

$$T^\mu \partial_\mu = \frac{1}{\Omega(\eta)} \partial_\eta - H x^i \partial_i = \partial_t - H x^i \partial_i; \tag{7.3.19}$$

where the second equality follow from $\Omega(\eta)\mathrm{d}\eta = \mathrm{d}t \Leftrightarrow \Omega^{-1}\partial_\eta = \partial_t$ (cf. equations (7.3.11) and (7.3.13)). This means we may take the flat spacetime limit by setting $H \to 0$, and hence identify $T^\mu \partial_\mu$ as the de Sitter analog of the generator of time translation symmetry in Minkowski spacetime. □

Problem 7.24. d-de Sitter as a hyperboloid in $(d+1)$-Minkowski In this problem, we shall see that d-dimensional de Sitter spacetime can be viewed as a hyperboloid in $(d+1)$-dimensional Minkowski spacetime. Let $x^\mu \equiv (\eta, x^i)$ be the coordinates in eq. (7.3.11) and let X^A

[71] As Weinberg [1] explains, maximally symmetric spacetimes are essentially unique, in that they are characterized by a single dimension-ful scale. We see that this scale is nothing but the cosmological constant Λ.

be coordinates in the ambient Minkowski spacetime (where A runs from 0 to d); and consider

$$\begin{aligned}
X^0 &= \frac{1}{2\eta}\left(\eta^2 - \vec{x}^2 - \frac{1}{H^2}\right), \\
X^d &= \frac{1}{2\eta}\left(-\eta^2 + \vec{x}^2 - \frac{1}{H^2}\right), \\
X^i &= \frac{x^i}{H\eta}, \qquad\qquad i \in \{1, 2, \dots, d-1\}.
\end{aligned} \tag{7.3.20}$$

First show that the induced metric is eq. (7.3.11):

$$(\mathrm{d}s^2)_{\text{Eq.(7.3.11)}} = \eta_{AB}\frac{\partial X^A}{\partial x^\mu}\frac{\partial X^B}{\partial x^\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{7.3.21}$$

Then verify that the embedding formulas in eq. (7.3.20) in fact satisfy the hyperboloid equation

$$-(X^0)^2 + \vec{X}^2 = -\eta_{AB}X^A X^B = \frac{1}{H^2}. \tag{7.3.22}$$

It turns out eq. (7.3.20), and hence eq. (7.3.11), really only describes half of the hyperboloid in eq. (7.3.22). □

Problem 7.25. de Sitter and Lorentz Invariance Notice the Lorentz invariance of eq. (7.3.22): any X'^A related to X^A via a Lorentz transformation $\Lambda^A{}_B$, i.e., all $X'^A = \Lambda^A{}_B X^B$, satisfies eq. (7.3.22). This in turns allows one to relate the isometries of de Sitter spacetime to this ambient Minkowski Lorentz symmetry. For example, equations (7.3.16) and (7.3.17) are related to boosts in this ambient Minkowski spacetime. Can you compute the Lorentz generators

$$J^{AB} \equiv x^{[A}\partial^{B]} \tag{7.3.23}$$

on the de Sitter hyperboloid, and recover all the $d(d+1)/2$ Killing vectors from Problem (7.23)? One way to parametrize a hyperboloid is to do

$$X^A = \rho\left(\sinh[\tau], \widehat{r}(\vec{\theta})\cosh[\tau]\right), \tag{7.3.24}$$

where $\widehat{r}$ is the unit spatial radial vector. de Sitter is simply the $\rho = 1/H$ surface. □

8 Weakly Curved Spacetimes and Gravitational Waves

In this section, we shall study the weakly curved spacetime generated by an isolated self-gravitating system situated in an otherwise empty Minkowski spacetime. This includes the very vibrations of spacetime itself – aka *gravitational waves* (GWs) – set up by the motion/internal dynamics of the system. These GWs carry energy-momentum away from the source to infinity, leading to time-reversal violating dissipation. Such weak field systems include our own Solar System, and the early in-spiral stages of compact binary neutron star/small black hole systems.[72] The results developed here, as well as their nonlinear generalizations, form much of the theoretical basis underlying the most precise experimental tests of General Relativity to date.

The geometry of a weakly curved spacetime may be described as a small deviation from its flat cousin:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{8.0.1}$$

Denoting the stress tensor of the isolated system as $T_{\mu\nu}$, we proceed to solve Einstein's equation (7.1.20), but without the cosmological constant term Λ,

$$G_{\mu\nu} = 8\pi G_{\rm N} T_{\mu\nu}. \tag{8.0.2}$$

The (heuristic) justification for neglecting the cosmological constant term goes as follows. By comparing a typical term $\partial_\mu \partial_\nu h_{\alpha\beta}$ in the linearized Einstein's equations (see below) with the cosmological constant term $\Lambda h_{\alpha\beta}$, and if we attribute all of Dark Energy to the presence of Λ, then $1/\sqrt{\Lambda} \sim \mathcal{O}(\text{few Gpc})$.

The fractional correction to the weak field metric generated by isolated self-gravitating systems due to the cosmological constant, is expected to be of the order

$$\epsilon_\Lambda \equiv \left(\frac{\text{larger of the characteristic time- or length-scale of source}}{\text{few Gpc}} \right)^2 . \tag{8.0.3}$$

For instance, Pluto takes 248 years to orbit around the Sun; this is much larger than its roughly 40 astronomical units distance from the Sun. But since 1 light year is roughly 3×10^{-10} Gpc, 248 years is roughly 10^{-7} Gpc, and hence $\epsilon_\Lambda[\text{Pluto-Sun}] \sim 10^{-14}$.

8.1 Linearized Einstein's Equations

We will begin by solving the linearized version of Einstein's equations in eq. (8.0.2).

$$\delta_1 G_{\mu\nu} = 8\pi G_{\rm N} \delta_0 T_{\mu\nu} \equiv 8\pi G_{\rm N} \overline{T}_{\mu\nu}; \tag{8.1.1}$$

[73]where all the indices are to be moved with $\eta_{\mu\nu}$. The linearized Einstein tensor can be gotten by setting $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ (and, hence, $\overline{\nabla}_\mu = \partial_\mu$) in eq. (2.6.37), while the zeroth order stress tensor

[72]Black holes are, by themselves, strong gravity systems. However, as long as they are separated by distances large compared to their individual sizes, the effective dynamics of the binary system and the gravitational waves they generate, may be modeled through the weak field analysis we are performing here.

[73]Note that both the Einstein tensor on the left hand side and the stress energy tensor on the right hand side are usually nonlinear functions of the full metric $g_{\mu\nu}$, and one should therefore expect both to admit an infinite series in $h_{\mu\nu}$.

$\overline{T}_{\mu\nu}$ is simply the matter stress tensor evaluated on $g_{\mu\nu} = \eta_{\mu\nu}$. Because the background metric is flat, the barred geometric tensors are zero.

$$\delta_1 G_{\mu\nu} \equiv -\frac{1}{2}\left(\partial^2 \bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial_\sigma\partial_\rho \bar{h}^{\sigma\rho} - \partial_{\{\mu}\partial^\sigma \bar{h}_{\nu\}\sigma}\right), \tag{8.1.2}$$

with (cf. eq. (2.6.35))

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h. \tag{8.1.3}$$

Under an infinitesimal coordinate transformation

$$x^\mu \to x^\mu + \xi^\mu, \tag{8.1.4}$$

we have already deduced in eq. (2.6.46) that we may attribute all the transformations to that of the metric perturbation, handing us the following 'gauge transformation' replacement rule

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \tag{8.1.5}$$

The trace of this gauge transformation rule is

$$h \to h + 2\partial_\sigma \xi^\sigma. \tag{8.1.6}$$

According to eq. (8.1.3), the barred graviton transforms as

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial_\sigma \xi^\sigma, \tag{8.1.7}$$

$$\bar{h} \to \bar{h} + (2-d)\partial^\sigma \xi_\sigma. \tag{8.1.8}$$

Because the Riemann tensor, Ricci tensor, Einstein tensor, and Ricci scalar are zero when evaluated on the 'background metric' $\eta_{\mu\nu}$, from Problem (2.56), we know that the linearized versions of these tensors – in particular, the Einstein tensor $\delta_1 G_{\mu\nu}$ in eq. (8.1.2) – must be gauge-invariant: they takes exactly the same expression after the replacement in eq. (8.1.7). Just as the gauge-invariance of the electromagnetic tensor $F_{\mu\nu}$ gave rise to an infinity of vector potential A_μ solutions for a given physical setup, we see that this gauge-invariance of linearized weak field gravitation also gives an infinity of graviton $\bar{h}_{\alpha\beta}$ solutions.

Problem 8.1. Graviton Wave Operator Cannot Be Inverted Just as gauge-invariance implied the differential (wave) operator acting on the vector potential cannot be inverted unless we 'fixed a gauge', such is the case for linearized gravitation. In Fourier spacetime ($\partial_\mu \to -ik_\mu$), first verify that eq. (8.1.1) reads:

$$K^{\mu\nu\alpha\beta}\widetilde{\bar{h}}_{\alpha\beta} = 8\pi G_{\rm N}\widetilde{\overline{T}}^{\mu\nu}, \tag{8.1.9}$$

$$K^{\mu\nu\alpha\beta} \equiv \frac{1}{2}\left(\frac{1}{2}\eta^{\mu\{\alpha}\eta^{\beta\}\nu}k^2 + \eta^{\mu\nu}k^\alpha k^\beta - \frac{1}{2}k^\alpha k^{\{\mu}\eta^{\nu\}\beta} - \frac{1}{2}k^\beta k^{\{\mu}\eta^{\nu\}\alpha}\right). \tag{8.1.10}$$

Then demonstrate that the 'pure gauge' piece on the right hand side of eq. (8.1.7) is annihilated by $K^{\mu\nu\alpha\beta}$, namely

$$K^{\mu\nu\alpha\beta}\mathrm{PG}_{\alpha\beta} = 0 \tag{8.1.11}$$

$$\mathrm{PG}_{\alpha\beta} \equiv k_\alpha \delta^{(\lambda)}_\beta + k_\beta \delta^{(\lambda)}_\alpha - \eta_{\alpha\beta}k^{(\lambda)}; \tag{8.1.12}$$

where λ here is to be viewed as a fixed number, labeling the λth null eigenvector of $K^{\mu\nu\alpha\beta}$. (There are $\lambda \in \{0, 1, 2, \dots, d-1\}$ such null vectors of the operator $K^{\mu\nu\alpha\beta}$.) This means it should not be possible to find the inverse K^{-1}:

$$K^{\mu\nu\alpha\beta}\left(K^{-1}\right)_{\alpha\beta\rho\sigma} = \frac{1}{2}\delta^{\mu}_{\{\rho}\delta^{\nu}_{\sigma\}} \qquad \text{(Not possible)}. \tag{8.1.13}$$

□

de Donder gauge Just as we 'fixed a gauge' for the vector potential A_μ to solve Maxwell's equations, we will now choose the de Donder gauge to solve the linearized Einstein's equations. In analogy with the Lorenz gauge of electromagnetism, it reads – in Fourier and real spacetime –

$$k^\mu \widetilde{\overline{h}}_{\mu\nu} = k^\mu \left(\widetilde{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\widetilde{h}\right) = 0, \tag{8.1.14}$$

$$\partial^\mu \overline{h}_{\mu\nu} = \partial^\mu \left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\right) = 0. \tag{8.1.15}$$

This choice renders all terms in eq. (8.1.2) zero except the $\partial^2\overline{h}_{\mu\nu}$ term. At this point, the linearized Einstein's equations in eq. (8.1.1) have been massaged into the form

$$\partial^2 \overline{h}_{\mu\nu} = -16\pi G_{\rm N}\overline{T}_{\mu\nu}. \tag{8.1.16}$$

This is a set of $d \times d$ scalar wave equations.

Existence If $h_{\mu\nu}$ were not in the de Donder gauge, we may refer to equations (8.1.5) and (8.1.6) and demand the new gravitational perturbation field variables

$$h'_{\mu\nu} \equiv h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu \tag{8.1.17}$$

obey the de Donder gauge

$$\partial^\mu h'_{\mu\nu} = \frac{1}{2}\partial_\nu h', \tag{8.1.18}$$

$$\Leftrightarrow \partial^\mu h_{\mu\nu} + \partial^2\xi_\nu + \partial_\nu\partial^\mu\xi_\mu = \frac{1}{2}\partial_\nu h + \partial_\nu\partial_\sigma\xi^\sigma \tag{8.1.19}$$

$$\Leftrightarrow \partial^2\xi_\nu = \partial^\mu\left(\frac{1}{2}\eta_{\mu\nu}h - h_{\mu\nu}\right). \tag{8.1.20}$$

In other words, by choosing the gauge transformation vector ξ_ν to be the solution to the wave equation (8.1.20), the gravitational perturbation $h_{\mu\nu}$ written in any gauge can be re-expressed in the de Donder one. Since $1/\partial^2$ exists, therefore, the de Donder gauge must also exist.

Problem 8.2. $h_{\mu\nu}$ as a Lorentz tensor The gravitational perturbation $h_{\mu\nu}$ is not a spacetime tensor, because it is the full metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ that transforms as a rank-2 tensor:

$$g_{\alpha'\beta'}(x') = g_{\mu\nu}(x)\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} \tag{8.1.21}$$

$$= \left(\eta_{\mu\nu} + h_{\mu\nu}(x)\right)\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} \tag{8.1.22}$$

$$\neq \eta_{\alpha\beta} + h_{\alpha'\beta'}(x'). \qquad (8.1.23)$$

However, if we restrict our attention to only Poincaré transformations

$$x^\mu = \Lambda^\mu{}_\nu x'^\nu + a^\mu, \qquad (8.1.24)$$

show that

$$h_{\alpha'\beta'}(x') = h_{\mu\nu}(x = \Lambda \cdot x' + a)\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta. \qquad (8.1.25)$$

That ia, the perturbation $h_{\mu\nu}$ can be regarded as a Lorentz tensor. Hint: You will find that the Lorentz invariance of $\eta_{\mu\nu}$ to be crucial here. □

Conservation of stress-energy At this zeroth order, and for material sources held together by non-gravitational forces, the conservation of stress-energy reads

$$\partial^\mu \overline{T}_{\mu\nu} = 0. \qquad (8.1.26)$$

This holds, for e.g., for stellar or planetary interiors; since these astrophysical bodies are held together *primarily* by electromagnetic and nuclear forces – not gravity – as long as they are not too massive (their sizes much larger than their Schwarzschild radii). This *does not* hold for astrophysical binary or $n-$body systems, where two or more bodies orbit around each other via their mutual gravity. In this latter case, to account for inter-body gravitational forces, the stress-tensor of the gravitational field itself must be just as important as that of the material stress-tensor; and, hence, divergence of $\overline{T}_{\mu\nu}$ alone cannot be zero.

Assuming eq. (8.1.26), let us define the total mass to be

$$M \equiv \int_{\mathbb{R}^D} \mathrm{d}^D\vec{x}\,\overline{T}^{00}(t, \vec{x}); \qquad (8.1.27)$$

the total dipole moment to be

$$d^i \equiv \int_{\mathbb{R}^D} \mathrm{d}^D\vec{x}\,x^i\overline{T}^{00}(t, \vec{x}); \qquad (8.1.28)$$

the total quadrupole moment to be

$$Q^{ij} \equiv \int_{\mathbb{R}^D} \mathrm{d}^D\vec{x}\,x^i x^j\overline{T}^{00}(t, \vec{x}); \qquad (8.1.29)$$

the total linear momentum to be

$$P^i \equiv \int_{\mathbb{R}^D} \mathrm{d}^D\vec{x}\,\overline{T}^{0i}(t, \vec{x}); \qquad (8.1.30)$$

and the total angular momentum to be

$$L^{ab} \equiv \int_{\mathbb{R}^D} \mathrm{d}^D\vec{x}\,x^{[a}\overline{T}^{b]0}(t, \vec{x}). \qquad (8.1.31)$$

We may readily show that, for instance, that total mass is a constant:

$$\dot{M} = \int_{\mathbb{R}^D} \mathrm{d}^D\vec{x}\partial_0\overline{T}^{00}(t,\vec{x}) \tag{8.1.32}$$

$$= -\int_{\mathbb{R}^D} \mathrm{d}^D\vec{x}\partial_i\overline{T}^{i0}(t,\vec{x}). \tag{8.1.33}$$

The last line may be converted into a surface integral at spatial infinity, which is zero as long as the matter source itself is isolated.

Problem 8.3. Conservation and time derivatives Show that the time derivative of the dipole is the linear momentum,

$$\dot{d}^i = P^i; \tag{8.1.34}$$

the first time derivative of the quadrupole yields

$$\dot{Q}^{ij} = \int_{\mathbb{R}^D} \mathrm{d}^D\vec{x}x^{\{i}\overline{T}^{j\}0}(t,\vec{x}); \tag{8.1.35}$$

the second time derivative of the quadrupole yields

$$\ddot{Q}^{ij} = 2\int_{\mathbb{R}^D} \mathrm{d}^D\vec{x}\overline{T}^{ij}(t,\vec{x}); \tag{8.1.36}$$

and, finally, the linear and angular momentum are constant,

$$\dot{P}^i = 0 = \dot{L}^{ab}. \tag{8.1.37}$$

Be sure to explain why the constancy of angular momentum depends crucially on the symmetric nature of the stress tensor; specifically $\overline{T}^{ij} = \overline{T}^{ji}$. □

8.2 Stationary, Non-Relativistic Limit

Before moving on to solving the full set of the linearized Einstein's equations, let us specialize to the time-independent (aka stationary) limit, where the zeroth order stress tensor of some isolated matter $\overline{T}_{\mu\nu}(\vec{x})$ is now assumed to be only dependent on $\vec{x}$ but not t. Furthermore, we shall also assume the matter is non-relativistic, so that the momentum density $\overline{T}^{i0}$ is suppressed relative to the energy density $\overline{T}^{00}$ as $v_m \ll 1$, the characteristic speed of the internal motion of the matter source itself. The shear-stress $\overline{T}^{ij}$ is suppressed relative to $\overline{T}^{00}$ by v_m^2. These assumptions tell us that its mass M, dipole d^i, quadrupole Q^{ij}, linear momentum P^i, and angular momentum L^{ab} are all constants. But dipole being constant means, by Problem (8.3), the linear momentum must be zero. Along the same lines,

$$\int_{\mathbb{R}^D} \mathrm{d}^D\vec{x}x^{\{i}\overline{T}^{j\}0}(t,\vec{x}) = 0 = \int_{\mathbb{R}^D} \mathrm{d}^D\vec{x}\overline{T}^{ij}(t,\vec{x}). \tag{8.2.1}$$

Additionally, we may also choose the origin of the spatial coordinate system such that the dipole is zero. Let $\vec{x}_0$ be the coordinate displacement vector joining one such system to another. Consider, then,

$$\int_{\mathbb{R}^D} \mathrm{d}^D \vec{x} \overline{T}_{00} \left(x^i - x_0^i \right) = d^i - M \cdot x_0^i. \tag{8.2.2}$$

The $\vec{d}$ is the dipole in the $\vec{x}$ system; whereas the $d^i - M \cdot x_0^i$ is the dipole in the $\vec{x}' \equiv \vec{x} - \vec{x}_0$ system. But since $\vec{x}_0$ was arbitrary, we may simply choose it such that $\vec{d} = M \cdot \vec{x}_0$.

Problem 8.4. Static Newtonian Spacetimes Even though this section's primary goal is the study of (time dependent) gravitational waves, in this problem, we shall examine how Newtonian gravity is recovered from taking the static limit of Einstein's equations.

As a start, we shall assume static means the $h-$independent piece of the matter stress tensor $\bar{T}_{\mu\nu}(\vec{x})$ does not depend on time. Show that, if the metric takes the following time-independent form

$$\mathrm{d}s^2 = (1 + 2\Phi(\vec{x}))\, \mathrm{d}t^2 - (1 - 2\Phi(\vec{x}))\, \mathrm{d}\vec{x} \cdot \mathrm{d}\vec{x} - 2A_i(\vec{x}) \mathrm{d}t \mathrm{d}x^i, \tag{8.2.3}$$

i.e., where Φ and A_i do not depend on t; then the linearized Einstein tensor reads

$$\delta_1 G^{00} = 2\vec{\nabla}^2 \Phi \tag{8.2.4}$$

$$\delta_1 G^{0i} = \delta_1 G^{i0} = \frac{1}{2} \vec{\nabla}^2 A_i(\vec{x}) \tag{8.2.5}$$

$$\delta_1 G^{ij} = 0; \tag{8.2.6}$$

with $\vec{\nabla}^2$ denoting the Euclidean space Laplacian. (You may need to argue that $\partial_i A_i = 0$.) Hence, the linearized Einstein's equations with $\Lambda = 0$ in turn becomes

$$\vec{\nabla}^2 \Phi = 4\pi G_{\mathrm{N}} \bar{T}^{00}(\vec{x}) \qquad \text{and} \qquad \vec{\nabla}^2 A_i = 16\pi G_{\mathrm{N}} \bar{T}^{0i}(\vec{x}); \tag{8.2.7}$$

whereas $\bar{T}^{ij}$ has to be trivial.

This calculation teaches us, Φ is the gravitational potential in Newtonian gravity; but even in this static limit, General Relativity introduces an extra 'vector potential' A_i that has no counterpart in Newtonian gravity, sourced by the static matter momentum current $\bar{T}_{0i}$.

Bonus: Can you generalize this analysis to the case where $\bar{T}_{ij}(\vec{x})$ is static but non-zero? □

The solutions to eq. (8.2.7) are given by convolutions against the Laplacian's Green's function $G[\vec{x} - \vec{x}] = -1/(4\pi|\vec{x} - \vec{x}'|)$:

$$\Phi(\vec{x}) = -G_{\mathrm{N}} \int_{\mathbb{R}^{\not\models}} \mathrm{d}^3 \vec{x}' \frac{\overline{T}^{00}(\vec{x}')}{|\vec{x} - \vec{x}'|}, \tag{8.2.8}$$

$$A_i(\vec{x}) = -4G_{\mathrm{N}} \int_{\mathbb{R}^{\not\models}} \mathrm{d}^3 \vec{x}' \frac{\overline{T}^{0i}(\vec{x}')}{|\vec{x} - \vec{x}'|}. \tag{8.2.9}$$

To gain some insight into this Newtonian spacetime, let us situate an observer outside the body and assume $\vec{d} = \vec{0}$, so that we may Taylor expand

$$\frac{1}{|\vec{x}-\vec{x}'|} = \exp\left(-\vec{x}'\cdot\vec{\nabla}_{\vec{x}}\right)\frac{1}{|\vec{x}|} \tag{8.2.10}$$

$$= \frac{1}{|\vec{x}|} + \vec{x}'\cdot\frac{\vec{x}}{|\vec{x}|^3} + \dots . \tag{8.2.11}$$

If we define $r \equiv |\vec{x}|$ and $\widehat{r} \equiv \vec{x}/|\vec{x}|$, we may therefore assert

$$\Phi(\vec{x}) = -\frac{G_{\rm N}}{r}\int \mathrm{d}^3\vec{x}'\left(1+\frac{\vec{x}'\cdot\vec{x}}{r^2}+\dots\right)\overline{T}^{00} = -\frac{G_{\rm N}}{r}\left(M+\frac{\vec{d}\cdot\widehat{r}}{r}+\dots\right) \tag{8.2.12}$$

$$= -\frac{G_{\rm N}M}{r}\left(1+\mathcal{O}[(r'/r)^2]\right) \tag{8.2.13}$$

$$A_i(\vec{x}) = -\frac{4G_{\rm N}}{r}\int \mathrm{d}^3\vec{x}'\left(1+\frac{\vec{x}'\cdot\vec{x}}{r^2}+\dots\right)\overline{T}^{0i} = -\frac{4G_{\rm N}}{r}\left(P^i+\frac{\widehat{r}^j}{2r}\left(x'^{\{j}\overline{T}^{i\}0}+x'^{[j}\overline{T}^{i]0}\right)+\dots\right)$$

$$= -\frac{2G_{\rm N}}{r^2}\widehat{r}^j L^{ji}\left(1+\mathcal{O}[(r'/r)]\right). \tag{8.2.14}$$

That is, in the $t-$independent limit and center-of-mass frame, the Newtonian scalar and vector potentials go as (total mass)$/r$ and (total angular momentum)$/r^2$ respectively.

Next, we may compute the 'electric' and 'magnetic' fields.

$$E^i = -\partial_i\Phi = -\frac{G_{\rm N}M}{r^2}\widehat{r}^i + \dots, \tag{8.2.15}$$

$$F_{jk} = \partial_j\left(-\frac{2G_{\rm N}\widehat{r}^l L^{lk}}{r^2}\right) - (j\leftrightarrow k) + \dots \tag{8.2.16}$$

$$= \frac{4G_{\rm N}\widehat{r}^l L^{lk}}{r^3}\widehat{r}^j - \frac{2G_{\rm N}P^{lj}L^{lk}}{r^3} - (j\leftrightarrow k) + \dots \tag{8.2.17}$$

$$= \frac{4G_{\rm N}\widehat{r}^l L^{l[k}\widehat{r}^{j]}}{r^3} - \frac{2G_{\rm N}}{r^3}\left(L^{[jk]} - \widehat{r}^l L^{l[k}\widehat{r}^{j]}\right) + \dots, \tag{8.2.18}$$

$$= -\frac{4G_{\rm N}}{r^3}\left(L^{jk}+\frac{3}{2}\widehat{r}^{[j}L^{k]l}\widehat{r}^l\right) + \dots; \tag{8.2.19}$$

where we have employed

$$\partial_i\widehat{r}^j = \frac{P^{ij}}{r}, \tag{8.2.20}$$

$$P^{ij} \equiv \delta^{ij} - \widehat{r}^i\widehat{r}^j. \tag{8.2.21}$$

Problem 8.5. Gravitoelectromagnetism Show that, in the non-relativistic limit, geodesics $z^i(t)$ in the spacetime of eq. (8.2.3) may be written in the form

$$\frac{\mathrm{d}^2\vec{z}}{\mathrm{d}t^2} = \vec{E} + \frac{\mathrm{d}\vec{z}}{\mathrm{d}t}\times\vec{B}; \tag{8.2.22}$$

where the 'electric' and 'magnetic' fields are

$$E^i = -\partial_i \Phi, \tag{8.2.23}$$

$$B^i = \frac{1}{2}\epsilon_{ijk} F_{jk}, \qquad F_{jk} \equiv \partial_j A_k - \partial_k A_j. \tag{8.2.24}$$

Here, $\epsilon_{123} \equiv 1$. Notice the resemblance of eq. (8.2.22) to the Lorentz force law of electromagnetism. □

Problem 8.6. Frame Dragging Argue that a geodesic whose velocity was initially radial – i.e., $\mathrm{d}\vec{z}(t_0)/\mathrm{d}t = v(t_0)\widehat{r}$, for some initial time $t = t_0$ – would not remain purely radial due to the presence of A_i in eq. (8.2.3). You should find that a rotating body that generates a non-zero A_i would therefore 'drag' free-falling objects with it. □

8.3 de Donder gauge in background (3+1)D Minkowski spacetime

In the physically important case of $d = 4$, we may use the retarded Green's function $1/\partial^2$ in eq. (5.1.1) to solve eq. (8.1.16).

$$\begin{aligned} \bar{h}_{\mu\nu}(t,\vec{x}) &= -16\pi G_{\rm N} \int_{\mathbb{R}^{3,1}} \mathrm{d}^4 x' G_4^+(x-x')\overline{T}_{\mu\nu}(x') \\ &= -4G_{\rm N} \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}' \frac{\overline{T}_{\mu\nu}\left(t-|\vec{x}-\vec{x}'|,\vec{x}'\right)}{|\vec{x}-\vec{x}'|} \end{aligned} \tag{8.3.1}$$

Causality and Retarded Green's Functions The choice of using the retarded Green's function ensures the gravitational perturbation solution of eq. (8.3.1) obeys causality – 'cause precedes effect'. To see this, note that constraint $t - t' = |\vec{x} - \vec{x}'|$ imposed by the δ-function in eq. (5.1.1), which in turn leads in eq. (8.3.1) to the stress tensor evaluated at retarded time

$$t_r \equiv t - |\vec{x} - \vec{x}'|. \tag{8.3.2}$$

The t' is the time of emission from $\vec{x}'$ while t is the observation time at $\vec{x}$. The null cone emanating from $(t', \vec{x}')$ is described by the locus of spacetime points $(t, \vec{x})$ obeying the constraints $(t - t')^2 = (\vec{x} - \vec{x}')^2$ – the *forward* light cone $t - t' = |\vec{x} - \vec{x}'| > 0$ is the half where emission precedes observation.

Vacuum Solutions & Spin-2 Just as the vacuum solutions of the Lorenz-gauge vector potential A_μ in electromagnetism led to the concept of massless spin-1 (i.e., helicity ± 1) waves, we shall see a similar discourse using the de Donder gauge will lead us to see that vacuum 4D linearized General Relativity yields massless spin-2 gravitational waves.

Let us set to zero the stress energy tensor $\overline{T}_{\mu\nu}$ in the linearized Einstein's equation (8.1.16).

$$\partial^2 \bar{h}_{\mu\nu} = 0 \tag{8.3.3}$$

Problem 8.7. Explain why eq. (8.3.3) implies

$$\partial^2 h_{\mu\nu} = 0. \tag{8.3.4}$$

Are equations (8.3.3) and (8.3.4) *equivalent* – i.e., does one imply the other? Recall the relationship between $\bar{h}_{\mu\nu}$ and $h_{\mu\nu}$ in eq. (8.1.3). □

If we follow the parallel discussion for spin-1 electromagnetic waves, we find that a single Fourier mode solution may be expressed as

$$h_{\mu\nu}(x) = 2 \text{ Re}\left\{\epsilon_{\mu\nu}(\vec{k}) \exp(-ik \cdot x)\right\} \tag{8.3.5}$$

$$k_\mu = (k, k_i), \qquad\qquad k \equiv |\vec{k}|. \tag{8.3.6}$$

The de Donder gauge condition in eq. (8.1.14) tells us

$$k^\mu \epsilon_{\mu\nu} = \frac{1}{2} k_\nu \epsilon, \qquad \epsilon \equiv \eta^{\sigma\rho} \epsilon_{\sigma\rho}. \tag{8.3.7}$$

Breaking it into $\nu = 0$ and $\nu = j$ components,

$$\frac{k^i}{k_0} \epsilon_{i0} = -\frac{1}{2}(\epsilon_{00} + \epsilon_{ll}), \tag{8.3.8}$$

$$\epsilon_{0j} = -\frac{k^i}{k^0}\epsilon_{ij} + \frac{1}{2}\frac{k_j}{k^0}(\epsilon_{00} - \epsilon_{ll}). \tag{8.3.9}$$

Problem 8.8. Further Constraints Show that

$$\frac{k^i k^j \epsilon_{ij}}{k_0^2} = \epsilon_{ll}. \tag{8.3.10}$$

Hint: Start by contracting eq. (8.3.9) with k^j. □

Eq. (8.3.7) yields 4 constraints for the 10 algebraically independent components of $\epsilon_{\mu\nu}$; hence, there are actually $10 - 4 = 6$ basis polarization tensors. The general homogeneous solution $h_{\mu\nu}$ is therefore a superposition of the $\vec{k}-$modes on the right hand side in eq. (8.3.5).

$$h_{\mu\nu}(x) = \sum_{\sigma=1}^{6} \left\{\epsilon_{\mu\nu}^{(\sigma)}(\vec{k}) \exp(-ik \cdot x) + \epsilon_{\mu\nu}^{(\sigma)}(\vec{k})^* \exp(+ik \cdot x)\right\}, \tag{8.3.11}$$

where $\epsilon_{\mu\nu}^{(\sigma)}$ is the $\sigma-$th basis polarization tensor.

Since each derivative ∂_μ acting on $h_{\mu\nu}$ corresponds to the replacement $\partial_\mu \to -ik_\mu$, we have from eq. (8.3.4) the massless condition

$$k_\sigma k^\sigma = k_0^2 - \vec{k}^2 = 0. \tag{8.3.12}$$

For example, let us consider a plane wave propagating along the positive 3−direction:

$$k_\mu = k(1, 0, 0, -1) \qquad \text{and} \qquad k^\mu = k(1, 0, 0, 1). \tag{8.3.13}$$

This choice of k_μ means our plane wave is proportional to $\exp(-ik(t - x^3))$, and is indeed traveling along the positive 3−direction. More generally, the general plane wave

$$\exp(-ik \cdot x) = \exp\left[ik\left(\widehat{k}^j x^j - t\right)\right], \tag{8.3.14}$$

$$\widehat{k}^i \equiv \frac{k^i}{k}, \qquad\qquad k \equiv k_0 = k^0, \tag{8.3.15}$$

indicates $\widehat{k}$ is the propagation direction in space.

Next, if we recall the discussion below eq. (8.1.7), that the perturbation

$$\Delta\bar{h}_{\mu\nu} \equiv \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial_\sigma \xi^\sigma \tag{8.3.16}$$

is 'pure gauge' in that the linearized geometric tensors built out of $\bar{h}_{\mu\nu} + \Delta\bar{h}_{\mu\nu}$ is the same as that built solely out of $\bar{h}_{\mu\nu}$. If these ξ^σ themselves obey the wave equation

$$\partial^2 \xi^\sigma = 0, \tag{8.3.17}$$

then we check that $\Delta\bar{h}_{\mu\nu}$ obeys the de Donder gauge as well:

$$\partial^\mu \Delta\bar{h}_{\mu\nu} = \partial^2 \xi_\nu + \partial_\nu \partial_\sigma \xi^\sigma - \partial_\nu \partial_\sigma \xi^\sigma = 0. \tag{8.3.18}$$

Synchronous gauge We will now proceed to add such a pure gauge term,

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \equiv h'_{\mu\nu}, \tag{8.3.19}$$

so our new metric perturbation has only non-trivial space-space components:

$$h'_{0\mu} = h'_{\mu 0} = 0. \tag{8.3.20}$$

This is known as the synchronous gauge – below, we will see why this is a useful gauge for understanding gravitational waves. Remember, however, we are still within the de Donder gauge as long as eq. (8.3.17) is obeyed. In fact, by eq. (8.3.12), this is guaranteed if we take

$$\xi_\mu(x) = 2\mathrm{Re}\left\{\widetilde{\xi}(\vec{k})e^{-ik\cdot x}\right\}. \tag{8.3.21}$$

Furthermore, since the only condition we used to obtain eq. (8.1.1) was the de Donder gauge itself, it must follow that $h'_{\mu\nu}$ obey the same homogeneous wave equation

$$\partial^2 h'_{\mu\nu} = 0; \tag{8.3.22}$$

which in turn means a single Fourier mode is

$$h'_{\mu\nu}(x) = 2\mathrm{Re}\left\{\epsilon'_{\mu\nu}(\vec{k})e^{-ik\cdot x}\right\} \tag{8.3.23}$$

As already alluded to, partial derivatives acting on equations (8.3.5) and (8.3.21) amount to the replacement $\partial_\mu \to -ik_\mu$. That ξ_ν in eq. (8.3.21) satisfies eq. (8.3.17), therefore follows from eq. (8.3.12). Along the same lines, and by referring to equations (8.3.5), (8.3.21), and (8.3.23), the gauge transformation rules can be phrased in terms of the shift in the polarization tensor $\epsilon_{\mu\nu}$:

$$\epsilon_{00} \to \epsilon_{00} - 2ik_0\widetilde{\xi}_0 = \epsilon'_{00}, \tag{8.3.24}$$

$$\epsilon_{0i} \to \epsilon_{0i} - ik_0\widetilde{\xi}_i - ik_i\widetilde{\xi}_0 = \epsilon'_{0i}, \tag{8.3.25}$$

$$\epsilon_{ij} \to \epsilon_{ij} - ik_i\widetilde{\xi}_j - ik_j\widetilde{\xi}_i = \epsilon'_{ij}. \tag{8.3.26}$$

Now, to set to zero h'_{00}, or equivalently $\epsilon'_{00} = 0$, we see that the right hand side of eq. (8.3.24) needs to be set to zero.

$$\widetilde{\xi}_0 = \frac{\epsilon_{00}}{2ik_0} \tag{8.3.27}$$

To set to zero h'_{0i} means the right hand side of (8.3.26) needs to be set to zero.

$$\widetilde{\xi}_i = \frac{\epsilon_{0i}}{ik_0} - \frac{k_i}{k_0}\widetilde{\xi}_0 \tag{8.3.28}$$

$$= \frac{\epsilon_{0i}}{ik_0} - \frac{k_i}{k_0^2}\frac{\epsilon_{00}}{2i} \tag{8.3.29}$$

By exploiting (8.3.9),

$$\widetilde{\xi}_i = \frac{1}{ik_0}\left(-\frac{k^l}{k^0}\epsilon_{li} + \frac{1}{2}\frac{k_i}{k^0}(\epsilon_{00} - \epsilon_{ll})\right) - \frac{k_i}{k_0^2}\frac{\epsilon_{00}}{2i} \tag{8.3.30}$$

$$= -\frac{1}{ik_0}\left(\frac{k^l}{k^0}\epsilon_{li} + \frac{1}{2}\frac{k_i}{k^0}\epsilon_{ll}\right), \tag{8.3.31}$$

so that now

$$\epsilon'_{ij} = \epsilon_{ij} - \widehat{k}_{\{i}\epsilon_{j\}l}\widehat{k}_l + \widehat{k}_i\widehat{k}_j\widehat{k}_m\widehat{k}_n\epsilon_{mn}, \tag{8.3.32}$$

where we have recognized $k^i/k_0 = k^i/|\vec{k}| = \widehat{k}^i = -\widehat{k}_i$ and, in the rightmost term, we have employed eq. (8.3.10).

Problem 8.9. Spatial Trace Is Zero in de Donder-Synchronous Gauge Use the relation in eq. (8.3.10) to show that the spatial trace of ϵ'_{ij} in eq. (8.3.32) is zero:

$$\epsilon'_{ll} \equiv \delta^{mn}\epsilon'_{mn} = 0. \tag{8.3.33}$$

□

Since we are still within the de Donder gauge, note that $\partial^\mu h'_{\mu\nu} = (1/2)\partial_\nu h'$ now becomes $-\partial_i h'_{i\nu} = (1/2)\partial_\nu(-h'_{ll}) = 0$, since $h'_{0\nu} = 0$ and h'_{ij} is traceless. In $k-$space and in terms of $\epsilon'_{\mu\nu}$, moreover,

$$k^\mu\epsilon'_{\mu\nu} = \frac{1}{2}k_\nu\epsilon' \qquad \Rightarrow \qquad k^i\epsilon'_{ij} = 0. \tag{8.3.34}$$

Namely, ϵ'_{ij} is orthogonal to the propagation direction $\vec{k}$. To sum:

> **Transverse-Traceless (TT)** Starting from the de Donder gauge homogeneous solution to the linearized Einstein's equations, by performing a de Donder gauge preserving gauge transformation to remove the 00 and $0i$ components of $h_{\mu\nu}$, we are led to a transverse-traceless metric perturbation. Denoting $h'_{\mu\nu} \equiv h^{\text{TT}}_{\mu\nu}$, we have
>
> $$\delta^{ij}h^{\text{TT}}_{ij} = 0 \qquad \text{(Traceless)}, \tag{8.3.35}$$
>
> $$\partial_i h^{\text{TT}}_{ij} = 0 \qquad \text{(Transverse)}. \tag{8.3.36}$$
>
> By linearity, these are equivalent to equations (8.3.33) and (8.3.34).

For simplicity let us again suppose our wave is a single mode propagating along the 3−direction. Then eq. (8.3.34) implies

$$\epsilon'_{i3} = \epsilon'_{3i} = 0, \qquad i = 1, 2, 3. \tag{8.3.37}$$

In words: only the 2×2 sub-matrix $\{h'_{\rm IJ} | \rm I, J = 1, 2\}$ is non-trivial. Additionally, (8.3.33) now tells us

$$\epsilon'_{11} + \epsilon'_{22} + \epsilon'_{33} = \epsilon'_{11} + \epsilon'_{22} = 0. \tag{8.3.38}$$

Namely, the 11 and 22 components are equal in magnitude but opposite in sign. At this point, we surmise from its symmetric-transverse-traceless character that the polarization tensor really has only two independent components:

$$\epsilon^{\rm TT}_{\mu\nu}(k^i = \delta^i_3) \equiv \epsilon'_{\mu\nu}(k^i = \delta^i_3) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \equiv h_+ \epsilon^+{}_{\mu\nu} + h_\times \epsilon^\times{}_{\mu\nu}. \tag{8.3.39}$$

General TT Homogeneous GWs More generally, for each plane wave

$$\exp(-ik \cdot x) = \exp\left[ik\left(\widehat{k}^j x^j - t\right)\right], \tag{8.3.40}$$

$$\widehat{k}^i \equiv \frac{k^i}{k}, \qquad k \equiv k_0 = k^0 > 0; \tag{8.3.41}$$

where k_i is no longer necessarily pointing along the 3−axis, we may first construct the null basis vectors

$$\ell^\pm{}_\mu \equiv \frac{1}{\sqrt{2}}\left(\delta^0_\mu \pm \delta^i_\mu \widehat{k}^i\right), \tag{8.3.42}$$

$$\eta^{\mu\nu}\ell^\pm{}_\mu \ell^\pm{}_\nu = 0, \qquad \eta^{\mu\nu}\ell^+{}_\mu \ell^-{}_\nu = 1; \tag{8.3.43}$$

followed by erecting 2 mutually orthogonal spatial vectors $\epsilon^{(1)}{}_\mu$ and $\epsilon^{(2)}{}_\mu$ that are also perpendicular to these $\ell^\pm$:

$$\eta^{\mu\nu}\epsilon^{(\rm I)}{}_\mu \epsilon^{(\rm J)}{}_\nu = -\delta^{\rm IJ} \qquad \text{and} \qquad \eta^{\mu\nu}\epsilon^{(\rm I)}{}_\mu \ell^\pm{}_\nu = 0. \tag{8.3.44}$$

The $\epsilon^+{}_{\mu\nu}$ and $\epsilon^\times{}_{\mu\nu}$ are then

$$\epsilon^+{}_{\mu\nu} \equiv \epsilon^{(1)}{}_\mu \epsilon^{(1)}{}_\nu - \epsilon^{(2)}{}_\mu \epsilon^{(2)}{}_\nu \tag{8.3.45}$$

$$\epsilon^\times{}_{\mu\nu} \equiv \epsilon^{(1)}{}_{\{\mu}\epsilon^{(2)}{}_{\nu\}}. \tag{8.3.46}$$

From eq. (8.3.42), note that since $\ell^+{}_\mu + \ell^-{}_\mu \propto \delta^0_\mu$ and since $\ell^\pm$ are perpendicular to the $\epsilon^{(1)}$ and $\epsilon^{(2)}$ (cf. eq. (8.3.44)), it must be that

$$\epsilon^{+,\times}{}_{0\nu} = 0 = \epsilon^{+,\times}{}_{\nu 0}. \tag{8.3.47}$$

The general homogeneous solution to the vacuum linearized General Relativity in eq. (8.3.4), within the de Donder-Synchronous gauge, can now be expressed as

$$h_{\mu\nu}^{\text{TT}} = \text{Re} \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \left\{ h_+(\vec{k}) \epsilon^+{}_{\mu\nu}(\vec{k}) + h_\times(\vec{k}) \epsilon^\times{}_{\mu\nu}(\vec{k}) \right\} e^{-ik\cdot x}, \tag{8.3.48}$$

$$h_{0\nu}^{\text{TT}} = h_{\nu 0}^{\text{TT}} = 0, \qquad \partial_i h_{ij}^{\text{TT}} = 0 = \delta^{ij} h_{ij}^{\text{TT}}. \tag{8.3.49}$$

Problem 8.10. TT GWs Define the projectors

$$\widetilde{P}_{ij} \equiv \delta_{ij} - \widehat{k}_i \widehat{k}_j \tag{8.3.50}$$

$$\widetilde{P}_{ijab} \equiv \frac{1}{2}\left(\widetilde{P}_{i\{a} \widetilde{P}_{b\}j} - \widetilde{P}_{ij} \widetilde{P}_{ab} \right). \tag{8.3.51}$$

Show that the ϵ'_{ij} in eq. (8.3.32) can be expressed as

$$\epsilon'_{ij} = \widetilde{P}_{ijab} \epsilon_{ab}. \tag{8.3.52}$$

(Hint: You may need to take into account eq. (8.3.10).) Verify the following projector properties.

$$\widetilde{P}_{ij} \widetilde{P}_{jk} = \widetilde{P}_{ik}, \qquad \widetilde{P}_{ii} = 2, \qquad \text{and} \qquad \widetilde{P}_{ij} \widehat{k}^i = 0; \tag{8.3.53}$$

and

$$\widetilde{P}_{ijab} \widetilde{P}_{abmn} = \widetilde{P}_{ijmn} \qquad \text{and} \qquad \widehat{k}^i \widetilde{P}_{ijab} = 0 = \delta^{ij} \widetilde{P}_{ijab}. \tag{8.3.54}$$

Explain why these projector properties guarantee ϵ'_{ij} is indeed transverse-traceless. □

Equation (8.3.48) tells us, an equivalent manner to phrase eq. (8.3.48) is

$$h_{ij}^{\text{TT}}(x) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{1}{2} \sum_{\sigma=\pm} \left\{ \epsilon_{ab}^{\text{TT}|\sigma}(\vec{k}) e^{-ik\cdot x} + \epsilon_{ab}^{\text{TT}|\sigma}(\vec{k})^* e^{+ik\cdot x} \right\}. \tag{8.3.55}$$

Problem 8.11. Circularly Polarized Gravitational Waves & Massless Spin-2 Recall the spin-1 polarization vectors of eq. (5.1.24). Verify that, if $k_\mu = (1, 0, 0, -1)$,

$$\epsilon^{++}{}_{\mu\nu} \equiv \epsilon^+{}_\mu \epsilon^+{}_\nu = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & -i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{8.3.56}$$

$$\epsilon^{--}{}_{\mu\nu} \equiv \epsilon^-{}_\mu \epsilon^-{}_\nu = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{8.3.57}$$

Proceed to re-write eq. (8.3.48) in terms of these polarization basis. Specifically, you should find that

$$h_+ \epsilon^+{}_{\mu\nu} + h_\times \epsilon^\times{}_{\mu\nu} = (h_+ + ih_\times) \epsilon^{++}{}_{\mu\nu} + (h_+ - ih_\times) \epsilon^{--}{}_{\mu\nu}. \tag{8.3.58}$$

If $R^\mu{}_\nu(\theta)$ implements a rotation by angle θ around the $\vec{k}$−axis – see eq. (5.1.25) – explain why

$$\epsilon^{++}{}_{\rho\sigma}(\vec{k}) R^\rho{}_\mu(\theta) R^\sigma{}_\nu(\theta) = \epsilon^{++}{}_{\mu\nu}(\vec{k}) e^{-i(+2)\theta}, \tag{8.3.59}$$

$$\epsilon^{--}{}_{\rho\sigma}(\vec{k}) R^\rho{}_\mu(\theta) R^\sigma{}_\nu(\theta) = \epsilon^{--}{}_{\mu\nu}(\vec{k}) e^{-i(-2)\theta}. \tag{8.3.60}$$

The $\pm$ depends on whether it is a counterclockwise or clockwise rotation about $\vec{k}$.

The ± 2 in the exponents may be regarded as the eigenvalues of the rotation operator $\widehat{k} \cdot \vec{J}$. This in turn implies the transverse-traceless homogeneous gravitational waves are massless spin−2 (i.e., helicity−2) states. □

Proper Spatial Distances In Synchronous Gauge Having just exploited the synchronous gauge to obtain the transverse-traceless homogeneous gravitational waves, we now turn to explaining its physical and geometric significance.

In the synchronous gauge, where $h_{00} = h_{0i} = h_{i0} = 0$, co-moving observers (with $\mathrm{d}\vec{x} = 0$) have clocks that coincide with the global time t since $\mathrm{d}\tau = \sqrt{g_{00}}\mathrm{d}t = \mathrm{d}t$. We may then measure proper spatial distances on this constant proper time surface; using eq. (2.6.86),

$$\begin{aligned} \sigma(\vec{x}, \vec{x}' | t = t' = \tau) = &-\frac{1}{2}|\vec{x} - \vec{x}'|^2 \\ &+ \frac{1}{2}(x - x')^i (x - x')^j \int_0^1 h_{ij}\left(\tau, \vec{x}' + \lambda(\vec{x} - \vec{x}')\right) \mathrm{d}\lambda + \mathcal{O}(h^2). \end{aligned} \tag{8.3.61}$$

We shall now consider a pair of test masses co-moving with such a metric:

$$X^\mu \equiv (t, \vec{x}) \qquad \text{and} \qquad X'^\mu = (t, \vec{x}'). \tag{8.3.62}$$

Because only the spatial components of the metric are non-trivial, recall from the discussion on the synchronous gauge that these test masses will remain still for all time – i.e., $(t, \vec{x})$ and $(t, \vec{x}')$ satisfy the geodesic equation. On a constant-time surface, and up to $\mathcal{O}(h)$, the proper length between the geodesic test masses at $\vec{x}$ and $\vec{x}'$ is

$$L(\tau) \equiv \sqrt{|2\sigma(\vec{x}, \vec{x}'|\tau)|} = |\vec{x} - \vec{x}'| \left(1 - \frac{1}{2}\widehat{n}^i \widehat{n}^j \int_0^1 h_{ij}\left(\tau, \vec{x}' + \lambda(\vec{x} - \vec{x}')\right) \mathrm{d}\lambda\right), \tag{8.3.63}$$

where the unit radial vector is

$$\widehat{n}^i \equiv \frac{x^i - x'^i}{|\vec{x} - \vec{x}'|} \tag{8.3.64}$$

and we have used the Taylor expansion result that $(1 - h)^{1/2} = 1 - (1/2)h + \mathcal{O}(h^2)$ for small h.

It is customary in the GR literature to write the effect of GWs on such co-moving test masses through the *strain*, defined as the fractional distortion $\delta L / L_0$, where $L_0 = |\vec{x} - \vec{x}'|$ is the proper distance between the $\vec{x}$ and $\vec{x}'$ well before the GW hit them (i.e., when $h_+ = h_\times = 0$). According to eq. (8.3.63),

$$\begin{aligned} \frac{\delta L(\tau)}{L_0} &\equiv \frac{L(\tau) - L_0}{L_0} \\ &= -\frac{\widehat{n}^i \widehat{n}^j}{2} \int_0^1 h_{ij}\left(\tau, \vec{x}' + \lambda(\vec{x} - \vec{x}')\right) \mathrm{d}\lambda. \end{aligned} \tag{8.3.65}$$

Let us examine how the proper spatial distance changes between them with the passage of a GW train. In an otherwise empty 4D spacetime filled with spin-2 gravitons

$$h_{ij} = h_{ij}^{\rm TT} = \mathrm{Re}\left\{\epsilon_{ij} e^{-ik\cdot x}\right\}, \tag{8.3.66}$$

consider a plane wave that has wave vector along the positive 3−axis: $k_\mu = (\omega, 0, 0, -\omega)$. The spin-2 wave polarization vector has to obey $k^i \epsilon_{ij} = 0 = \delta^{ij}\epsilon_{ij}$. For the case at hand, this implies $\epsilon_{3i} = \epsilon_{i3} = 0$ and $\epsilon_{11} + \epsilon_{22} + \epsilon_{33} = \epsilon_{11} + \epsilon_{22} = 0$.

$$\epsilon_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{8.3.67}$$

To gain some insight into the pattern of gravitational waves, let us parametrize the unit vector $\widehat{n}$ using spherical coordinates. Its Cartesian components are therefore

$$\begin{aligned} \widehat{n}^i &= (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)) \\ &\equiv \sin(\theta)\cos(\phi)\widehat{e}^i_1 + \sin(\theta)\sin(\phi)\widehat{e}^i_2 + \cos(\theta)\widehat{e}^i_3; \end{aligned} \tag{8.3.68}$$

where $\widehat{e}^i_{\rm I}$ is the unit vector parallel to the I-th Cartesian axis. The strain eq. (8.3.65) now becomes

$$\begin{aligned} \frac{\delta L(\tau)}{L_0} &= -\frac{\sin(\theta)^2}{2}\mathrm{Re}\int_0^1 \left(c_\phi^2 h_{11} + s_\phi^2 h_{22} + c_\phi s_\phi (h_{12} + h_{21})\right) e^{-ik\cdot x}\mathrm{d}\lambda \qquad (8.3.69) \\ &= -\frac{\sin(\theta)^2}{2}\mathrm{Re}\int_0^1 \left(c_\phi^2 h_+ - s_\phi^2 h_+ + 2c_\phi s_\phi h_\times\right) e^{-ik\cdot x}\mathrm{d}\lambda \qquad (8.3.70) \\ &= -\frac{\sin(\theta)^2}{2}\mathrm{Re}\int_0^1 \left(\cos(2\phi)h_+ + \sin(2\phi)h_\times\right) e^{-ik\cdot x}\mathrm{d}\lambda. \qquad (8.3.71) \end{aligned}$$

Remember the wave is now assumed to be traveling along the 3−axis, so that its polarization tensor takes the form in eq. (8.3.67).

Long wavelength limit When the GW wavelength is large enough compared to the proper distance between the co-moving observers at $\vec{x}$ and $\vec{x}'$, such that $h_+ \exp(-ik\cdot x)$ and $h_\times \exp(-ik\cdot x)$ are approximately constant between them, we may effectively discard the integral $\int_0^1 \dots \mathrm{d}\lambda$ around them.

$$\frac{\delta L(\tau)}{L_0} = -\frac{\sin(\theta)^2}{2}\mathrm{Re}\left\{\left(\cos(2\phi)h_+ + \sin(2\phi)h_\times\right) e^{-ik\cdot x}\right\} \tag{8.3.72}$$

We may make the following observations.

- When the pair of test masses lie along the direction of the wave propagation, the space between them does not get distorted. (This is where $\theta = \pi/2$ in eq. (8.3.71).)
- There are two distinct gravitational wave polarizations h_+ and $h_\times$.
- For the h_+: there is zero distortion when the (projected) orientation of the test masses lie on $\phi = \pm\pi/4$ lines on the 2D spatial plane perpendicular to the wave propagation; while there is maximum distortion when the (projected) orientation of the test masses lie on $\phi = 0$ and $\phi = \pi/2$ lines on the 2D spatial plane perpendicular to the wave propagation.

- For the $h_\times$: there is zero distortion when the (projected) orientation of the test masses lie on $\phi = 0$ and $\phi = \pi/2$ lines on the 2D spatial plane perpendicular to the wave propagation; while there is maximum distortion when the (projected) orientation of the test masses lie on $\phi = \pm\pi/4$ lines on the 2D spatial plane perpendicular to the wave propagation.
- In other words, the h_+ polarization becomes the $h_\times$ polarization upon a rotation of $\pi/4$ around the axis parallel to the direction of wave propagation.

Problem 8.12. Gravitational Wave Pattern for Linear Polarizations See the Wikipedia article here for an rather exaggerated animation of $\delta L/L_0$ in the presence of only h_+ type or only $h_\times$ type (long wavelength) gravitational waves in eq. (8.3.71). Can you reproduce this pattern? □

Problem 8.13. Gravitational Wave Pattern for Circular Polarizations Describe the time evolution of $\delta L/L_0$ in the presence of only $++$ polarized $\text{Re}\{h_{++}\epsilon^{++}{}_{\mu\nu}\exp(-ik\cdot x)\}$ or $--$ polarized $\text{Re}\{h_{--}\epsilon^{--}{}_{\mu\nu}\exp(-ik\cdot x)\}$ gravitational waves (cf. (8.3.58)). □

8.4 Gravitational Radiation in (3+1)D Minkowski Spacetime

In this section we will assume that an observer has set up a gravitational wave detector at some very large distance r away from some isolated source of gravitational radiation. That is, if r_c is the characteristic spatial size of the source, then $r_c/r \ll 1$; as well as $\tau_c/r \ll 1$, where τ_c is the characteristic time scale of the source.

Far Zone From the de Donder gauge solution in eq. (8.3.1),

$$\bar{h}_{\mu\nu}(t,\vec{x}) = -4G_{\text{N}} \int_{\mathbb{R}^3} \text{d}^3\vec{x}' \exp\left(-x'^j \partial_{x^j}\right) \left(\frac{\overline{T}_{\mu\nu}\left(t - |\vec{x}|, \vec{x}'\right)}{|\vec{x}|}\right), \tag{8.4.1}$$

where $\exp\left(-x'^j\partial_j\right)$ is the operator that performs a translation in space by $-\vec{x}'$. Let us place the spatial origin $\vec{0}$ of our coordinate system within the source. By Taylor expanding the exponential, we see that every additional derivative acting on the $1/r$, for $r \equiv |\vec{x}|$, would yield a higher power of $1/r$, which in turn becomes more suppressed as $r \to \infty$. On the other hand, the first ∂_j acting on the $T_{\mu\nu}$ would yield

$$\partial_j T_{\mu\nu}(t - r, \vec{x}') = \partial_j(-r)\partial_0 T_{\mu\nu}(t - r, \vec{x}'), \qquad r \equiv |\vec{x}|. \tag{8.4.2}$$

The second derivative would hand us

$$\partial_i\partial_j T_{\mu\nu}(t - r, \vec{x}') = \partial_i(-r)\partial_j(-r)\partial_0^2 T_{\mu\nu}(t - r, \vec{x}') + \partial_i\partial_j(-r)\partial_0 T_{\mu\nu}(t - r, \vec{x}') \tag{8.4.3}$$

$$= \widehat{r}^i\widehat{r}^j\partial_0^2 T_{\mu\nu}(t - r, \vec{x}')\left(1 + \mathcal{O}\left(\frac{\tau_c}{r}\right)\right), \qquad \widehat{r}^i \equiv \frac{x^i}{|\vec{x}|} = \partial_i r. \tag{8.4.4}$$

By assumption, the $\mathcal{O}(\tau_c/r)$ correction is small. Therefore, the dominant contribution of the translation operator in the far zone is when all the spatial derivatives with respect to $\vec{x}$ are acting on the $T_{\mu\nu}$.

$\bar{h}_{\mu\nu}(t, \vec{x}; r \gg r_c, \tau_c)$

$$= -4G_{\rm N}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}' \sum_{\ell=0}^{\infty} \frac{(-x')^{i_1}\dots(-x')^{i_\ell}}{\ell!}(-\widehat{r}^{i_1})\dots(-\widehat{r}^{i_\ell})\frac{\partial_0^\ell \overline{T}_{\mu\nu}\left(t-|\vec{x}|,\vec{x}'\right)}{|\vec{x}|}\left(1+\mathcal{O}\left(\frac{\tau_c}{r}\right)\right)$$

$$= -4G_{\rm N}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}' \sum_{\ell=0}^{\infty} \frac{(\vec{x}'\cdot\widehat{r})^\ell}{\ell!}\frac{\partial_0^\ell \overline{T}_{\mu\nu}\left(t-r,\vec{x}'\right)}{|\vec{x}|}\left(1+\mathcal{O}\left(\frac{\tau_c}{r}\right)\right). \tag{8.4.5}$$

That is, at leading $1/r$ order (i.e., in the far zone)

$$\bar{h}_{\mu\nu}(t,\vec{x};r\gg r_c,\tau_c) \approx -\frac{4G_{\rm N}}{r}\int_{\mathbb{R}^3}\mathrm{d}^3\vec{x}'\overline{T}_{\mu\nu}\left(t-r+\vec{x}'\cdot\widehat{r},\vec{x}'\right). \tag{8.4.6}$$

Note that this formula is exact at $\mathcal{O}(1/r)$, even when the source is relativistic – as long as gravity is weak throughout spacetime, including within the interior of the matter distribution itself.

Next, by examining each term in the Taylor series of eq. (8.4.5), we see each factor of $(\vec{x}'\cdot\widehat{r})\partial_0$ scales heuristically as r_c/τ_c, since ∂_0 acts on the stress tensor describing the matter source. The non-relativistic limit is precisely the regime where this ratio is small, which tells us when such a scenario holds, the dominant contribution to the gravitational signal is therefore the one with zero time derivatives:

$$\bar{h}_{\mu\nu}(t,\vec{x};r\gg r_c,\tau_c;r_c/\tau_c\ll 1) \approx -4G_{\rm N}\frac{A_{\mu\nu}(t-r)}{r}\left(1+\mathcal{O}\left(\frac{\tau_c}{r},\frac{r_c}{\tau_c}\right)\right), \tag{8.4.7}$$

$$A_{\mu\nu}(t-r) \equiv \int_{\mathbb{R}^3}\mathrm{d}^3\vec{x}'\overline{T}_{\mu\nu}\left(t-r,\vec{x}'\right). \tag{8.4.8}$$

Let us decompose the stress tensor of the source as a superposition of its individual frequencies

$$\overline{T}_{\mu\nu}\left(t,\vec{x}'\right) = \int_{-\infty}^{+\infty}\frac{\mathrm{d}\Omega}{2\pi}e^{-i\Omega t}\widetilde{\overline{T}}_{\mu\nu}\left(\Omega,\vec{x}'\right). \tag{8.4.9}$$

Then,

$$\bar{h}_{\mu\nu}(t,\vec{x};r\gg r_c,\tau_c;r_c/\tau_c\ll 1) \approx \int_{-\infty}^{+\infty}\frac{\mathrm{d}\Omega}{2\pi}\frac{e^{-i\Omega(t-r)}}{r}\widetilde{A}_{\mu\nu}(\Omega), \tag{8.4.10}$$

$$\widetilde{A}_{\mu\nu} \equiv -4G_{\rm N}\int_{\mathbb{R}^3}\mathrm{d}^3\vec{x}'\widetilde{\overline{T}}_{\mu\nu}\left(\Omega,\vec{x}'\right). \tag{8.4.11}$$

Problem 8.14. Outgoing Spherical Waves By writing down the frequency-transform representation of the retarded Green's function in eq. (5.1.1), and by placing the spacetime point source at the origin $(t',\vec{x}')=0^\mu$, show directly that the $\{\exp(-i\Omega(t-r))/r\}$ are indeed homogeneous solutions when $r\gg r_c$. Specifically,

$$\partial^2\left(\frac{\exp(-i\Omega(t-r))}{r}\right) = -\left(\Omega^2+\vec{\nabla}^2\right)\left(\frac{\exp(-i\Omega(t-r))}{r}\right) = 4\pi\delta^{(3)}\left(\vec{x}\right)e^{-i\Omega t}. \tag{8.4.12}$$

Note: the 'outgoing' $A(r-t)/r$ nature of these spherical waves is a direct consequence of choosing the 'retarded' as opposed to advanced Green's function. □

Relativity & Gauge Condition There is a subtlety in taking the non-relativistic limit too early, however. In particular, let us verify that the de Donder gauge $\partial^\mu \bar{h}_{\mu\nu} = 0$ is no longer obeyed; at order $1/r$,

$$
\begin{aligned}
\partial^\mu \bar{h}_{\mu\nu} &\approx -\frac{4G_{\rm N}}{r}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}' \partial^\mu \overline{T}_{\mu\nu}(t-r,\vec{x}') \\
&= -\frac{4G_{\rm N}}{r}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}' \left(\partial_0 \overline{T}_{\mu\nu}(t-r,\vec{x}') - \frac{\partial}{\partial x^i}\overline{T}_{i\nu}(t-r,\vec{x}')\right) \\
&= -\frac{4G_{\rm N}}{r}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}' \left(\partial_{i'} \overline{T}_{i\nu}(t-r,\vec{x}') + \widehat{r}^i \partial_0 \overline{T}_{i\nu}(t-r,\vec{x}')\right) \\
&= -\frac{4G_{\rm N}}{r}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}' \widehat{r}^i \partial_0 \overline{T}_{i\nu}(t-r,\vec{x}') = \widehat{r}^i \partial_0 \bar{h}_{i\nu}.
\end{aligned} \tag{8.4.13}
$$

Going from the third to last equality, we have assumed the stress tensor is sufficiently localized so that the spatial volume integral of $\partial_{i'} T_{i\nu}$ may be converted to a surface integral at infinity, where it is identically zero. In what follows, we will continue to make use of this assumption in similar ways.

Let us instead take the divergence of the $\mathcal{O}(1/r)$-exact relativistic result in eq. (8.4.6):

$$
\begin{aligned}
\partial^\mu \bar{h}_{\mu\nu} &\approx -\frac{4G_{\rm N}}{r}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}' \partial^\mu \overline{T}_{\mu\nu}(t-r+\widehat{r}\cdot\vec{x}',\vec{x}') \\
&= -\frac{4G_{\rm N}}{r}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}' \left(\partial_0 \overline{T}_{0\nu}(t-r+\widehat{r}\cdot\vec{x}',\vec{x}') - \partial_{x^i} \overline{T}_{i\nu}(t-r+\widehat{r}\cdot\vec{x}',\vec{x}')\right) \\
&\approx -\frac{4G_{\rm N}}{r}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}' \left((\partial_{x'^i} \overline{T}_{i\nu}(t-r+\widehat{r}\cdot\vec{x}',\vec{x}'))_{t-r+\widehat{r}\cdot\vec{x}'} + \widehat{r}^i \partial_0 \overline{T}_{i\nu}(t-r+\widehat{r}\cdot\vec{x}',\vec{x}')\right) \\
&= -\frac{4G_{\rm N}}{r}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}' \Big(\partial_{x'^i} \overline{T}_{i\nu}(t-r+\widehat{r}\cdot\vec{x}',\vec{x}') - \partial_{x'^i}(t-r+\widehat{r}\cdot\vec{x}')\partial_0 \overline{T}_{i\nu}(t-r+\widehat{r}\cdot\vec{x}',\vec{x}') \\
&\qquad\qquad + \widehat{r}^i \partial_0 \overline{T}_{i\nu}(t-r+\widehat{r}\cdot\vec{x}',\vec{x}')\Big) \\
&= -\frac{4G_{\rm N}}{r}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}' \left(-\widehat{r}^i \partial_0 \overline{T}_{i\nu} + \widehat{r}^i \partial_0 \overline{T}_{i\nu}\right) = 0.
\end{aligned} \tag{8.4.14}
$$

Note that we have dropped the terms that involved $\partial_i \widehat{r} \sim \widehat{r}/r$, because they are suppressed as $r \to \infty$. Moreover, the $(\partial_{x'^i} \overline{T}_{i\nu}(t-r+\widehat{r}\cdot\vec{x}',\vec{x}'))_{t-r+\widehat{r}\cdot\vec{x}'}$ means we are taking the x'^i derivative holding the time component $t-r+\widehat{r}\cdot\vec{x}'$ fixed, which can be converted into a unrestricted derivative with respect to $\partial_{x'^i}$ minus the variation of the time component with respect to x'^i. The main point here is, including the relativistic correction $\widehat{r}\cdot\vec{x}'$ in the time argument of our matter source ensures our $\mathcal{O}(1/r)$-accurate gravitational field respects the de Donder gauge.

In frequency space, eq. (8.4.6) reads

$$
\begin{aligned}
\bar{h}_{\mu\nu}(t,\vec{x}) &= -\frac{4G_{\rm N}}{r}\int_{\mathbb{R}} \frac{\mathrm{d}\Omega}{2\pi}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}' e^{-i\Omega(t-r+\widehat{r}\cdot\vec{x}')} \widetilde{\overline{T}}_{\mu\nu}(\Omega,\vec{x}') \\
&= -\frac{4G_{\rm N}}{r}\int_{\mathbb{R}} \frac{\mathrm{d}\Omega}{2\pi} e^{-i\Omega(t-r)} \widetilde{\widetilde{\overline{T}}}_{\mu\nu}(\Omega,\Omega\widehat{r})
\end{aligned} \tag{8.4.15}
$$

where $\widetilde{\widetilde{\overline{T}}}_{\mu\nu}(\Omega,\Omega\widehat{r})$ is now the spacetime Fourier transform of the matter stress tensor, evaluated at frequency Ω and reciprocal space position $\Omega\widehat{r}$. The de Donder gauge condition now translates to, at order $1/r$,

$$\partial^\mu \bar{h}_{\mu\nu}(t,\vec{x}) \approx -\frac{4G_{\rm N}}{r}\int_{\mathbb{R}} \frac{{\rm d}\Omega}{2\pi} e^{-i\Omega(t-r)}(-i\Omega)\partial^\mu(t-r)\widetilde{\overline{T}}_{\mu\nu}\left(\Omega,\Omega\widehat{r}\right) = 0. \tag{8.4.16}$$

Since this holds for each linearly independent $\exp(-i\Omega(t-r))$,

$$q^\mu \widetilde{\overline{T}}_{\mu\nu} = \widetilde{\overline{T}}_{0\nu}\left(\Omega,\Omega\widehat{r}\right) + \widehat{r}^i \widetilde{\overline{T}}_{i\nu}\left(\Omega,\Omega\widehat{r}\right) = 0, \tag{8.4.17}$$

$$q^\mu \equiv \partial^\mu(t-r) = \delta^\mu_0 + \delta^\mu_i \widehat{r}^i, \qquad q_\mu = \delta^0_\mu - \delta^i_\mu \widehat{r}^i. \tag{8.4.18}$$

We may write down the zeroth component

$$\widetilde{\overline{T}}_{00}\left(\Omega,\Omega\widehat{r}\right) + \widehat{r}^i \widetilde{\overline{T}}_{i0}\left(\Omega,\Omega\widehat{r}\right) = 0; \tag{8.4.19}$$

and the jth component

$$\widetilde{\overline{T}}_{0j}\left(\Omega,\Omega\widehat{r}\right) = -\widehat{r}^i \widetilde{\overline{T}}_{ij}\left(\Omega,\Omega\widehat{r}\right). \tag{8.4.20}$$

If we dot both sides with $\widehat{r}^j$ and employ eq. (8.4.19), namely $\widehat{r}^j \widetilde{\overline{T}}_{0j} = -\widetilde{\overline{T}}_{00}$,

$$\widetilde{\overline{T}}_{00}\left(\Omega,\Omega\widehat{r}\right) = \widehat{r}^i \widehat{r}^j \widetilde{\overline{T}}_{ij}\left(\Omega,\Omega\widehat{r}\right). \tag{8.4.21}$$

JWKB Spherical Waves We will now focus on a single frequency from the superposition in eq. (8.4.15):

$$\widetilde{\overline{h}}_{\mu\nu} = -4G_{\rm N}\widetilde{\overline{T}}_{\mu\nu}(\Omega,\Omega\widehat{r})\frac{e^{i\Omega(r-t)}}{r}. \tag{8.4.22}$$

Notice this is an outgoing spherical wave, where the 'outgoing-wave' $\exp(i\Omega(r-t))$ dependence on spacetime is a direct consequence of employing the retarded, i.e., causal, Green's function. Focusing on a pure frequency is not an over-idealization because there are astrophysical systems – such as the compact binary systems we have heard from – that are nearly periodic for at least part of their evolution. Moreover, gravitational wave detectors themselves are only sensitive to a limited bandwidth.

Exploiting the conservation results in equations (8.4.20) and (8.4.21), note that we may decompose eq. (8.4.22) into

$$\widetilde{\overline{h}}_{00} = -4G_{\rm N}\widehat{r}^m \widehat{r}^n \widetilde{\overline{T}}_{mn}(\Omega,\Omega\widehat{r})\frac{e^{i\Omega(r-t)}}{r}, \tag{8.4.23}$$

$$\widetilde{\overline{h}}_{0i} = -4G_{\rm N}(-)\widehat{r}^l \widetilde{\overline{T}}_{il}(\Omega,\Omega\widehat{r})\frac{e^{i\Omega(r-t)}}{r}, \tag{8.4.24}$$

$$\widetilde{\overline{h}}_{ij} = -4G_{\rm N}\widetilde{\overline{T}}_{ij}(\Omega,\Omega\widehat{r})\frac{e^{i\Omega(r-t)}}{r}. \tag{8.4.25}$$

Now, even though the background Minkowski spacetime has zero geometric curvature – i.e., the associated wavelength of its Riemann tensor is infinite – it is still helpful to draw an analogy between the JWKB analysis in §(6.3). In particular, the gravitational analog of eq. (6.3.1) is

$$\widetilde{\overline{h}}_{\mu\nu} = {\rm Re}\left\{a_{\mu\nu}\exp(iS)\right\} \tag{8.4.26}$$

where comparison with (the real part of) eq. (8.4.22) informs us the slowly varying amplitude is

$$a_{\mu\nu} = -8G_{\rm N} \frac{\widetilde{\overline{T}}_{\mu\nu}(\Omega, \Omega\widehat{r})}{r}; \tag{8.4.27}$$

while the rapidly oscillating phase is

$$\exp(iS) = \exp(-i\Omega(t-r)). \tag{8.4.28}$$

In spherical coordinates,

$$\eta_{\mu\nu}{\rm d}x^\mu {\rm d}x^\nu = {\rm d}t^2 - {\rm d}r^2 - r^2\left({\rm d}\theta^2 + \sin(\theta)^2{\rm d}\phi^2\right), \tag{8.4.29}$$

we may recognize the null vector $k_\mu \equiv \nabla_\mu S$ as

$$\begin{aligned} k_\mu &= (\partial_t, \partial_r, \partial_\theta, \partial_\phi)\left(-i\Omega(t-r)\right) \\ &= -i\Omega\,(1,-1,0,0) \equiv -i\Omega q_\mu, \end{aligned} \tag{8.4.30}$$

where q^μ has already been defined in eq. (8.4.18). Its null character is clear from

$$k_\mu k^\mu = (-i\Omega)^2 q_\mu q^\mu = (-i\Omega)^2(\eta^{00} + \eta^{rr}) = 0. \tag{8.4.31}$$

Problem 8.15. Length/Time Scales & Gauge Transformations in JWKB Approximation Argue using the spherical wave solution in eq. (8.4.22) that in the far zone its derivatives go as

$$\partial_\alpha \widetilde{\overline{h}}_{\mu\nu} = -i\Omega q_\alpha \widetilde{\overline{h}}_{\mu\nu}\left(1 + \mathcal{O}\left(\frac{\tau_c}{r}\right)\right). \tag{8.4.32}$$

Be sure to explain how the corrections scale with the time/length scales in the problem, i.e., the $\mathcal{O}(\tau_c/r)$.

Next, suppose we perform a gauge transformation of the form $x^\mu \to x^\mu + \xi^\mu$, where ξ_μ is itself a spherical wave solution

$$\xi_\mu = -4G_{\rm N}\frac{\ell_\mu}{r}e^{-i\Omega(t-r)}, \tag{8.4.33}$$

where ℓ_μ may depend on Ω but not on spacetime. Verify that, like $\partial_\alpha \widetilde{\overline{h}}_{\mu\nu}$ in eq. (8.4.32) at $\mathcal{O}(1/r)$, the gauge vector ξ_μ itself obeys the wave equation

$$\partial^2 \xi_\mu(\vec{x} \neq \vec{0}) = 0; \tag{8.4.34}$$

and explain why it induces a de Donder gauge preserving infinitesimal coordinate transformation. Finally, explain why, up to order $1/r$, the gauge transformation of $h_{\mu\nu}$ reads

$$\widetilde{h}_{00} \to \widetilde{h}_{00} - 2i\Omega\widetilde{\xi}_0 \equiv \widetilde{h}'_{00}, \tag{8.4.35}$$

$$\widetilde{h}_{0i} \to \widetilde{h}_{0i} - i\Omega\widetilde{\xi}_i - i\Omega\widehat{r}_i\widetilde{\xi}_0 \equiv \widetilde{h}'_{0i}, \tag{8.4.36}$$

$$\widetilde{h}_{ij} \to \widetilde{h}_{ij} - i\Omega\widehat{r}_i\widetilde{\xi}_j - i\Omega\widehat{r}_j\widetilde{\xi}_i \equiv \widetilde{h}'_{ij}. \tag{8.4.37}$$

□

de Donder-Synchronous Gauge In eq. (8.3.65) we have seen how switching to the synchronous gauge $h_{0\nu} = h_{\nu 0} = 0$ allowed us to not only readily define co-moving observers but also, through Synge's world function, compute the proper spatial distance between free-falling test masses. To this end, let us remind ourselves of the relationship between $\bar{h}_{\mu\nu}$ and $h_{\mu\nu}$ in eq. (2.6.34) when $d = 4$ and $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$.

$$h_{00} = \bar{h}_{00} - \frac{1}{2}\left(\bar{h}_{00} - \bar{h}_{ll}\right) = \frac{1}{2}\bar{h}_{00} + \frac{1}{2}\bar{h}_{ll}, \tag{8.4.38}$$

$$h_{0i} = h_{i0} = \bar{h}_{0i}, \tag{8.4.39}$$

$$h_{ij} = h_{ji} = \bar{h}_{ij} + \frac{1}{2}\delta_{ij}\left(\bar{h}_{00} - \bar{h}_{ll}\right). \tag{8.4.40}$$

Now, in Fourier spacetime,

$$\widetilde{h}_{00} = -4G_{\rm N}\frac{e^{-i\Omega(t-r)}}{r}\left(\frac{\delta^{mn} + \widehat{r}^m\widehat{r}^n}{2}\widetilde{\overline{T}}_{mn}\right), \tag{8.4.41}$$

$$\widetilde{h}_{0i} = -4G_{\rm N}\frac{e^{-i\Omega(t-r)}}{r}\left(-\widehat{r}^m\widetilde{\overline{T}}_{im}\right), \tag{8.4.42}$$

$$\widetilde{h}_{ij} = -4G_{\rm N}\frac{e^{-i\Omega(t-r)}}{r}\left(\widetilde{\overline{T}}_{ij} + \frac{1}{2}\delta_{ij}\left(\widehat{r}^m\widehat{r}^n - \delta^{mn}\right)\widetilde{\overline{T}}_{mn}\right). \tag{8.4.43}$$

To render $\widetilde{h}'_{00} = 0$, we must therefore put

$$\ell_0 = \frac{1}{i\Omega}\frac{\delta^{mn} + \widehat{r}^m\widehat{r}^n}{4}\widetilde{\overline{T}}_{mn}. \tag{8.4.44}$$

And, thus, to put $\widetilde{h}'_{0i} = 0$,

$$\ell_i = -\frac{\widehat{r}^m\widetilde{\overline{T}}_{im}}{i\Omega} - \widehat{r}_i\ell_0 \tag{8.4.45}$$

$$= -\frac{\widehat{r}^m\widetilde{\overline{T}}_{im}}{i\Omega} - \frac{\widehat{r}_i}{i\Omega}\frac{\delta^{mn} + \widehat{r}^m\widehat{r}^n}{4}\widetilde{\overline{T}}_{mn}. \tag{8.4.46}$$

Finally, we have

$$\widetilde{h}'_{ij} = -4G_{\rm N}\frac{e^{-i\Omega(t-r)}}{r}\left(\widetilde{\overline{T}}_{ij} + \frac{1}{2}\delta_{ij}\left(\widehat{r}_m\widehat{r}_n - \delta_{mn}\right)\widetilde{\overline{T}}_{mn} - \widehat{r}_m\widetilde{\overline{T}}_{m\{i}\widehat{r}_{j\}} + \widehat{r}_i\widehat{r}_j\frac{\delta_{mn} + \widehat{r}_m\widehat{r}_n}{2}\widetilde{\overline{T}}_{mn}\right). \tag{8.4.47}$$

Problem 8.16. tt Graviton Define the objects

$$P_{ij} \equiv \delta_{ij} - \widehat{r}_i\widehat{r}_j = \delta^{ij} - \widehat{r}^i\widehat{r}^j, \tag{8.4.48}$$

$$P^{ijab} \equiv \frac{1}{2}P^{i\{a}P^{b\}j} - \frac{1}{2}P^{ij}P^{ab} = \frac{1}{2}P_{i\{a}P_{b\}j} - \frac{1}{2}P_{ij}P_{ab}. \tag{8.4.49}$$

Show that P_{ijab} is transverse, in that

$$\widehat{r}^iP_{ijab} = 0 = \widehat{r}^jP_{ijab}; \tag{8.4.50}$$

and it is traceless, in that

$$\delta^{ij} P_{ijab} = 0. \tag{8.4.51}$$

Then show that eq. (8.4.47) can be packaged into

$$\widetilde{h}'_{ij} = P_{ijab} \widetilde{h}_{ab}. \tag{8.4.52}$$

To sum: By starting with the spherical gravitational wave solutions in eq. (8.4.22), and transforming to the synchronous gauge – while preserving the de Donder one – we have arrived at the 'tt' (transverse-to-$\widehat{r}$ and traceless) gravitational wave $h^{\rm tt}_{ij} \equiv P_{ijab} \widetilde{\overline{h}}_{ab}$. □

Summary: 'tt' Gravitational Waves Let us collect the results thus far. The 'tt' gravitational wave solution in eq. (8.4.22), whose source(s) $\overline{T}_{\mu\nu}(t, \vec{x})$ oscillate at some non-zero frequency $\Omega \neq 0$, are now given by[74]

$$\widetilde{\overline{h}}_{\nu 0} = \widetilde{\overline{h}}_{0\nu} = \widetilde{h}_{\nu 0} = \widetilde{h}_{0\nu} = 0, \tag{8.4.53}$$

$$\widetilde{\overline{h}}^{\rm tt}_{ij} = \widetilde{h}^{\rm tt}_{ij} = -4 G_{\rm N} \frac{\widetilde{\overline{T}}^{\rm tt}_{ij}(\Omega, \Omega\widehat{r})}{r} \exp\left(-i\Omega(t - r)\right), \tag{8.4.54}$$

$$\widetilde{\overline{T}}^{\rm tt}_{ij} \equiv P_{ijab} \widetilde{\overline{T}}_{ab}. \tag{8.4.55}$$

with the property that $\widetilde{h}^{\rm tt}_{ij}$ is transverse to the direction of propagation $\widehat{r}$,

$$\widehat{r}^i \widetilde{h}^{\rm tt}_{ij} = \widehat{r}^i \widetilde{\overline{T}}^{\rm tt}_{ij} = \widetilde{h}^{\rm tt}_{ij} \widehat{r}^j = \widetilde{\overline{T}}^{\rm tt}_{ij} \widehat{r}^j = 0; \tag{8.4.56}$$

as well as being spatially traceless

$$\delta^{ij} \widetilde{h}^{\rm tt}_{ij} = \delta^{ij} \widetilde{\overline{T}}^{\rm tt}_{ij} = 0. \tag{8.4.57}$$

A gravitational wave detector measuring $\delta L / L_0$ sensitive to such a frequency Ω would suffer a distortion given by eq. (8.3.72) whenever its size is much smaller than the wavelength of the gravitational radiation itself $\lambda_{\rm GW} \sim 1/\Omega$. □

Problem 8.17. Non-Relativistic Limits: Quadrupole Formula The quadrupole moment is defined as

$$Q_{ij}(s) \equiv \int_{\mathbb{R}^3} \mathrm{d}^3 \vec{x}' (x' - a)^i (x' - a)^j \overline{T}_{00}(s, \vec{x}'), \tag{8.4.58}$$

where $\vec{a}$ is related to the choice of the spatial coordinate system's origin. If the matter stress-energy tensor $\overline{T}_{\mu\nu}$ is conserved, show that its acceleration yields twice of the total shear-stress, namely

$$\ddot{Q}_{ij}(s) = 2 \int_{\mathbb{R}^3} \mathrm{d}^3 \vec{x}' \overline{T}_{ij}(s, \vec{x}'). \tag{8.4.59}$$

[74]A side note: if one takes the non-relativistic limit (cf. eq. (8.4.7)) too early, before going to synchronous gauge, one would *not* find the resulting $h^{\rm tt}_{ij}$; this, we believe, may be attributed to the non-relativistic limit no longer obeying the de Donder gauge condition.

(Does this result depend on the choice of $\vec{a}$?) Use this result to show that, in the far zone and non-relativistic limit, eq. (8.4.54) turns into

$$\bar{h}^{\rm tt}_{ij}(t,\vec{x}) \approx -\frac{2G_{\rm N}}{r}\ddot{Q}^{\rm (tt)}_{ij}(t-r), \tag{8.4.60}$$

$$Q^{\rm (tt)}_{ij}(s) \equiv P_{ijab} Q_{ab}(s). \tag{8.4.61}$$

This is a key formula for computing the gravitational wave signature as well as its energy-momentum content. □

'tt' versus TT Spin/Helicity-2 Waves Because these 'tt' waves split naturally into a very slowly varying amplitude $\widetilde{A}_{\mu\nu}/r$ multiplied by a rapidly oscillating 'wave' $\exp(-i\Omega(t-r))$ – we may in fact draw an approximate equivalence between $\widetilde{\bar{h}}_{\mu\nu}$ and the plane wave (Fourier space) spin-2 solutions in eq. (8.3.48):

$$\widetilde{h}^{\rm TT}_{ij} = \epsilon^{\pm}{}_{ij}(\vec{k})e^{-ik_\mu x^\mu}, \qquad\qquad k_\sigma k^\sigma = 0. \tag{8.4.62}$$

Namely, we have the following identification between the spherical 'tt' waves (LHS) and plane 'TT' waves (RHS):

$$\partial_\mu S \equiv \partial_\mu (t-r) \leftrightarrow k_\mu, \tag{8.4.63}$$

$$\frac{A^{\rm tt}_{ij}(\Omega)}{r} \leftrightarrow \epsilon^{\pm}{}_{ij}(\vec{k}), \tag{8.4.64}$$

$$\exp(-i\Omega(t-r)) \leftrightarrow \exp(-ik_\mu x^\mu). \tag{8.4.65}$$

The spatial traceless property is an exact correspondence between the two sides – $\delta^{ij}A^{\rm tt}_{ij}/r = \delta^{ij}\epsilon^{\pm}{}_{ij} = 0$ – but the transverse property on the left hand side is an orthogonality/algebraic relationship (cf. (**??**)) on the left hand side whereas it is a differential relationship in real space $\partial_i h^{\rm TT}_{ij} = 0$ (it is orthogonal-algebraic in Fourier space). However, since our spherical waves $\widetilde{h}^{\rm tt}_{ij}$ do take the JWKB form in the $r/r_c, r/\tau_c \to \infty$ limit, we employ eq. (8.4.32) to verify

$$\partial_i \widetilde{h}^{\rm tt}_{ij} = -i\Omega\widehat{r}^i\frac{\widetilde{A}^{\rm tt}_{ij}}{r}\left(1+\mathcal{O}\left(\frac{\tau_c}{r}\right)\right)e^{-i\Omega(t-r)} \approx 0. \tag{8.4.66}$$

In words: over a small region of space very far from the source – where the radius of curvature of the constant phase surfaces of $h^{\rm tt}_{ij}$ is very large – these spherical 'tt' waves are approximately plane 'TT' spin-2 waves propagating in the radial direction $\widehat{r}^i \leftrightarrow k^i/k^0$, with the two notions of transversality coinciding as $r/\tau_c, r/r_c \to \infty$.

In the following section we will perform a scalar-vector-tensor decomposition of General Relativity linearized about flat spacetime, just as we performed a scalar-vector decomposition for Maxwell's equation in §(5.2), in order to develop further the notion of a 'TT' gravitational wave and to gain a deeper understanding of which aspects of the gravitational perturbations are actually coordinate independent.

Problem 8.18. Energy-Momentum Conservation & Time-Independence of $\widetilde{\bar{h}}_{\nu 0} = \widetilde{\bar{h}}_{0\nu}$ In this far zone non-relativistic limit, by recalling the solution in eq. (8.4.6), use the local conservation of the stress tensor of matter

$$\partial^\mu \overline{T}_{\mu\nu}(t,\vec{x}) = 0, \tag{8.4.67}$$

to deduce that the components $\bar{h}_{\mu 0} = \bar{h}_{0\mu}$ are time independent. Make sure you explain why this is related to the conservation of total mass

$$M(t) \equiv \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}' \overline{T}_{00}\left(t, \vec{x}'\right) \tag{8.4.68}$$

and of total spatial momentum

$$P_i(t) \equiv \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}' \overline{T}_{0i}\left(t, \vec{x}'\right). \tag{8.4.69}$$

Why do these facts tell us (cf. (**??**)), for $\Omega \neq 0$,

$$\widetilde{A}_{00} = \widetilde{A}_{0\nu} = \widetilde{A}_{0\nu} = 0? \tag{8.4.70}$$

Therefore, only the spatial components $\widetilde{A}_{ij}$ of the (reduced) amplitude are non-zero for active sources. Moreover, show that equations (**??**) and (8.4.70) together imply the orthogonality property

$$\widehat{r}^i \widetilde{A}_{ij} = 0. \tag{8.4.71}$$

As we shall witness shortly, it is this energy-momentum conservation that is responsible for gravitational radiation to begin at the quadrupole order. □

What is Radiation? Electromagnetic and gravitational radiation are, respectively, the piece of electromagnetic field and metric perturbation that are capable of carrying energy-momentum from the material source to infinity. In $d-$dimensional flat spacetime, the volume measure in spherical coordinates $(t, r, \vec{\theta})$ takes the form

$$\mathrm{d}^d x \sqrt{|g|} \to \mathrm{d}t \mathrm{d}r r^{d-2} \mathrm{d}\Omega_{\mathbb{S}^{d-2}}. \tag{8.4.72}$$

That means, if $T^{\mu\nu}$ and $t^{\mu\nu}$ denote the stress tensor of electromagnetism and gravitation itself, whatever outgoing flux of momentum

$$\frac{\mathrm{d}E}{\mathrm{d}t\mathrm{d}\Omega} = \widehat{r}^i \frac{\mathrm{d}P^i}{\mathrm{d}\Omega} = r^{d-2}\widehat{r}^i t^{0i} \qquad \text{or} \qquad r^{d-2}\widehat{r}^i T^{0i} \tag{8.4.73}$$

we wish to associate with its radiation, the $T^{\mu\nu}$ and $t^{\mu\nu}$ must themselves scale as $1/r^{d-2}$, for any other power of r would either lead to a growing or decreasing total momentum at increasing distance from the source. For the electromagnetic case, the stress tensor is purely quadratic in the electromagnetic fields, $T_{\mu\nu} = -F_{\mu\alpha}F_{\nu\beta}\eta^{\alpha\beta} + (1/4)\eta_{\mu\nu}F^2$, and therefore $F_{\mu\nu}[\text{radiation}] \sim 1/r^{(d/2)-1}$. For (3+1)D gravitation, we have found the dominant part of the far zone $h_{\mu\nu}$ to go as $1/r$. That suggests the piece of Einstein's tensor containing precisely n powers of $h_{\mu\nu}$ would scale at leading order as $1/r^n$. Since $1/r^{d-2} = 1/r^2$ in 4D, this means only the quadratic ($n = 2$) piece of Einstein's tensor contributes to gravitational energy-momentum at large distances from the source. Any higher power – say $2 + n$ for $n \geq 1$ – would yield an amplitude that scales at most as $1/r^{2+n}$ and therefore decay to zero as $r \to \infty$.

Pseudo-Stress Tensor for GWs: Non-Relativistic Limit If tidal forces – specifically, undulations of the Riemann tensor generated by distant astrophysical systems – are capable of squeezing and stretching a Weber bar on Earth, it must be that they carry energy-momentum, for otherwise this production of mechanical energy would not be possible.[75] Motivated by this consideration, we shall now work out the energy-momentum carried by gravitational waves, so that we may understand how astrophysical systems dissipate their internal energy through gravitational radiation.

Let's now turn to the perturbative, i.e., infinite series, formulation of General Relativity in eq. (8.0.2).

$$\delta_1 G_{\mu\nu} = 8\pi G_{\rm N} \sum_{\ell=0}^{\infty} \left(\delta_\ell T_{\mu\nu} - \frac{\delta_{\ell+2} G_{\mu\nu}}{8\pi G_{\rm N}} \right). \tag{8.4.74}$$

Here, $\delta_\ell T_{\mu\nu}$ and $\delta_\ell G_{\mu\nu}$ are, respectively, the terms within the stress-energy of matter and Einstein tensor containing exactly ℓ powers of $h_{\mu\nu}$; so, for instance, $\delta_0 T_{\mu\nu} = \overline{T}_{\mu\nu}$. Since the linearized Einstein tensor $\delta_1 G_{\mu\nu}$ obeys the corresponding Bianchi identity $\partial^\mu \delta_1 G_{\mu\nu} = 0$, that implies

$$\partial^\mu \delta_1 G_{\mu\nu} = 0 = 8\pi G_{\rm N} \sum_{\ell=0}^{\infty} \partial^\mu \left(\delta_\ell T_{\mu\nu} - \frac{\delta_{\ell+2} G_{\mu\nu}}{8\pi G_{\rm N}} \right). \tag{8.4.75}$$

In the far zone, $T_{\mu\nu} = 0$, and only the higher order terms in the Einstein tensor on the right hand side are non-zero. That means we must have the conservation law:

$$\partial^\mu \sum_{\ell=0}^{\infty} \frac{\delta_{\ell+2} G_{\mu\nu}}{-8\pi G_{\rm N}} = 0. \tag{8.4.76}$$

In this far zone, the conservation law within the de Donder gauge condition would instead read

$$\partial^\mu \left(-\frac{1}{2} \partial^2 \bar{h}_{\mu\nu} \right) = -\frac{1}{2} \partial^2 \partial^\mu \bar{h}_{\mu\nu} = 0 = \partial^\mu \left(-8\pi G_{\rm N} \sum_{\ell=0}^{\infty} \frac{\delta_{\ell+2} G_{\mu\nu}}{8\pi G_{\rm N}} \right). \tag{8.4.77}$$

The preservation of the de Donder gauge condition is, hence, intimately tied to the conservation of the gravitational energy-momentum. Because these $\delta_{\ell \geq 2} G_{\mu\nu}$ are composed purely of gravitons, and contains an infinite sequence involving higher powers of $h_{\mu\nu}$, we may interpret gravitational wave energy as the result of gravitation interacting with itself.

As already argued earlier, the far zone stress tensor $t_{\mu\nu}$ of gravitational waves has to contain exactly two powers of $h_{\mu\nu}$. We may now identify $t_{\mu\nu}$ to be the quadratic piece of this conserved infinite series:

$$t_{\mu\nu} = -\frac{\delta_2 G_{\mu\nu}}{8\pi G_{\rm N}} \qquad \text{(Stress tensor for GWs)}; \tag{8.4.78}$$

where the second order piece of Einstein $G_{\mu\nu} = R_{\mu\nu} - (1/2) g_{\mu\nu} \mathcal{R}$ is itself

$$\delta_2 G_{\mu\nu} = \delta_2 R_{\mu\nu} - \frac{1}{2} \left(\eta_{\mu\nu} \delta_2 \mathcal{R} + h_{\mu\nu} \delta_1 \mathcal{R} \right). \tag{8.4.79}$$

[75] Of course, as of 2023, we have not yet detected gravitational waves using Weber bars. However, there are several such detectors around the world, either in the planning or already in operation.

If we insert into eq. (8.4.78) the non-relativistic quadrupole formula in (8.4.60), we will of course find $\delta_1 R_{\mu\nu} = 0$ in the far zone and hence

$$\delta_2 G_{\mu\nu} = \delta_2 R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\delta_2\mathcal{R}. \tag{8.4.80}$$

More explicitly, evaluated on the linearized solutions, one would find – after considerable work! – that

$$\delta_2 t_{\mu\nu} = -\frac{2G_{\rm N}}{\pi r^2}\partial_\mu(t-r)\partial_\nu(t-r)\left(\frac{\dddot{Q}^{(\rm tt)}_{ij}(t-r)\dddot{Q}^{(\rm tt)}_{ij}(t-r)}{-16} + \frac{\partial}{\partial t}\frac{\ddot{Q}^{(\rm tt)}_{ij}(t-r)\dddot{Q}^{(\rm tt)}_{ij}(t-r)}{8}\right)$$
$$+ \text{ terms linear in the quadrupole moments } \partial_t^4 Q_{ab}(t-r). \tag{8.4.81}$$

Notice the second term and the second line contains terms that are total time derivatives. Remember $\ddot{Q}_{ab}(t-r)$ is proportional to the total shear-stress; so if we assume it begins with and settles to some constant at the infinite past and future – namely,

$$\partial_t^3 Q^{(\rm tt)}_{ij}[\pm\infty] = 0 \tag{8.4.82}$$

– then we may say that these total derivative terms do not contribute to the total energy, since

$$\frac{\mathrm{d}E}{\mathrm{d}\Omega} = \int_{-\infty}^{+\infty} r^2\widehat{r}^i\delta_2 t^{i0}(t,\vec{x})\mathrm{d}t \tag{8.4.83}$$

$$= \dots + (\text{const.})\left[\frac{\ddot{Q}_{ab}(s)\dddot{Q}_{ab}(s)}{8}\right]_{s=-\infty}^{s-+\infty} + (\text{const.})_{ab}\left[\dddot{Q}_{ab}(s)\right]_{s=-\infty}^{s-+\infty}. \tag{8.4.84}$$

Therefore, we may now identify the *effective* pseudo stress tensor of gravitational waves to be

$$\delta_2 t^{(\rm eff)}_{\mu\nu}(t,\vec{x}) = \frac{G_{\rm N}}{8\pi r^2}\partial_\mu(t-r)\partial_\nu(t-r)\dddot{Q}^{(\rm tt)}_{ij}(t-r)\dddot{Q}^{(\rm tt)}_{ij}(t-r). \tag{8.4.85}$$

Note, too, its null form: $\delta_2 t_{\mu\nu} \propto \partial_\mu(t-r)\partial_\nu(t-r)$, where $\partial^\mu(t-r)\partial_\mu(t-r) = 0$.

Problem 8.19. Derive the results

$$\int_{\mathbb{S}^2}\mathrm{d}^2\Omega_{\widehat{r}}\widehat{r}^i\widehat{r}^j = \frac{4\pi}{3}\delta^{ij}, \tag{8.4.86}$$

$$\int_{\mathbb{S}^2}\mathrm{d}^2\Omega_{\widehat{r}}\widehat{r}^i\widehat{r}^j\widehat{r}^a\widehat{r}^b = \frac{4\pi}{15}\left(\delta^{ij}\delta^{ab} + \delta^{ia}\delta^{jb} + \delta^{ib}\delta^{ja}\right). \tag{8.4.87}$$

Hint: Start with the integral $\int_{\mathbb{S}^2}\mathrm{d}^2\Omega\exp\left(i\vec{k}\cdot\widehat{r}\right)$. Next, use these results to derive the famous quadrupole radiation formula

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_{\mathbb{S}^2}\mathrm{d}^2\Omega r^2\widehat{r}^i\delta_2 t^{0i}_{(\rm eff)}(t,\vec{x}) \tag{8.4.88}$$

$$= \frac{G_{\rm N}}{5}\dddot{Q}^{(\rm t)}_{ab}(t-r)\dddot{Q}^{(\rm t)}_{ab}(t-r); \tag{8.4.89}$$

where the traceless quadrupole is defined as

$$Q^{(\mathrm{t})}_{ab}(s) \equiv \left(\frac{1}{2}\delta_{a\{i}\delta_{j\}b} - \frac{1}{3}\delta_{ab}\delta_{ij}\right) Q_{ij}(s). \tag{8.4.90}$$

Note: $P_{abij}P_{ijmn} = P_{abmn}$ – why? – so you only need to integrate one factor of this projector over the solid angle. □

Compact Binary Systems We now turn to applying our formalism thus far to a quasi-periodic system of astrophysical importance: the compact binary system. It is the astrophysical system composed of a pair of black holes and/or small stars orbiting around each other.[76] Because such a system loses energy and angular momentum to gravitational radiation, its orbital radius will shrink and the two bodies will eventually merge to become a single object. (In fact, this orbital decay due to gravitational radiation was first observed in the Hulse-Taylor binary system, decades before gravitational waves were directly heard by LIGO.) The whole process – from their orbital in-spiral while well separated, to their coalescence, to the settling down of the final single body – generates perturbations of spacetime that are capable of carrying energy-momentum to infinity. We will focus on the simpler case of circular orbits.

Problem 8.20. Non-relativistic Binary Systems: Circular Orbits You have probably solved the Kelper problem in a classical mechanics course, but let us remind ourselves of its solution. Consider a binary system which will we model as a non-relativistic system of two point masses m_1 and m_2, with respective trajectories $\vec{x}_1(t)$ and $\vec{x}_2(t)$. It's Lagrangian is

$$L_{\text{binary}} = \frac{1}{2}m_1\dot{\vec{x}}_1^2 + \frac{1}{2}m_2\dot{\vec{x}}_2^2 + \frac{G_{\mathrm{N}} m_1 m_2}{|\vec{x}_1 - \vec{x}_2|}. \tag{8.4.91}$$

Consider the following change-of-variables:

$$\vec{x}_+ \equiv \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2}, \tag{8.4.92}$$

$$\vec{x}_- \equiv \vec{x}_1 - \vec{x}_2. \tag{8.4.93}$$

Show that the binary system Lagrangian becomes

$$L_{\text{binary}} = \frac{1}{2}M\dot{\vec{x}}_+^2 + \mu\left(\frac{1}{2}\dot{\vec{x}}_-^2 + \frac{G_{\mathrm{N}} M}{|\vec{x}_-|}\right), \tag{8.4.94}$$

where the total and reduced masses are, respectively,

$$M \equiv m_1 + m_2 \qquad \text{and} \qquad \mu \equiv \frac{m_1 m_2}{m_1 + m_2}. \tag{8.4.95}$$

What is $\vec{x}_+$? (Notice it behaves like a free particle.) Observe that $\vec{x}_-$ is the trajectory of a particle subject to a central Newtonian gravitational force. Now let us, for simplicity, assume $\vec{x}_-$ sweeps out a circular trajectory with radius $\bar{r}$. Show that such a trajectory, which – by the

[76] As of December 2017, all gravitational wave events detected to date have involved either black hole (GW150914, GW151226, GW170104, GW170814, GW170608) or neutron star (GW170817) binaries; see the LIGO site here for more detailed information.

rotational symmetry of Euclidean space – we may assume takes place on the $(1,2)$-plane, has the following Cartesian components:

$$\vec{x}_- = \bar{r}\left(\cos\left(\Omega t + \varphi\right), \sin\left(\Omega t + \varphi\right), 0\right), \tag{8.4.96}$$

$$\Omega(\bar{r}) = \frac{\sqrt{G_{\text{N}} M}}{\bar{r}^{3/2}} \qquad \Leftrightarrow \qquad \bar{r} = \sqrt[3]{\frac{G_{\text{N}} M}{\Omega^2}}. \tag{8.4.97}$$

(The φ is an arbitrary angle, set by the initial condition.) Next, explain why we may choose $\vec{x}_+ = \vec{0}$. With this choice, show that

$$\vec{x}_1 = \frac{m_2}{M}\vec{x}_- \qquad \text{and} \qquad \vec{x}_2 = -\frac{m_1}{M}\vec{x}_-. \tag{8.4.98}$$

Finally, from the matter energy density

$$\begin{aligned} \overline{T}_{00} &\equiv {}_{(1)}\overline{T}_{00} + {}_{(2)}\overline{T}_{00} \\ &= m_1 \delta^{(3)}\left(\vec{x} - \vec{x}_1\right) + m_2 \delta^{(3)}\left(\vec{x} - \vec{x}_2\right), \end{aligned} \tag{8.4.99}$$

show that the quadrupole moment and its acceleration are

$$Q_{ij}(t) = \mu x_-^i x_-^j \tag{8.4.100}$$

$$= \frac{G_{\text{N}}^{2/3}}{2\Omega^{4/3}} \frac{m_1 m_2}{(m_1+m_2)^{1/3}} \left[\begin{array}{ccc} 1+\cos[2(\Omega t+\varphi)] & \sin[2(\Omega t+\varphi)] & 0 \\ \sin[2(\Omega t+\varphi)] & 1-\cos[2(\Omega t+\varphi)] & 0 \\ 0 & 0 & 0 \end{array}\right] \tag{8.4.101}$$

$$\begin{aligned} \ddot{Q}_{ij}(t) &= -2G_{\text{N}}^{2/3}\Omega^{2/3}\frac{m_1 m_2}{(m_1+m_2)^{1/3}} \\ &\quad \times \left\{\left(\widehat{x}^i\widehat{x}^j - \widehat{y}^i\widehat{y}^j\right)\cos[2(\Omega t+\varphi)] + \left(\widehat{x}^i\widehat{y}^j + \widehat{y}^i\widehat{x}^j\right)\sin[2(\Omega t+\varphi)]\right\}. \end{aligned} \tag{8.4.102}$$

An observer at some large distance r away from the binary system, which we shall place at $\vec{0}$, will detect gravitational radiation described by eq. (8.4.60). The gravitational wave propagates outwards along $\widehat{r}$, which means the transverse directions are spanned by the unit vectors in spherical coordinates $\widehat{\theta}$ and $\widehat{\phi}$. Explain why the linear polarization basis for the helicity-two radiation are

$$e_{ij}^{+} \equiv \widehat{\theta}^i\widehat{\theta}^j - \widehat{\phi}^i\widehat{\phi}^j, \tag{8.4.103}$$

$$e_{ij}^{\times} \equiv \widehat{\theta}^i\widehat{\phi}^j + \widehat{\phi}^i\widehat{\theta}^j; \tag{8.4.104}$$

and why, therefore, the transverse-traceless wave must expressible as

$$h_{ij}^{\text{tt}}[t,\vec{x}] = h_+ \epsilon_{ij}^+ + h_\times \epsilon_{ij}^\times. \tag{8.4.105}$$

Demonstrate that

$$h_+ = \frac{G_{\text{N}}\mu}{r}(G_{\text{N}} M \Omega)^{2/3}(\cos(2\theta)+3)\cos(2(\Omega t - \phi)), \tag{8.4.106}$$

$$h_\times = \frac{4G_{\text{N}}\mu}{r}(G_{\text{N}} M \Omega)^{2/3}\cos(\theta)\sin(2(\Omega t - \phi)). \tag{8.4.107}$$

(That the angular frequency of the GW is twice that of the orbital motion – i.e., the $\Omega_{\text{GW}} \equiv 2\Omega$ in $\cos[2\Omega t + \dots]$ and $\sin[2\Omega t + \dots]$ – is because of the quadratic $x^i_- x^j_-$ dependence in the quadrupole momentum's acceleration.) Finally, compute the power radiated by the compact binary via gravitational waves and show that it is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{32}{5} G_{\text{N}}^{7/3} \Omega^{10/3} \left(\frac{m_1 m_2}{(m_1 + m_2)^{1/3}} \right)^2 . \tag{8.4.108}$$

[77]We have inserted a − sign to indicate it is a *loss* of energy by the compact binary system due to gravitational radiation. □

Problem 8.21. Frequency Evolution of Compact Binaries Now, the total energy of a company binary system is simply the non-relativistic kinetic plus potential energy:

$$E = \frac{\mu}{2} \dot{x}_-^2 - \frac{G_{\text{N}} \mu M}{|\vec{x}_-|}. \tag{8.4.109}$$

Show that it is

$$E(\Omega) = -\frac{G_{\text{N}}^{2/3} \Omega^{2/3}}{2} \frac{m_1 m_2}{(m_1 + m_2)^{1/3}}. \tag{8.4.110}$$

We have assumed Ω to be constant thus far. But the loss of total system energy due to gravitational waves must be consistent only if the angular frequency *increases* with time – so that E grows more negative – albeit very slowly for non-relativistic motion. Use eq. (8.4.108) in the previous problem and the fact that the *observed* angular frequency $\Omega_{\text{GW}} = 2\pi f_{\text{GW}}$ of the gravitational wave is twice that of the binary system's $\Omega \equiv 2\pi f$, to derive an equation for the time evolution of Ω:

$$\frac{\mathrm{d}f_{\text{GW}}}{\mathrm{d}t} = \frac{96}{5} \pi^{8/3} \left(G_{\text{N}} M_c \right)^{5/3} f_{\text{GW}}^{11/3}, \tag{8.4.111}$$

where the *chirp mass* M_c is defined as

$$M_c \equiv \left(\frac{m_1 m_2}{(m_1 + m_2)^{1/3}} \right)^{3/5} . \tag{8.4.112}$$

Now, the orbital period of the binary is simply $P_b = \Omega/(2\pi)$. Convert eq. (8.4.111) into

$$\dot{P}_b = -\frac{192\pi (G_{\text{N}} M_c)^{5/3}}{5} \left(\frac{2\pi}{P_b} \right)^{5/3} . \tag{8.4.113}$$

This *decrease* of the orbital frequency of a binary system due to emission of gravitational radiation has been verified through the Hulse-Taylor (see Fig. (2) of [22]) and other similar systems – long before the direct detection of gravitational waves by LIGO and VIRGO.

[77]The formula for the more general Keplerian orbit with a non-zero eccentricity e can be found in eq. (16) of Peters and Matthews [21]: $(\mathrm{d}E/\mathrm{d}t)_e = (\mathrm{d}E/\mathrm{d}t)_{e=0}(1 + (73/24)e^2 + (37/96)e^4)/(1 - e^2)^{7/2}$, where $(\mathrm{d}E/\mathrm{d}t)_{e=0}$ is the power emitted in eq. (8.4.108) when eccentricity is zero. Non-zero e enhances the power radiated due to greater accelerations.

Moreover, the dramatic *increase* of the orbital frequency – and, hence, that of the gravitational waves – predicted by eq. (8.4.111) during the end stages of the binary inspiral can be seen from the LIGO and VIRGO data – see, for instance, here. For a pedagogical discussion of the basic physics behind the gravitational waves generated by the inspiral and merger of binary black hole systems, see [23]. □

Including Eccentricity A generic Keplerian orbit is not perfectly circular, of course. Hence, let us briefly discuss the bound orbit case where the eccentricity e of the orbit lies within $[0, 1)$. If we continue to define the $(1, 2)-$plane to be the plane containing the orbit, the displacement vector joining the two masses is

$$\vec{x}_- = \left(\frac{G_{\rm N} M}{\Omega^2}\right)^{\frac{1}{3}} \frac{1-e^2}{1+e\cos[\psi]} (\cos\psi, \sin\psi, 0) . \tag{8.4.114}$$

quadrupole moment is now

$$Q_{ij}(0 \leq e < 1) = \mu \left(\frac{G_{\rm N} M}{\Omega^2}\right)^{\frac{2}{3}} \left(\frac{1-e^2}{1+e\cos[\psi]}\right)^2 \begin{bmatrix} \cos^2[\psi] & \sin[\psi]\cos[\psi] & 0 \\ \sin[\psi]\cos[\psi] & \sin^2[\psi] & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{8.4.115}$$

The acceleration and jerk – with angular momentum conservation imposed – are, respectively,

$$\begin{aligned} \ddot{Q}_{ij}(0 \leq e < 1) &= \mu \frac{(G_{\rm N} M \Omega)^{2/3}}{1-e^2} \qquad (8.4.116) \\ \times &\begin{bmatrix} 2\sin^2(\psi) - 2\cos^2(\psi)(e\cos(\psi)+1) & \sin(\psi)(-(e(\cos(2\psi)+3)+4\cos(\psi))) & 0 \\ \sin(\psi)(-(e(\cos(2\psi)+3)+4\cos(\psi))) & \frac{1}{2}(7e\cos(\psi)+e(4e+\cos(3\psi))+4\cos(2\psi)) & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \dddot{Q}_{ij}(0 \leq e < 1) &= \mu \frac{(G_{\rm N} M)^{2/3} \Omega^{5/3}}{(1-e^2)^{5/2}} (1-e\cos\psi)^2 \qquad (8.4.117) \\ \times &\begin{bmatrix} \sin(2\psi)(3e\cos(\psi)+4) & \frac{1}{2}(-5e\cos(\psi)-3e\cos(3\psi)-8\cos(2\psi)) & 0 \\ \frac{1}{2}(-5e\cos(\psi)-3e\cos(3\psi)-8\cos(2\psi)) & \sin(\psi)(-(e(3\cos(2\psi)+5)+8\cos(\psi))) & 0 \\ 0 & 0 & 0 \end{bmatrix} . \end{aligned}$$

The far zone gravitational radiation $h_{ij}^{\rm tt} = -(2G_{\rm N}/r)\ddot{Q}_{ij}^{(\rm tt)}$ is therefore

$$h_{ij}^{(\rm tt)} = h_+ e_{ij}^+ + h_\times e_{ij}^\times , \tag{8.4.118}$$

where

$$\begin{aligned} h_+ = \frac{G_{\rm N}\mu}{r} \cdot \frac{(G_{\rm N} M \Omega)^{2/3}}{8(1-e^2)} \Big(& 8e\sin^2(\theta)(e+\cos(\psi)) \\ & + 2(\cos(2\theta)+3)\cos(2\phi)(5e\cos(\psi)+e(2e+\cos(3\psi))+4\cos(2\psi)) \\ & + 4(\cos(2\theta)+3)\sin(\psi)\sin(2\phi)(e(\cos(2\psi)+3)+4\cos(\psi)) \Big), \end{aligned} \tag{8.4.119}$$

$$\begin{aligned} h_\times = -\frac{G_{\rm N}\mu}{r} \cdot & \frac{(G_{\rm N} M \Omega)^{2/3}}{1-e^2} \cos(\theta) \\ & \times \Big(e(2e\sin(2\phi)+\sin(2\phi-3\psi)+5\sin(2\phi-\psi)) + 4\sin(2(\phi-\psi)) \Big). \end{aligned} \tag{8.4.120}$$

Application of eq. (8.4.89) tells us, the power radiated is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{8G_{\mathrm{N}}\mu^2\Omega^{10/3}(G_{\mathrm{N}}M)^{4/3}}{15\left(1-e^2\right)^5}(1+e\cos[\psi])^4\left(e^2\sin^2(\psi)+12(e\cos(\psi)+1)^2\right). \tag{8.4.121}$$

The ψ is the angular position of the orbit at retarded time. If we integrate it over one orbital period T,

$$\left\langle\frac{\mathrm{d}E}{\mathrm{d}t}\right\rangle \equiv \frac{1}{T}\int_0^T \frac{\mathrm{d}E}{\mathrm{d}t}\mathrm{d}t = \frac{\Omega}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}E}{\mathrm{d}t}\frac{\mathrm{d}\psi}{(\mathrm{d}\psi/\mathrm{d}t)} \tag{8.4.122}$$

$$= \frac{32}{5}\frac{G_{\mathrm{N}}\mu^2(G_{\mathrm{N}}M)^{4/3}\Omega^{10/3}}{(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right); \tag{8.4.123}$$

where angular momentum conservation was employed to convert $\dot{\psi}$ into a function of $r[\psi]$.

Angular Momentum of Gravitational Waves Starting from $t_{\mu\nu} = -\delta_2 G_{\mu\nu}/(8\pi G_{\mathrm{N}})$, we may identify the flux of angular momentum of gravitational radiation as[78]

$$\frac{\mathrm{d}L^i}{\mathrm{d}t\mathrm{d}\Omega} = \lim_{r\to\infty} r^2\epsilon_{ijk}x^j t^{km}\widehat{r}^m. \tag{8.4.124}$$

Let us focus on an isolated non-relativistic astrophysical system with quadrupole moment Q_{ij}, and carry out an analysis similar to that leading up to eq. (8.4.89), but one that would require the far zone $h_{\mu\nu}$ to be developed to order $1/r^2$ – i.e., $1/r$ relative to the leading order – due to the extra power of r in the cross-product $x^j t^{km}$. This would then reveal the rate of loss of angular momentum by the matter system due to emission of gravitational waves is

$$\frac{\mathrm{d}L^i}{\mathrm{d}t} = -\frac{2}{3}G_{\mathrm{N}}\epsilon_{ijk}\ddot{Q}_{aj}(t-r)\dddot{Q}_{ak}(t-r). \tag{8.4.125}$$

For the compact binary system, a direct calculation yields the only non-zero component to be

$$\frac{\mathrm{d}L^3}{\mathrm{d}t} = -\frac{4\mu^2(G_{\mathrm{N}}M)^{4/3}\Omega^{7/3}}{3\left(1-e^2\right)^{7/2}} \tag{8.4.126}$$

$$\times(1+e\cos\psi)^3\left(e^2+3e(e\cos(2\psi)+4\cos(\psi))+8\right).$$

Averaging over one period,[79]

$$\left\langle\frac{\mathrm{d}L^3}{\mathrm{d}t}\right\rangle = -\frac{32\mu^2(G_{\mathrm{N}}M)^{4/3}\Omega^{7/3}}{3\left(1-e^2\right)^2}\left(1+\frac{7}{8}e^2\right). \tag{8.4.127}$$

For the compact binary system, such angular momentum radiated would lead to a *decrease* in its eccentricity: their orbit would *circularize*. This suggests many of the binary systems detected by LIGO and VIRGO, because they are at the end stages of their evolution, would likely have circular orbits – they had enough time to radiate their eccentricity away.

[78]Remember that $J^{\mu\nu\alpha} = x^{[\mu}T^{\nu]\alpha}$ is the conserved Noether current of Lorentz symmetry whenever the conserved stress tensor $T^{\mu\nu}$ is the Noether current of spacetime translation symmetry; we have $\partial_\mu T^{\mu\nu} = 0 = \partial_\alpha J^{\mu\nu\alpha}$.

[79]**YZ: My answer below differs from Peter's.**

A Warning & Nonlinear 'Completion' of GR I pulled a fast one on you the reader! For gravitationally bound systens like the compact binary star system, both the kinetic and gravitational potential energies are important in describing their dynamics. Hence, it is erroneous to demand only the matter stress tensor to be divergence-free; i.e., it is the sum of both matter *and* gravitational energy-momentum that needs to be conserved.

Specifically, for the point masses we used to approximate the compact stars, if only their matter stress tensors are divergence-free, $\partial^\mu \bar{T}_{\mu\nu} = 0$; then this would in fact imply their respective trajectories follow geodesics in the background Minkowski spacetimes – namely, straight lines – when in fact they follow (at leading order) elliptical orbits because they are primarily bound by $1/r$ attractive potentials. It is this latter $1/r$ gravitational energy that needs to be included in the conservation equations, for the dynamics to be self-consistent. Fortunately, the final formulas we have derived for the GW form and its energy-momentum loss in terms of the quadrupole moments are believed to be correct; but it is the intermediate steps that need a proper treatment. I hope to return to fixing them in the near future.[80]

Linear GW Memory S.

8.5 Gauge-Invariant General Relativity Linearized On Minkowski Background

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I thank Leo Stein for clarifying how xAct [20] carries out perturbation theory.

B Conventions

Function argument There is a notational ambiguity whenever we write "f is a function of the variable x" as $f(x)$. If you did not know f were meant to be a function, what is $f(x+\sin(\theta))$? Is it some number f times $x + \sin\theta$? For this reason, in my personal notes and research papers I reserve square brackets exclusively to denote the argument of functions – I would always write $f[x + \sin[\theta]]$, for instance. (This is a notation I borrowed from the software Mathematica.) However, in these lecture notes I will stick to the usual convention of using parenthesis; but I wish to raise awareness of this imprecision in our mathematical notation.

Einstein summation and index notation Repeated indices are always summed over, unless otherwise stated:

$$\xi^i p_i \equiv \sum_i \xi^i p_i. \tag{B.0.1}$$

Often I will remain agnostic about the range of summation, unless absolutely necessary.

[80] The primary issue is, I have yet to find a book, set of lecture notes, or a review article that treats this problem accurately and pedagogically, taking into account both matter and gravitational stress tensors.

In such contexts when the Einstein summation is in force – unless otherwise stated – both the superscript and subscript are enumeration labels. ξ^i is the ith component of $(\xi^1, \xi^2, \xi^3, \dots)$, not some variable ξ raised to the ith power. The position of the index, whether it is super- or sub-script, usually represents how it transforms under the change of basis or coordinate system used. For instance, instead of calling the 3D Cartesian coordinates (x, y, z), we may now denote them collectively as x^i, where $i = 1, 2, 3$. When you rotate your coordinate system $x^i \to R^i{}_j y^j$, the derivative transforms as $\partial_i \equiv \partial/\partial x^i \to (R^{-1})^j{}_i \partial_j$.

Dimensions Unless stated explicitly, the number of space dimensions is D; it is an arbitrary positive integer greater or equal to one. Unless stated explicitly, the number of spacetime dimensions is $d = D + 1$; it is an arbitrary positive integer greater or equal to 2.

Spatial vs. spacetime indices I will employ the common notation that spatial indices are denoted with Latin/English alphabets whereas spacetime ones with Greek letters. Spacetime indices begin with 0; the 0th index is in fact time. Spatial indices start at 1. I will also use the "mostly minus" convention for the metric; for e.g., the flat spacetime geometry in Cartesian coordinates reads

$$\eta_{\mu\nu} = \text{diag}\left[1, -1, \dots, -1\right], \tag{B.0.2}$$

where "diag$[a_1, \dots, a_N]$" refers to the diagonal matrix, whose diagonal elements (from the top left to the bottom right) are respectively a_1, a_2, $\dots$, a_N. Spatial derivatives are $\partial_i \equiv \partial/\partial x^i$; and spacetime ones are $\partial_\mu \equiv \partial/\partial x^\mu$. The scalar wave operator in flat spacetime, in Cartesian coordinates, read

$$\partial^2 = \Box = \eta^{\mu\nu}\partial_\mu\partial_\nu. \tag{B.0.3}$$

The Laplacian in flat space, in Cartesian coordinates, read instead

$$\vec{\nabla}^2 = \delta^{ij}\partial_i\partial_i, \tag{B.0.4}$$

where δ_{ij} is the Kronecker delta, the unit $D \times D$ matrix $\mathbb{I}$:

$$\begin{aligned} \delta_{ij} &= 1, \qquad i = j \\ &= 0, \qquad i \neq j. \end{aligned} \tag{B.0.5}$$

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