Gradients & Equipotential Surfaces

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Let us consider taking the dot product of the gradient of some potential V with an infinitesimal displacement $\mathrm{d}\vec{s}$:

$$\vec{\nabla} V \cdot \mathrm{d}\vec{s} = \partial_x V \mathrm{d}x + \partial_y V \mathrm{d}y + \partial_z V \mathrm{d}z \tag{0.0.1}$$

$$= \mathrm{d}V = V(x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z) - V(x, y, z). \tag{0.0.2}$$

In words: $\vec{\nabla} V \cdot \mathrm{d}\vec{s}$ is the change in the potential V induced upon moving from (x, y, z) to $(x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z)$. On the other hand, we may use the property of the dot product to deduce

$$\vec{\nabla} V \cdot \mathrm{d}\vec{s} = |\vec{\nabla} V||\mathrm{d}\vec{s}| \cos\theta = \mathrm{d}V. \tag{0.0.3}$$

Thus, the change in V is greatest when $\theta = 0$ or $\theta = \pi$, i.e., when the infinitesimal displacement is parallel or anti-parallel to $\vec{\nabla} V$ itself. There is no change in V when the infinitesimal displacement is perpendicular to $\vec{\nabla} V$, where $\theta = \pi/2$.

> $\vec{\nabla} V$ is perpendicular to the equipotential surfaces of V; it is (anti-)parallel to the direction of greatest rate of change of V.