Magnetic Field Due to Infinite Straight Wire

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There is a Gauss' law for magnetism, just as there is one for the electric field, except there are no magnetic charges in Nature:

$$\vec{\nabla} \cdot \vec{B} = 0. \tag{0.0.1}$$

(Compare with the electric case: $\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0$.) The integral version can be stated as:

> The flux of the magnetic field through any closed surface is zero.

When the physical system is static, so that we may assume the time derivative of the electric field is zero, we have Ampere's law:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}, \tag{0.0.2}$$

where $\vec{J}$ is the current density of the system. The integral version of this same law goes as follows. Consider performing a line integral of the magnetic field $\oint \vec{B} \cdot \mathrm{d}\vec{s}$ around a closed loop in 3D space. Pick *any* 2D surface $\mathfrak{D}$ whose boundary $\partial\mathfrak{D}$ is this loop; for example, if the loop is a circle lying on the xy plane, the surface can simply be the interior of this circle or the hemisphere whose equator is the circle. Next, define a unit normal $\widehat{n}$ to this 2D surface in accordance to the right hand rule – when looking 'down' on the surface (i.e., the $\widehat{n}$ is pointing at you) the line integral will be in the counter-clockwise fashion. Then, Ampere's law states

$$\oint_{\partial\mathfrak{D}} \vec{B} \cdot \mathrm{d}\vec{s} = \mu_0 \oint_{\mathfrak{D}} \vec{J} \cdot \mathrm{d}^2\vec{A} = \mu_0 \oint_{\mathfrak{D}} \left(\vec{J} \cdot \widehat{n}\right) \mathrm{d}^2 A. \tag{0.0.3}$$

The $\oint_{\mathfrak{D}} \left(\vec{J} \cdot \widehat{n}\right) \mathrm{d}^2 A$ is of course the total current flowing through the surface.

> Ampere's law: The line integral of the magnetic field around a closed loop C is equal to μ_0 times the total electric current flowing through *any* surface whose boundary is C.

Note that Ampere's law would be inconsistent if charge were not conserved.

Infinite Straight Wire We are now ready to use the magnetic Gauss' law and Ampere's law to deduce the magnetic field around an infinite straight current

$$\vec{I} = I\widehat{z} \tag{0.0.4}$$

running along the z axis, i.e., $(0, 0, z \in \mathbb{R})$. This is a system that is invariant under rotation about and translation along the $z-$axis; so it is advantageous to employ the cylindrical coordinate system

$$(x, y, z) = (r\cos\phi, r\sin\phi, z). \tag{0.0.5}$$

Furthermore, because of the axial and translation symmetry, the $\vec{B}$ field can only depend on r. At any point in space, the magnetic field would have radial B_r, azimuthal B_ϕ and z components B_z:

$$\vec{B} = B_r(r)\widehat{r} + B_\phi(r)\widehat{\phi} + B_z(r)\widehat{z}. \tag{0.0.6}$$

We will now argue that the only non-zero component is the azimuthal one.

First consider a closed Ampere loop that runs parallel the z axis at $r = r_1$ from $z = z_1$ to $z = z_2$, then radially outwards to $r = r_2$, then anti-parallel to z, then radially inwards from $(z = z_1,\ r = r_2)$ to $(z = z_1,\ r = r_1)$. Along the radial segments, $\vec{B} \cdot \mathrm{d}\vec{s} = \pm B_r \mathrm{d}r$, where the $+$ sign is for the outward pointing (top) segment and the $-$ sign is for the inward pointing one. For a fixed r, these two infinitesimal contributions to the line integral cancel. Therefore the entire inward plus outward radial segments do not contribute to the line integral. Since there are no currents going through our Ampere loop, at this point we have

$$\int_{z_1}^{z_2} \vec{B}(r_1) \cdot \widehat{z}\mathrm{d}z - \int_{z_1}^{z_2} \vec{B}(r_2) \cdot \widehat{z}\mathrm{d}z = 0, \tag{0.0.7}$$

$$(B_z(r_1) - B_z(r_2))(z_1 - z_2) = 0, \tag{0.0.8}$$

$$B_z(r_1) = B_z(r_2). \tag{0.0.9}$$

In words: the z component of the B field is the same at any radius r. It can be show that B_z vanishes infinitely far away from the current, $B_z(r = \infty) = 0$; therefore we conclude B_z is zero everywhere.[1]

Next, we consider a 'Gaussian cylinder' of radius r and height h whose axis coincides with the current $\vec{I}$ itself. Let us compute the magnetic flux through it. We have just argued that $B_z = 0$ everywhere, so there is no flux through the top and bottom surfaces of the Gaussian cylinder; there is only flux on the side:

$$2\pi r h B_r(r) = 0. \tag{0.0.10}$$

Therefore, the radial component is zero everywhere.

Finally, let us consider an Ampere circle of radius r lying on a constant z plane, centered at $(x, y) = (0, 0)$. Let the line integral run counterclockwise as viewed from the positive z axis, i.e., looking down in the negative z direction. The total current passing through this loop is I. (If we had chosen a clockwise loop the total current would be $-I$.)

$$\oint \vec{B} \cdot \mathrm{d}\vec{s} = \int_0^{2\pi} \vec{B} \cdot \widehat{\phi}(r\mathrm{d}\phi) = 2\pi r B_\phi(r) = \mu_0 I. \tag{0.0.11}$$

[1]That $B_z(r = \infty) = 0$ is true can be seen using the Biot-Savart law.

We have arrived at

$$\vec{B}(r) = \frac{\mu_0 I}{2\pi r}\widehat{\phi}. \tag{0.0.12}$$

Remark If you step through this derivation, can you see why the result in eq. (0.0.12) really holds outside any axially symmetric static current distribution?