HW 2 Problem 1

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The electric field $\vec{E}(x,y)$ generated by a dipole $\vec{p}$ located at $(0,0)$ is

$$\begin{aligned}\vec{E}(x,y) &= \frac{kp}{(x^2+y^2)^{5/2}}\left(3xy, 2y^2-x^2\right) \\ &= \frac{kp}{r^3}\left(\frac{3}{2}\sin(2\theta), \frac{1+3\cos(2\theta)}{2}\right);\end{aligned} \tag{0.0.1}$$

where $(x,y) = r(\sin\theta, \cos\theta)$ and $k \equiv 1/(4\pi\varepsilon_0)$.

Show that, if $\vec{p}$ were located at $\vec{r}'$, the following general expression for the electric field $\vec{E}(\vec{r})$ is consistent with the result in eq. (0.0.1).

$$\vec{E}(\vec{r}) = \frac{k}{|\vec{r}-\vec{r}'|^3}\left(3\left(\vec{p}\cdot\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}\right)\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|} - \vec{p}\right) \tag{0.0.2}$$

For simplicity, assume that $\vec{p}_1$ and $\vec{p}_2$ lie on the same plane.

(Hint: Simply set $\vec{r}' = (0,0)$, $\vec{r} = (x,y)$ and $\vec{p} = (0,p)$.) Next, show that the force on a static dipole $\vec{p}_2$ at $\vec{r}$ due the electric field generated by a dipole $\vec{p}_1$ located at $\vec{r}'$, is given by

$$\begin{aligned}&\vec{p}_2\cdot\vec{\nabla}\vec{E}[\text{due to }\vec{p}_1] \\ &= \frac{k}{R^4}\left(3\left\{(\vec{p}_1\cdot\vec{p}_2) - 5(\vec{p}_1\cdot\widehat{R})(\vec{p}_2\cdot\widehat{R})\right\}\widehat{R} + 3\vec{p}_1(\vec{p}_2\cdot\widehat{R}) + 3\vec{p}_2(\vec{p}_1\cdot\widehat{R})\right),\end{aligned} \tag{0.0.3}$$

where

$$\widehat{R} = \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}, \qquad R \equiv |\vec{r}-\vec{r}'|. \tag{0.0.4}$$

Hint: Again, set $\vec{r}' = (0,0)$, $\vec{r} = (x,y)$ and $\vec{p}_1 = (0,p_1)$, $\vec{p}_2 = (p_{2x}, p_{2y})$. You may then evaluate $(\vec{p}_2\cdot\vec{\nabla})\vec{E}(\text{due to }\vec{p}_1)$ using equations (0.0.1) and (0.0.3) and simply verify they yield the same result.

Finally, what is the torque exerted by $\vec{p}_1$ at $\vec{r}'$ on $\vec{p}_2$ at $\vec{r}$?