Linear Dielectrics In Electrostatics

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Induced dipole & bound charge densities In the presence of an external (static) electric field $\vec{E}_0$, the positive charges within the neutral atoms/molecules of a given material would be pushed along the direction of $\vec{E}_0$ and the negative charges will be pulled towards $-\vec{E}_0$. This causes a minute separation between positive and negative charges – i.e., a net electric dipole moment is induced for each atom or molecule. Averaged over many atoms/molecules, we may associate a dipole density $\vec{P}(\vec{x})$ with such a material. This dipole density $\vec{P}$ will in turn alter the total electric field at a given location within the material.

The primary physics associated with the dipole density $\vec{P}$ are as follows.

- We may associate a 'bound-charge' density ρ_b of the dielectric with the negative divergence of the dipole density:

$$\rho_b(\vec{x}) = -\vec{\nabla} \cdot \vec{P}(\vec{x}). \tag{0.0.1}$$

- At the boundary of the dielectric, there is a surface 'bound-charge' density σ_b given by the dot product of the dipole density with the outward unit normal $\widehat{n}$:

$$\sigma_b(\vec{x}) = \vec{P}(\vec{x}) \cdot \widehat{n}(\vec{x}). \tag{0.0.2}$$

With these identifications, Gauss' law continue to hold:

$$\vec{\nabla} \cdot \left(\varepsilon_0 \vec{E}\right) = \rho, \tag{0.0.3}$$

except ρ is now the total electric charge density – the bound-charges of equations (0.0.1) and (0.0.2) together with everything else (e.g., the 'free' charges flowing through a conductor).

Linear dielectrics The key assumption that defines a linear dielectric is that the total dipole density $\vec{P}$ at a given point is space is proportional to the electric field itself:

$$\vec{P} = \varepsilon_0 \chi_D \vec{E}, \tag{0.0.4}$$

where χ_D is a constant that characterizes the particular dielectric at hand. More general linear dielectrics would involve a polarization response tensor $(\chi_D)_{ij}$ (i.e., a 3×3 matrix)

$$P_i = \varepsilon_0 \sum_{j=1}^{3} (\chi_D)_{ij} E_j, \tag{0.0.5}$$

but we will work with the simpler version in eq. (0.0.4).[1] With the assumption encoded in eq. (0.0.4), equations (0.0.1) and (0.0.2) translate to

$$\rho_b = -\varepsilon_0 \chi_D \vec{\nabla} \cdot \vec{E}, \tag{0.0.6}$$

$$\sigma_b = \varepsilon_0 \chi_D \vec{E} \cdot \widehat{n}. \tag{0.0.7}$$

Example Referring to Benson's Figure 26.19, we may draw a 'Gaussian cylinder' with one end protruding perpendicularly into the top capacitor conducting plate. If the bottom end is less than $(d-t)/2$ away from the conductor, it is in vacuum and the integral version of Gauss law in eq. (0.0.3) would tell us

$$E[\text{inside conductor}] \cdot A + E[\text{vacuum}] \cdot A = \frac{\sigma \cdot A}{\varepsilon_0}; \tag{0.0.8}$$

where the first term on the left hand side is zero because $\vec{E} = 0$ inside a conductor; while the σ on the right hand side is the charge density on the top capacitor plate. Therefore the electric field with the vacuum region of the capacitor is

$$E[\text{vacuum}] = \frac{\sigma}{\varepsilon_0}. \tag{0.0.9}$$

Now, if the bottom end of the 'Gaussian cylinder' is inside the dielectric, Gauss' law in eq. (0.0.3) tells us the total charge enclosed would now receive additional contributions from the dielectric's surface $\rho_b = \varepsilon_0 \chi_D \vec{E} \cdot \widehat{n}$ and potentially from the interior $\rho_b = -\varepsilon_0 \chi_D \vec{\nabla} \cdot \vec{E}$. If the electric field inside the dielectric is constant, however, then $\vec{\nabla} \cdot \vec{E} = 0$ and $\rho_b = 0$ – as we shall see very shortly, that we will be able to solve for $\vec{E}$ consistently justifies this assumption. Applying Gauss' law in eq. (0.0.3),

$$E[\text{inside dielectric}] \cdot A = \varepsilon_0^{-1} \left(\sigma \cdot A - \varepsilon_0 \chi_D E[\text{inside dielectric}] \cdot A \right). \tag{0.0.10}$$

There is a $-$ sign on the second term of the right hand side because we are assuming $\vec{E}$ points downwards whereas the outward normal on the top side of the dielectric points upwards.

$$E[\text{inside dielectric}](1 + \chi_D) = \frac{\sigma}{\varepsilon_0} \tag{0.0.11}$$

Expressed in terms of the electric field in vacuum (eq. (0.0.9)), the electric field inside the dielectric is reduced by $\kappa \equiv 1 + \chi_D$ if $\chi_D > 0$.

$$E[\text{inside dielectric}] = \frac{\sigma}{\varepsilon_0 \kappa} = \frac{E[\text{vacuum}]}{\kappa}, \tag{0.0.12}$$

$$\kappa \equiv 1 + \chi_D. \tag{0.0.13}$$

There is a slightly different manner to arrive at the same answer. Instead of placing the top end of our 'Gaussian cylinder' inside the top conducting plate, we shall lower it to the vacuum

[1] Even more generally, $\vec{P}$ may be a complicated function of $\vec{E}$; and eq. (0.0.5) may be viewed as the first term in its Taylor Series in the weak field limit.

region, where we know the electric field is $E[\text{vacuum}] = \sigma/\varepsilon_0$. Then, Gauss' law tells us the total charge enclosed now arises entirely from the surface bound-charge:

$$-E[\text{vacuum}] \cdot A + E[\text{dielectric}] \cdot A = \varepsilon_0^{-1} \left(-\varepsilon_0 \chi_D E[\text{dielectric}] \cdot A\right), \tag{0.0.14}$$

$$E[\text{dielectric}] \left(1 + \chi_D\right) = E[\text{vacuum}], \tag{0.0.15}$$

$$E[\text{dielectric}] = \frac{E[\text{vacuum}]}{\kappa}. \tag{0.0.16}$$

Capacitance is the ratio of the charge stored $(\sigma \cdot A)$ to the potential difference V; for constant electric fields, the latter is the electric field times the distance (work done by unit charge), which in turn is

$$V = E[\text{vacuum}] \cdot (d-t) + E[\text{dielectric}] \cdot t = \frac{\sigma}{\varepsilon_0} \left((d-t) + \frac{t}{\kappa}\right). \tag{0.0.17}$$

Therefore,

$$C = \frac{\sigma \cdot A}{V} = A\varepsilon_0 \left(d - t + \frac{t}{\kappa}\right)^{-1}. \tag{0.0.18}$$