Notes on Field Theory

Yi-Zen Chu

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1 Classical Scalar Fields in Minkowski Spacetime

Field theory in Minkowski spacetime indicates we wish to construct partial differential equations obeyed by fields such that they take the same form in all inertial frames – i.e., the PDEs are Lorentz covariant. As a warm-up, we shall in this section study the case of scalar fields.

A scalar field $\varphi(x)$ is an object that transforms, under Poincaré transformations

$$x^\mu = \Lambda^\mu{}_\nu x'^\nu + a^\mu \tag{1.0.1}$$

as simply

$$\varphi(x(x')) = \varphi\left(x^\mu = \Lambda^\mu{}_\nu x'^\nu + a^\mu\right) \equiv \varphi(x'). \tag{1.0.2}$$

To ensure that this is the case, we would like the PDE it obeys to take the same form in the two inertial frames $\{x^\mu\}$ and $\{x'^\mu\}$ related by eq. (1.0.1). The simplest example is the wave equation with some external scalar source $J(x)$. Let's first write it in the x^μ coordinate system.

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\varphi(x) = J(x), \qquad \partial_\mu \equiv \partial/\partial x^\mu. \tag{1.0.3}$$

If putting a prime on the index denotes derivative with respect to x'^μ, namely $\partial_{\mu'} \equiv \partial/\partial x'^\mu$, then by the chain rule,

$$\partial_{\mu'} = \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial}{\partial x^\sigma} = \partial_{\mu'}\left(\Lambda^\sigma{}_\rho x'^\rho + a^\sigma\right)\partial_\sigma \tag{1.0.4}$$

$$= \Lambda^\sigma{}_\mu \partial_\sigma. \tag{1.0.5}$$

Therefore the wave operator indeed takes the same form in both coordinate systems:

$$\eta^{\mu\nu}\partial_{\mu'}\partial_{\nu'} = \eta^{\mu\nu}\Lambda^\sigma{}_\mu\Lambda^\rho{}_\nu\partial_\sigma\partial_\rho \tag{1.0.6}$$

$$= \eta^{\sigma\rho}\partial_\sigma\partial_\rho. \tag{1.0.7}$$

because of Lorentz invariance

$$\eta^{\mu\nu}\Lambda^\sigma{}_\mu\Lambda^\rho{}_\nu = \eta^{\sigma\rho}. \tag{1.0.8}$$

A generalization of the wave equation in eq. (1.0.3) is to add a potential $V(\varphi)$:

$$\partial^2\varphi + V'(\varphi) = J, \tag{1.0.9}$$

where $\partial^2 \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$ and the prime is a derivative with respect to the argument.

1.1 Action Principle and Symmetries

There is in fact an efficient means to define a theory such that it would enjoy the symmetries one desires. This is the action principle. You may encountered it in (non-relativistic) Classical Mechanics, where Newton's second law emerges from demanding the integral

$$S \equiv \int_{t_\mathrm{i}}^{t_\mathrm{f}} L\mathrm{d}t, \tag{1.1.1}$$

$$L \equiv \frac{1}{2}m\dot{\vec{x}}(t)^2 - V(\vec{x}(t)). \tag{1.1.2}$$

Here, L is called the Lagrangian, and in this context is the difference between the particle's kinetic and potential energy. The action of a field theory also plays a central role in its quantum theory when phrased in the path integral formulation; roughly speaking, $\exp(iS)$ is related to the infinitesimal quantum transition amplitude. For these reasons, we shall study the classical field theories – leading up to General Relativity itself – through the principle of stationary action.

Lorentz covariance In field theory one defines an object similar to the one in eq. (1.1.1), except the integrand $\mathcal{L}$ is now a Lagrangian *density* (per unit spacetime volume). To obtain Lorentz covariant equations, we now demand that the Lagrangian density is, possibly up to a total divergence, a scalar under spacetime Lorentz transformations and other symmetry transformations relevant to the problem at hand.

$$S \equiv \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathcal{L} \mathrm{d}^d x \tag{1.1.3}$$

One then demands that the action is extremized under the boundary conditions that the field configurations at some initial t_i and final time t_f are fixed. If the spatial boundaries of the spacetime are a finite distance away, one would also have to impose appropriate boundary conditions there; otherwise, if space is infinite, the fields are usually assumed to fall off to zero sufficiently quickly at spatial infinity – below, we will assume the latter for technical simplicity. (In particle mechanics, the action principle also assumes the initial and final positions of the particle are specified.)

Let us begin with a scalar field φ. For concreteness, we shall form its Lagrangian density $\mathcal{L}(\varphi, \partial_\alpha \varphi)$ out of φ and its first derivatives $\partial_\alpha \varphi$. Demanding the resulting action be extremized means its first order variation need to vanish. That is, we shall replace $\varphi \to \varphi + \delta\varphi$ (which also means $\partial_\alpha \varphi \to \partial_\alpha \varphi + \partial_\alpha \delta\varphi$) and demand that the portion of the action linear in $\delta\varphi$ be zero.

$$\begin{aligned} \delta_\varphi S &= \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}^d x \left(\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \varphi)} \partial_\alpha \delta\varphi \right) \\ &= \left[\int \mathrm{d}^{d-1} \Sigma_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \varphi)} \delta\varphi \right]_{t_\mathrm{i}}^{t_\mathrm{f}} + \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}^d x \delta\varphi \left(\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \varphi)} \right) \end{aligned} \tag{1.1.4}$$

Because the initial and final field configurations $\varphi(t_\mathrm{i})$ and $\varphi(t_\mathrm{f})$ are assumed fixed, their respective variations are zero by definition: $\delta\varphi(t_\mathrm{i}) = \delta\varphi(t_\mathrm{f}) = 0$. This sets to zero the first term on the second equality. At this point, the requirement that the action be stationary means $\delta_\varphi S$ be zero for any small but arbitrary $\delta\varphi$, which in turn implies the coefficient of $\delta\varphi$ must be zero. That leaves us with the Euler-Lagrangian equations

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \varphi)}. \tag{1.1.5}$$

We may now consider a coordinate transformation $x(x')$. Assuming $\mathcal{L}$ is a coordinate scalar, this means the only ingredient that is not a scalar is the derivative with respect to $\partial_\alpha \varphi$. Since

$$\frac{\partial x^\alpha}{\partial x'^\mu} \partial_\alpha \varphi(x) = \partial_{\mu'} \varphi(x') \equiv \partial_{\mu'} \varphi\left(x(x')\right), \tag{1.1.6}$$

we have

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \varphi(x))} = \frac{\partial(\partial_{\mu'}\varphi(x'))}{\partial(\partial_\alpha \varphi(x))}\frac{\partial \mathcal{L}}{\partial(\partial_{\mu'}\varphi(x'))} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial \mathcal{L}}{\partial(\partial_{\mu'}\varphi(x'))}. \tag{1.1.7}$$

That is, $\partial\mathcal{L}/\partial(\partial_\alpha\varphi(x))$ transforms as a rank-1 vector; and $\partial_\alpha\{\partial\mathcal{L}/\partial(\partial_\alpha\varphi(x))\}$ is its divergence, i.e., a scalar. Altogether, we have thus demonstrated that the Euler-Lagrange equations in eq. (1.1.5), for a scalar field φ, is itself a scalar. This is a direct consequence of the fact that $\mathcal{L}$ is a coordinate scalar by construction. A common example of such a scalar action is

$$S[\varphi] \equiv \int \mathrm{d}^d x \left(\frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right), \tag{1.1.8}$$

where V is its scalar potential.

Problem 1.1. Show from eq. (1.1.5) that the equations derived from the action in eq. (1.1.8) is $\partial^2\varphi = -V'(\phi)$. □

Internal Global O_N Symmetry To provide an example of a symmetry other than the invariance under coordinate transformations, let us consider the following action involving $N > 1$ scalar fields $\{\varphi^{\mathrm{I}}|\mathrm{I} = 1, 2, 3, \dots, N\}$:

$$S \equiv \int \mathrm{d}^d x \mathcal{L}\left(\eta^{\mu\nu}\partial_\mu\varphi^{\mathrm{I}}\partial_\nu\varphi^{\mathrm{I}}, \varphi^{\mathrm{I}}\varphi^{\mathrm{I}}\right). \tag{1.1.9}$$

With summation covention in force, we see that the sum over the scalar field label 'I' is simply a dot product in 'field space'. This in turn leads us to observe that the action is invariant under a global rotation:

$$\varphi^{\mathrm{I}} \equiv \widehat{R}^{\mathrm{I}}{}_{\mathrm{J}}\varphi'^{\mathrm{J}}, \tag{1.1.10}$$

where $\widehat{R}^{\mathrm{I}}{}_{\mathrm{A}}\widehat{R}^{\mathrm{J}}{}_{\mathrm{B}}\delta_{\mathrm{IJ}} = \delta_{\mathrm{AB}}$. (By 'global' rotation, we mean the rotation matrices $\{\widehat{R}^{\mathrm{I}}{}_{\mathrm{J}}\}$ do not depend on spacetime.) Explicitly,

$$\int \mathrm{d}^d x \mathcal{L}\left(\eta^{\mu\nu}\partial_\mu\varphi^{\mathrm{I}}\partial_\nu\varphi^{\mathrm{I}}, \varphi^{\mathrm{I}}\varphi^{\mathrm{I}}\right) = \int \mathrm{d}^d x \mathcal{L}\left(\eta^{\mu\nu}\partial_\mu\varphi'^{\mathrm{I}}\partial_\nu\varphi'^{\mathrm{I}}, \varphi'^{\mathrm{I}}\varphi'^{\mathrm{I}}\right). \tag{1.1.11}$$

Let us now witness, because we have constructed a Lagrangian density that is invariant under such an internal O_N symmetry, the resulting equations of motion transform covariantly under rotations. Firstly, the I-th Euler-Lagrange equation, gotten by varying eq. (1.1.9) with respect to φ^{I}, reads

$$\frac{\partial\mathcal{L}}{\partial\varphi^{\mathrm{I}}} = \partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\varphi^{\mathrm{I}})}. \tag{1.1.12}$$

Under rotation, eq. (1.1.10) is equivalent to

$$\left(\widehat{R}^{-1}\right)^{\mathrm{J}}{}_{\mathrm{I}}\varphi^{\mathrm{I}} = \varphi'^{\mathrm{J}}, \tag{1.1.13}$$

which in turn tells us

$$\left(\widehat{R}^{-1}\right)^{\text{J}}{}_{\text{I}}\partial_\alpha\varphi^{\text{I}} = \partial_\alpha\varphi'^{\text{J}}. \tag{1.1.14}$$

Therefore eq. (1.1.12) becomes

$$\frac{\partial\varphi'^{\text{J}}}{\partial\varphi^{\text{I}}}\frac{\partial\mathcal{L}}{\partial\varphi'^{\text{J}}} = \frac{\partial\partial_\alpha\varphi'^{\text{J}}}{\partial\partial_\alpha\varphi^{\text{I}}}\partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\varphi^{\text{J}})}, \tag{1.1.15}$$

$$\left(\widehat{R}^{-1}\right)^{\text{J}}{}_{\text{I}}\frac{\partial\mathcal{L}}{\partial\varphi'^{\text{J}}} = \left(\widehat{R}^{-1}\right)^{\text{J}}{}_{\text{I}}\partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\varphi^{\text{J}})}. \tag{1.1.16}$$

The PDEs for our O_N-invariant scalar field theory transforms covariantly as a vector under global rotation of the fields $\{\varphi^{\text{I}}\}$.

1.2 Local Conservation Laws

Non-relativistic You would be rightly shocked if you had stored a sealed tank of water on your rooftop only to find its contents gradually disappearing over time – the total mass of water ought to be a constant. Assuming a flat space geometry, if you had instead connected the tank to two pipes, one that pumps water into the tank and the other pumping water out of it, the rate of change of the total mass of the water

$$M \equiv \int_{\text{tank}} \rho(t,\vec{x})\text{d}^3\vec{x} \tag{1.2.1}$$

in the tank – where t is time, $\vec{x}$ are Cartesian coordinates, and $\rho(t,\vec{x})$ is the water's mass density – is

$$\frac{\text{d}}{\text{d}t}\int_{\text{tank}}\rho\text{d}^3\vec{x} = -\left(\int_{\text{cross section of 'in' pipe}} + \int_{\text{cross section of 'out' pipe}}\right)\text{d}^2\vec{\Sigma}\cdot(\rho\vec{v}). \tag{1.2.2}$$

Note that $\text{d}^2\vec{\Sigma}$ points *outwards* from the tank, so at the 'in' pipe-tank interface, if the water were indeed following into the pipe, $-\text{d}^2\vec{\Sigma}\cdot(\rho\vec{v}) > 0$ and its contribution to the rate of increase is positive. At the 'out' pipe-tank interface, if the water were indeed following out of the pipe, $-\text{d}^2\vec{\Sigma}\cdot(\rho\vec{v}) < 0$. If we apply Gauss' theorem,

$$\int_{\text{tank}}\dot{\rho}\text{d}^3\vec{x} = -\int_{\text{tank}}\text{d}^3\vec{x}\vec{\nabla}\cdot(\rho\vec{v}). \tag{1.2.3}$$

If we applied the same sort of reasoning to any infinitesimal packet of fluid, with some local mass density ρ, we would find the following local conservation law

$$\dot{\rho} = -\partial_i\left(\rho\cdot v^i\right). \tag{1.2.4}$$

This is a "local" conservation law in the sense that mass cannot simply vanish from one location and re-appear a finite distance away, without first flowing to a neighboring location.

Relativistic We have implicitly assumed a non-relativistic system, where $|\vec{v}| \ll 1$. This is an excellent approximation for most hydrodynamics problems. Strictly speaking, however, relativistic effects – length contraction, in particular – imply that mass density is not a Lorentz scalar. If we define $\rho(t,\vec{x})$ to be the mass density at $(t,\vec{x})$ in a frame instantaneously at rest (aka 'co-moving') with the fluid packet, then the mass density current that is a locally conserved Lorentz vector is given by

$$J^\mu(t,\vec{x}) \equiv \rho(t,\vec{x}) v^\mu(t,\vec{x}). \tag{1.2.5}$$

Along its integral curve v^μ should be viewed as the proper velocity $\mathrm{d}(t,\vec{x})^\mu/\mathrm{d}\tau$ of the fluid packet, where τ is the latter's proper time. Moreover, as long as the velocity v^μ is timelike, which is certainly true for fluids, let us recall it is always possible to find a (local) Lorentz transformation ${\Lambda^\mu}_\nu(t,\vec{x})$ such that

$$(1,\vec{0})^\mu \equiv v'^\mu = {\Lambda^\mu}_\nu(t,\vec{x}) v^\nu(t,\vec{x}). \tag{1.2.6}$$

and the mass density-current is now

$$J'^\mu = \rho(t',\vec{x}') v'^\mu = \rho(x') \cdot \delta^\mu_0. \tag{1.2.7}$$

The local conservation law obeyed by this relativistically covariant current J^μ is now (in Cartesian coordinates)

$$\partial_\mu J^\mu = 0; \tag{1.2.8}$$

which in turn is a Lorentz invariant statement. Total mass M in a given global inertial frame at a fixed time t is

$$M \equiv \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} J^0. \tag{1.2.9}$$

To show it is a constant, we take the time derivative, and employ eq. (1.2.8):

$$\dot{M} = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} \partial_0 J^0 = -\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} \partial_i J^i. \tag{1.2.10}$$

The divergence theorem tells us that this is equal to the flux of J^i at spatial infinity. But there is no J^i at spatial infinity for physically realistic – i.e., isolated – systems.

1.3 Noether: Continuous Symmetries and Conserved Currents

A field is a substance permeating spacetime. In this section, we shall attempt to associate with it energy-momentum at every location in spacetime, by identifying the Noether's currents associated with the symmetries of Minkowski spacetime. Specifically, the conservation of energy is due to the time translation symmetry of the system at hand. The conservation of linear momentum is due to its spatial translation symmetry; whereas the conservation of angular momentum is due to rotational symmetry. Throughout this discussion, we will assume the dynamics of the field theory is governed by some Lorentz invariant Lagrangian density that depends on the field, and on its first derivatives – but no higher.

Spacetime Translations and Stress-Energy Tensor The physical interpretation delineated here for the components of $T^{\widehat{\mu}\widehat{\nu}}$ is really an assertion. Let us attempt to justify it partially, by appealing to the flat spacetime limit, where the momentum of a classical field theory may be viewed as the conserved Noether current of spacetime translation symmetry. Specifically, let us analyze the canonical scalar field theory of eq. (1.1.8) but with $g_{\mu\nu} = \eta_{\mu\nu}$.

$$\mathcal{L}(x) = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi(x)\partial_\nu\varphi(x) - V\left(\varphi(x)\right). \tag{1.3.1}$$

Since $\mathcal{L}$ is Lorentz invariant, we may consider an infinitesimal spacetime displacement,

$$x^\mu = x'^\mu + a^\mu, \tag{1.3.2}$$

for constant but 'small' a^μ.

$$\mathcal{L}(x) = \mathcal{L}(x') + a^\mu\partial_{\mu'}\mathcal{L}(x') + \mathcal{O}\left(a^2\right). \tag{1.3.3}$$

On the other hand, $\partial/\partial x^\mu = \partial_\mu = \partial_{\mu'} = \partial/\partial x'^\mu$ and

$$\mathcal{L}(x'+a) = \mathcal{L}(x') + \frac{\partial\mathcal{L}}{\partial\varphi(x')}a^\nu\partial_{\nu'}\varphi(x') + \frac{\partial\mathcal{L}}{\partial\partial_{\mu'}\varphi(x')}a^\nu\partial_{\nu'}\partial_{\mu'}\varphi(x') + \mathcal{O}\left(a^2\right) \tag{1.3.4}$$

$$= \mathcal{L}(x') + a^\nu\partial_{\nu'}\varphi(x')\left\{\frac{\partial\mathcal{L}}{\partial\varphi(x')} - \partial_{\mu'}\frac{\partial\mathcal{L}}{\partial\partial_{\mu'}\varphi(x')}\right\} + a^\nu\partial_{\mu'}\left(\partial_{\nu'}\varphi(x')\frac{\partial\mathcal{L}}{\partial\partial_{\mu'}\varphi(x')}\right) + \mathcal{O}\left(a^2\right). \tag{1.3.5}$$

Using the equations-of-motion for the scalar field

$$\frac{\partial\mathcal{L}}{\partial\varphi(x')} - \partial_{\mu'}\frac{\partial\mathcal{L}}{\partial\partial_{\mu'}\varphi(x')} = 0, \tag{1.3.6}$$

eq. (1.3.5) becomes

$$\mathcal{L}(x'+a) = \mathcal{L}(x') + a^\nu\partial_{\mu'}\left(\partial_{\nu'}\varphi(x')\frac{\partial\mathcal{L}}{\partial\partial_{\mu'}\varphi(x')}\right). \tag{1.3.7}$$

We may now equate the linear-in-a^ν terms on the right hand sides of equations (1.3.3) and (1.3.7), and find the following conservation law:

$$\partial_{\mu'}\left\{a^\gamma\left(\partial_{\gamma'}\varphi(x')\frac{\partial\mathcal{L}}{\partial\partial_{\mu'}\varphi(x')} - \delta^\mu_\gamma\mathcal{L}(x')\right)\right\} = 0. \tag{1.3.8}$$

By setting $a^\gamma = \delta^\gamma_\nu$, for a fixed ν, we may identify the conserved quantity inside the $\{\dots\}$ as the Noether momentum p_ν due to translation symmetry along the ν-th direction.[1] Doing so now allows us to identify the conserved stress tensor

$$T^\mu{}_\nu = \partial_{\nu'}\varphi(x')\frac{\partial\mathcal{L}}{\partial\partial_{\mu'}\varphi(x')} - \delta^\mu_\nu\mathcal{L}(x'). \tag{1.3.9}$$

[1] As a simple parallel to the situation here: in classical mechanics, because the free Lagrangian $L = (1/2)\dot{\vec{x}}^2$ is space-translation invariant, $\partial L/\partial x^i = 0$, we may identify the momentum $p_i \equiv \partial L/\partial \dot{x}^i$ as the corresponding Noether charge.

Applying this to eq. (1.3.1), we obtain

$$T^\mu{}_\nu = \partial_\nu\varphi\partial^\mu\varphi - \delta^\mu_\nu\left(\frac{1}{2}(\partial\varphi)^2 - V(\varphi)\right). \tag{1.3.10}$$

It is possible to obtain the same result by first writing the scalar action in curved spacetime, and reading off $T^{\mu\nu}$ as the coefficient of $-(1/2)\delta g_{\mu\nu}$ upon perturbing $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$. Unfortunately, this procedure does not yield a unique $T^{\mu\nu}$, let alone a necessarily gauge-invariant one. (This issue has a long history, starting from at least [**?**, **?**].)

Interpretation of $J^\mu{}_\nu a^\nu$. Just as the local conservation of mass or electric charge leads to their appropriate currents J^μ and their divergence free properties $\partial_\mu J^\mu$, we shall see here that the conservation of energy and momentum leads us to the divergence-less energy-momentum-shear-stress tensor – or, more commonly, the energy-momentum or stress-energy tensor.

In a given inertial frame, we associate time translation symmetry with the conservation of energy. Hence, by choosing $a^\nu\partial_\nu = \partial_t$, we may associate $J^\mu{}_0$ as the energy current. Whenever it is timelike, the zeroth component $J^0{}_0$ may be associated with the co-moving energy density; whereas J^{i0} is the momentum density (i.e., energy per time per area perpendicular to J^{i0}).

In the same inertial frame, we associate translation symmetry in the ith spatial direction with the conservation of the ith component of momentum. By choosing $a^\nu\partial_\nu = \partial_i$, we may associate $J^\mu{}_i$ with the current associated with the ith component of the (spatial) momentum. The zeroth component $J^0{}_i$ is the density of the ith component of momentum; this tells us $J^{0i} \sim J^{i0}$ (up to an overall sign). Whereas $J^k{}_i$ is the ith component of momentum per unit time across the spatial surface perpendicular to the kth spatial direction. In particular, when $k = i$, this would be the momentum per unit time through the surface perpendicular to the $i = k$th direction – but this is simply the pressure (force per unit time) acting on an infinitesimal slab between x^i and $x^i + \mathrm{d}x^i$ in the $i = k$ direction. For $i \neq k$, the $J^k{}_i$ is shear: force in the ith direction per unit area perpendicular to the kth direction. Now, if the force in the kth direction per unit area perpendicular to the ith direction were not equal to the $J^k{}_i$, there will be a torque generated on the (i, k) plane on an infinitesimal area. Hence, we expect $J^{ki} = J^{ik}$.

All these considerations allow us to identify the components $J^{\alpha\beta} \equiv T^{\alpha\beta}$ as those of the energy-momentum-shear-stress (note: stress $\equiv$ pressure) tensor, the flux of the μth component of energy-momentum across the hypersurface orthogonal to the νth direction.

- T^{00} is the energy density ($\equiv$ energy per unit spatial volume).
- $T^{0i} = T^{i0}$ is the linear momentum density ($\equiv$ energy per unit time per unit area perpendicular to the ith direction).
- $T^{ij} = T^{ji}$ for $i \neq j$ is the shear density; the flow of the ith component of momentum per unit time per unit area perpendicular to the jth surface.
- T^{ii} is the pressure/stress ($\equiv$ force per unit area) in the ith direction.

Noether: General Case Let us suppose that a small change is induced on some real scalar field $\delta\varphi$ that leaves the Lagrangian invariant up to a total derivative.

$$\varphi \to \varphi + \delta\varphi, \tag{1.3.11}$$
$$\mathcal{L} \to \mathcal{L} + \partial_\mu K^\mu. \tag{1.3.12}$$

On the other hand, we may expand

$$\partial_\mu K^\mu = \frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi}\partial_\mu \delta\varphi \tag{1.3.13}$$

$$= \left(\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi}\right)\delta\varphi + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi}\delta\varphi\right) \tag{1.3.14}$$

The first group of terms on the right hand side of the second equality is the Euler-Lagrange operation on the Lagrangian. In particular, we see that – if the EoM of the scalar is satisfied – then we may identify

$$\partial_\mu J^\mu = 0 \tag{1.3.15}$$

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\delta\varphi - K^\mu. \tag{1.3.16}$$

Noether's Currents: Ambiguities Notice we read off the Noether current from a divergence equation of the form $\partial_\mu J^\mu = 0$. That means we may add an identically conserved current to the LHS and, hence, yield a different Noether J^μ. It turns out, the most general identically conserved current takes the form $\partial_\mu \partial_\nu K^{\mu\nu}$ for arbitrary but anti-symmetric $K^{\mu\nu} = -K^{\nu\mu}$. Hence, Noether currents are always ambiguous up to this additive $\partial_\nu K^{\mu\nu}$ term. Furthermore, if J^μ is conserved, so is a constant A times it. To sum,

> If, due to some continuous symmetry, J^μ is conserved when evaluated on the solutions to the Euler-Lagrange equations, so is
>
> $$J'^\mu \equiv A \cdot (J^\mu + \partial_\nu K^{\mu\nu}) \tag{1.3.17}$$
>
> for arbitrary constant A and anti-symmetric $K^{\mu\nu} = -K^{\nu\mu}$.

This means the stress energy tensor we 'derived' earlier is, likewise, ambiguous in the same manner. In a given physical situation, therefore, we need additional criteria to pin down the precise physical meanings of the components of the Noether currents.

Internal SO$_D$ Example If a Lagrangian involves 3 scalar fields $\varphi^{\rm I}$ such that the former is invariant under global SO$_D$ rotations of the latter:

$$\mathcal{L} = \frac{1}{2}\partial_\alpha \varphi^{\rm I}\partial^\alpha \varphi^{\rm I} - V\left(\varphi^{\rm I}\varphi^{\rm I}\right), \tag{1.3.18}$$

$$\varphi^{\rm I} \to \widehat{R}^{\rm I}{}_{\rm J}\varphi^{\rm J}, \tag{1.3.19}$$

$$\mathcal{L} \to \mathcal{L}. \tag{1.3.20}$$

Under infinitesimal rotations, we may rotate the pairs $(1,2)$, $(1,3)$ and $(2,3)$.

$$\varphi^{\rm I} \to \varphi^{\rm I} - i\theta\left(\widehat{J}^{\rm AB}\right)_{\rm IJ}\varphi^{\rm J} \tag{1.3.21}$$

Recalling that $i(\widehat{J}^{\rm AB})_{\rm IJ} = \delta^{\rm A}_{[\rm I}\delta^{\rm B}_{\rm J]}$, we may recognize

$$\delta\varphi^{\rm I} = \delta^{\rm I[A}\varphi^{\rm B]}. \tag{1.3.22}$$

The Noether currents – one for each rotation generator – are

$$J^\mu_{\rm K} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^{\rm I}} \delta^{\rm I[A} \varphi^{\rm B]} \tag{1.3.23}$$

$$= \partial^\mu \varphi^{[\rm A} \cdot \varphi^{\rm B]}. \tag{1.3.24}$$

We may check explicitly that this is conserved. Firstly, the EoMs are

$$\partial^2 \varphi^{\rm I} = -2\varphi^{\rm I} V'(\vec{\varphi}^2). \tag{1.3.25}$$

Hence,

$$\partial_\mu J^\mu = \partial^2 \varphi^{[\rm A} \cdot \varphi^{\rm B]} + \partial^\mu \varphi^{[\rm I} \cdot \partial_\mu \varphi^{\rm B]} \tag{1.3.26}$$

$$= -2V' \cdot \varphi^{[\rm A} \cdot \varphi^{\rm B]} = 0. \tag{1.3.27}$$

Problem 1.2. Noether, Lorentz and Angular Momentum Above, we consider the Noether current ${T^\mu}_\alpha$, obeying $\partial_\mu {T^\mu}_\alpha = 0$. corresponding to spacetime translation symmetry.

Let us now consider the Noether current ${J^\mu}_{\alpha\beta}$, obeying $\partial_\mu {J^\mu}_{\alpha\beta} = 0$, from the Lorentz transformation $x^\alpha \to {\Lambda^\alpha}_\beta x^\beta$. (Why are there two extra indices ($\alpha\beta$) on the Noether current? Hint: How are the Lorentz generators labeled?) Show that it is possible to obtain

$$J^{\mu\alpha\beta} = T^{\mu[\alpha} x^{\beta]}, \tag{1.3.28}$$

where $T^{\mu\nu}$ is the Noether current of spacetime translations in eq. (1.3.9). Interpret the components of $J^{\mu\alpha\beta}$; i.e., what is the Noether current of spatial rotations? And of boosts? □

Problem 1.3. Symmetric Noether Currents Explain why, if $\partial_\mu J^{\mu\alpha\beta} = 0$, where $J^{\mu\alpha\beta}$ is given by eq. (1.3.28), then $T^{\mu\nu} = T^{\nu\mu}$.

In other words, if we can obtain the Noether current of Lorentz transformations to be related to that of spacetime translations in the form of eq. (1.3.28), then the Noether current of spacetime translations must be a symmetric tensor. This symmetry property is important for interpreting $T^{\mu\nu}$ as the energy-momentum-stress tensor. □

1.4 Hamiltonian Formulation

1D Particle Mechanics: Review In particle mechanics, from the Lagrangian $L(q, \dot{q})$, we may define the momentum conjugate to the (generalized) position q as

$$p \equiv \left(\frac{\partial L(q, \dot{q})}{\partial \dot{q}} \right)_q . \tag{1.4.1}$$

This relation between p and $(q, \dot{q})$ usually allows us to invert $\dot{q}$ for the pair (q, p); so that every variable can now be expressed in terms of this pair – this allows the interpretation of (q, p) as *independent* "phase space" variables in what follows. The Hamiltonian itself is

$$H(q, p) \equiv p \cdot \dot{q}(q, p) - L(q, p), \tag{1.4.2}$$

$$L(q, p) \equiv L(q, \dot{q}(q, p)). \tag{1.4.3}$$

Hamilton's equations now reads

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \text{and} \qquad \dot{p} = -\frac{\partial H}{\partial q}. \tag{1.4.4}$$

Field Theory For field theory, an analogous discussion follows. The Lagrangian $\mathcal{L}(\varphi, \partial\varphi) = \mathcal{L}(\varphi, \dot{\varphi}, \vec{\nabla}\varphi)$ depends on the field (which is analogous to the position q) and its partial derivatives. The time derivative $\dot{\varphi} \equiv \partial_0\varphi$ is analogous to $\dot{q}$; this, in turn, allows us to define the momentum conjugate to φ as

$$\Pi(x) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0\varphi(x))}. \tag{1.4.5}$$

Like the particle mechanics case, we shall assume it is possible to solve $\dot{\varphi}$ in terms of φ and Π. We then define the Hamiltonian density via the Legendre transform:

$$\mathcal{H} \equiv \Pi \cdot \partial_0\varphi - \mathcal{L}. \tag{1.4.6}$$

We may vary this Legendre transform,

$$\begin{aligned}
\delta\mathcal{H} &= \delta\Pi \cdot \partial_0\varphi + \Pi \cdot \partial_0\delta\varphi - \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi - \frac{\partial\mathcal{L}}{\partial\partial_0\varphi}\partial_0\delta\varphi - \frac{\partial\mathcal{L}}{\partial\partial_i\varphi}\partial_i\delta\varphi && (1.4.7)\\
&= \delta\Pi \cdot \partial_0\varphi + \Pi \cdot \partial_0\delta\varphi - \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi - \Pi \cdot \partial_0\delta\varphi - \frac{\partial\mathcal{L}}{\partial\partial_i\varphi}\partial_i\delta\varphi && (1.4.8)\\
&= \delta\Pi \cdot \partial_0\varphi - \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \partial_i\frac{\partial\mathcal{L}}{\partial\partial_i\varphi}\delta\varphi - \partial_i\left(\frac{\partial\mathcal{L}}{\partial\partial_i\varphi}\delta\varphi\right). && (1.4.9)
\end{aligned}$$

Applying the Euler-Lagrange equations

$$\begin{aligned}
\partial_0\frac{\partial\mathcal{L}}{\partial\partial_0\varphi} + \partial_i\frac{\partial\mathcal{L}}{\partial\partial_i\varphi} &= \frac{\partial\mathcal{L}}{\partial\varphi}, && (1.4.10)\\
\partial_i\frac{\partial\mathcal{L}}{\partial\partial_i\varphi} &= \frac{\partial\mathcal{L}}{\partial\varphi} - \partial_0\Pi. && (1.4.11)
\end{aligned}$$

to eq. (1.4.9), we find that

$$\begin{aligned}
\delta\mathcal{H} &= \delta\Pi \cdot \partial_0\varphi - \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \left(\frac{\partial\mathcal{L}}{\partial\varphi} - \dot{\Pi}\right)\delta\varphi - \partial_i\left(\frac{\partial\mathcal{L}}{\partial\partial_i\varphi}\delta\varphi\right) && (1.4.12)\\
&= \delta\Pi \cdot \dot{\varphi} - \dot{\Pi} \cdot \delta\varphi - \partial_i\left(\frac{\partial\mathcal{L}}{\partial\partial_i\varphi}\delta\varphi\right). && (1.4.13)
\end{aligned}$$

On the other hand, we may vary the Hamiltonian as a function of the field, its spatial gradient, and the conjugate momentum:

$$\begin{aligned}
\varphi &\to \varphi + \delta\varphi, && (1.4.14)\\
\Pi &\to \Pi + \delta\Pi; && (1.4.15)
\end{aligned}$$

and discover

$$\delta\mathcal{H} = \frac{\partial\mathcal{H}}{\partial\Pi}\cdot\delta\Pi + \frac{\partial\mathcal{H}}{\partial\varphi}\cdot\delta\varphi + \frac{\partial\mathcal{H}}{\partial\partial_i\varphi}\cdot\partial_i\delta\varphi \tag{1.4.16}$$

$$= \frac{\partial\mathcal{H}}{\partial\Pi}\cdot\delta\Pi + \left(\frac{\partial\mathcal{H}}{\partial\varphi} - \partial_i\frac{\partial\mathcal{H}}{\partial\partial_i\varphi}\right)\cdot\delta\varphi + \partial_i\left(\frac{\partial\mathcal{H}}{\partial\partial_i\varphi}\cdot\delta\varphi\right). \tag{1.4.17}$$

Comparing the two variation results,

$$\delta\Pi\cdot\left(\dot{\varphi} - \frac{\partial\mathcal{H}}{\partial\Pi}\right) - \left(\dot{\Pi} + \frac{\partial\mathcal{H}}{\partial\varphi} - \partial_i\frac{\partial\mathcal{H}}{\partial\partial_i\varphi}\right)\delta\varphi = \partial_i\left\{\left(\frac{\partial\mathcal{H}}{\partial\partial_i\varphi} + \frac{\partial\mathcal{L}}{\partial\partial_i\varphi}\right)\delta\varphi\right\}. \tag{1.4.18}$$

If we integrate both sides over space, the right hand side will be converted into a surface integral at spatial infinity, which we may argue should fall off to zero as long as $\delta\varphi$ does.

$$\int_{\mathbb{R}^D} \mathrm{d}^D\vec{x}\left\{\delta\Pi\cdot\left(\dot{\varphi} - \frac{\partial\mathcal{H}}{\partial\Pi}\right) - \left(\dot{\Pi} + \frac{\partial\mathcal{H}}{\partial\varphi} - \partial_i\frac{\partial\mathcal{H}}{\partial\partial_i\varphi}\right)\delta\varphi\right\} = 0 \tag{1.4.19}$$

By viewing Π and φ as independent variables, the coefficients of their variations on the left hand side must therefore be individually zero because $\delta\Pi$ and $\delta\varphi$ are arbitrary at every point in space.

$$\dot{\varphi} = \frac{\partial\mathcal{H}}{\partial\Pi} \tag{1.4.20}$$

$$\dot{\Pi} = \partial_i\frac{\partial\mathcal{H}}{\partial\partial_i\varphi} - \frac{\partial\mathcal{H}}{\partial\varphi} \tag{1.4.21}$$

This in turn implies, the right hand side of eq. (1.4.18) must be zero too. And since $\delta\varphi$ was arbitrary,

$$\frac{\partial\mathcal{H}}{\partial(\partial_i\varphi)} = -\frac{\partial\mathcal{L}}{\partial(\partial_i\varphi)}. \tag{1.4.22}$$

Example Let us work out Hamilton's equations for the canonical scalar field in eq. (1.3.1).

$$\Pi = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \dot{\varphi} \tag{1.4.23}$$

The Lagrangian is thus

$$\mathcal{L} = \frac{1}{2}\Pi^2 - \frac{1}{2}(\vec{\nabla}\varphi)^2 - V. \tag{1.4.24}$$

The Legendre transform now reads

$$\mathcal{H} = \Pi^2 - \left(\frac{1}{2}\Pi^2 - \frac{1}{2}(\vec{\nabla}\varphi)^2 - V\right) \tag{1.4.25}$$

$$= \frac{1}{2}\Pi^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + V. \tag{1.4.26}$$

Hamilton's equations are

$$\dot{\varphi} = \Pi, \tag{1.4.27}$$

$$\dot{\Pi} = \vec{\nabla}^2\varphi - V'(\varphi). \tag{1.4.28}$$

When combined, they simply yield $\ddot{\varphi} - \vec{\nabla}^2\varphi = -V'$ as before. We may also readily verify

$$\frac{\partial \mathcal{H}}{\partial(\partial_i \varphi)} = \partial_i \varphi = -\frac{\partial \mathcal{L}}{\partial(\partial_i \varphi)}. \tag{1.4.29}$$

Problem 1.4. Lagrangians from Hamiltonians Starting from the relationship between the Hamiltonian and the Lagrangian in eq. (1.4.6) – but in terms of Π, $\partial_i\varphi$ and φ – show that Hamilton's equations in equations (1.4.20) and (1.4.21) imply the Euler-Lagrange equation $\partial_\mu(\partial\mathcal{L}/\partial(\partial_\mu\varphi)) = \partial\mathcal{L}/\partial\varphi$. □

Problem 1.5. Hamilton's Equations for O_N model Work out Hamilton's equations from the Lagrangian in eq. (1.3.18). □

Problem 1.6. Poisson Brackets and Time Evolution Define the 'equal-time' functional derivatives

$$\frac{\delta\varphi(t,\vec{x})}{\delta\varphi(t,\vec{y})} = \delta^{(D)}(\vec{x}-\vec{y}), \tag{1.4.30}$$

$$\frac{\delta\Pi(t,\vec{x})}{\delta\Pi(t,\vec{y})} = \delta^{(D)}(\vec{x}-\vec{y}), \tag{1.4.31}$$

$$\frac{\delta\varphi(t,\vec{x})}{\delta\Pi(t,\vec{y})} = 0 = \frac{\delta\Pi(t,\vec{x})}{\delta\varphi(t,\vec{y})}; \tag{1.4.32}$$

as well as the 'equal-time' Poisson bracket

$$\begin{aligned} &\{f(\varphi(t,\vec{x}),\Pi(t,\vec{x})), g(\varphi(t,\vec{x}'),\Pi(t,\vec{x}'))\} \\ &\equiv \int_{\mathbb{R}^D} \mathrm{d}^D\vec{y} \left(\frac{\delta f(t,\vec{x})}{\delta\varphi(t,\vec{y})}\frac{\delta g(t,\vec{x}')}{\delta\Pi(t,\vec{y})} - \frac{\delta g(t,\vec{x})}{\delta\varphi(t,\vec{y})}\frac{\delta f(t,\vec{x})}{\delta\Pi(t,\vec{y})}\right). \end{aligned} \tag{1.4.33}$$

Show that, for the canonical Hamiltonian in eq. (1.4.26), the Poisson bracket equations

$$\partial_t\varphi(t,\vec{x}) = \{\varphi(t,\vec{x}), H(t)\}, \tag{1.4.34}$$

$$\partial_t\Pi(t,\vec{x}) = \{\Pi(t,\vec{x}), H(t)\} \tag{1.4.35}$$

involving the total Hamiltonian

$$H(t) \equiv \int_{\mathbb{R}^D} \mathrm{d}^D\vec{x}''\mathcal{H}(t,\vec{x}''); \tag{1.4.36}$$

is in fact equivalent to Hamilton's equations. □

1.5 Fourier Space(time)

One of the key insights that we need for doing perturbative field theory in Minkowski spacetime, is that the linear massive wave equation is really an infinite collection of simple harmonic oscillators (SHOs) in Fourier space. That is, if we decompose

$$\varphi(t,\vec{x}) = \int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D}\widetilde{\varphi}(t,\vec{k})e^{i\vec{k}\cdot\vec{x}} = \int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D}\widetilde{\varphi}(t,\vec{k})e^{-ik_jx^j}, \tag{1.5.1}$$

we see that

$$\partial^2\varphi(x) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\left(\ddot{\widetilde{\varphi}} + \vec{k}^2\widetilde{\varphi}\right)e^{i\vec{k}\cdot\vec{x}} \tag{1.5.2}$$

because each spatial derivative acting on the exponential amounts to the replacement rule $\partial_j \to -ik_j$ because

$$\partial_j e^{i\vec{k}\cdot\vec{x}} = \partial_j\left(-ik_lx^l\right)e^{i\vec{k}\cdot\vec{x}} \tag{1.5.3}$$

$$= \left(-ik_l\delta^l_j\right)e^{i\vec{k}\cdot\vec{x}} \tag{1.5.4}$$

$$= \left(-ik_j\right)e^{i\vec{k}\cdot\vec{x}}. \tag{1.5.5}$$

The wave equation sourced by some external J,

$$\ddot{\varphi}(t,\vec{x}) - \vec{\nabla}^2\varphi(t,\vec{x}) + m^2\varphi(t,\vec{x}) = J(t,\vec{x}) \tag{1.5.6}$$

becomes the driven SHO equation

$$\ddot{\widetilde{\varphi}}(t,\vec{k}) + E^2_{\vec{k}}\widetilde{\varphi}(t,\vec{k}) = \widetilde{J}(t,\vec{k}), \tag{1.5.7}$$

where the SHO frequency is the energy

$$E_{\vec{k}} \equiv \sqrt{\vec{k}^2 + m^2}. \tag{1.5.8}$$

This insight is particularly important when we later quantize φ, because we may then view each Fourier mode of the scalar field as a *quantum* SHO. But let's first turn to two key physical features of our scalar field. First, for the linear wave equation sourced by J, let us witness that the presence or absence of the mass m determines whether φ itself is short- or long-ranged.

Classical Linear Solutions Remember that the retarded Green's function of the SHO with frequency Ω is

$$G_{\text{SHO}}(T) = \Theta(T)\frac{\sin(\Omega T)}{\Omega}, \tag{1.5.9}$$

where

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}T^2} + \Omega^2\right)G_{\text{SHO}}(T) = \delta(T). \tag{1.5.10}$$

The classical causality-obeying solution to eq. (1.5.7) is thus

$$\widetilde{\varphi}(t,\vec{k}) = \int_{-\infty}^{t} \frac{\sin(E_{\vec{k}}(t-t'))}{E_{\rm k}} \cdot \widetilde{J}\left(t',\vec{k}\right) \mathrm{d}t', \tag{1.5.11}$$

$$\varphi(x) = \int_{\mathbb{R}^D} \mathrm{d}^D\vec{x}' \int_{\mathbb{R}} \mathrm{d}t' \left(\Theta(t-t') \int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \frac{\sin(E_{\vec{k}}(t-t'))}{E_k}\right) J\left(t',\vec{x}'\right) \tag{1.5.12}$$

$$\equiv \int_{\mathbb{R}^D} \mathrm{d}^D\vec{x}' \int_{\mathbb{R}} \mathrm{d}t' G_{\rm ret}(x-x') J\left(x'\right). \tag{1.5.13}$$

Problem 1.7. Long or Short Range? Massive versus Massless Consider a static point mass resting at $\vec{x}=0$ in the $\{x^\mu\}$ inertial frame, namely

$$J(t,\vec{x}) = J_0 \delta^{(3)}\left(\vec{x}\right), \qquad J_0 \text{ constant}. \tag{1.5.14}$$

Solve eq. (1.5.6). Hint: You may assume the time derivatives in eq. (1.0.9) can be neglected. Then go to Fourier $\vec{k}-$space. You should find

$$\widetilde{\varphi}(\vec{k}) = \frac{J_0}{\vec{k}^2 + m^2}. \tag{1.5.15}$$

You should find

$$\varphi(t, r \equiv |\vec{x}|, \theta, \phi) = J_0 \frac{\exp(-mr)}{4\pi r}. \tag{1.5.16}$$

That is, φ describes a short-range force (with range $1/m$) that, when $m \to 0$, recovers the long-range Coulomb/Newtonian $1/r$ potential. (Hint: The 3D Fourier integral can be reduced to a 1D integral, which can then be tackled by closing the contour on the complex plane.)

Next, consider an inertial frame $\{x'^\mu\}$ that is moving relative to the $\{x^\mu\}$ frame at velocity v along the positive x^3 axis. What is $\varphi(x')$ in the new frame? □

Problem 1.8. Nonlinearities as Self-Coupled SHOs By going to Fourier space, consider a potential that is a polynomial of degree n in the field φ, with minimum at $\varphi = 0$,

$$V(\varphi) = \sum_{\ell=2}^{n} \frac{p_\ell}{\ell} \varphi^\ell. \tag{1.5.17}$$

Show that the Fourier space version of eq. (1.0.9) is:

$$\ddot{\widetilde{\varphi}}(t,\vec{k}) + (k^2 + p_2)\widetilde{\varphi}(t,\vec{k}) = \widetilde{J}(t,\vec{k}) \\ - \sum_{\ell=2}^{n-1} p_{\ell+1} \prod_{s=1}^{\ell} \left(\int \frac{\mathrm{d}^3\vec{k}_s}{(2\pi)^3} \widetilde{\varphi}(t,\vec{k}_s) \right) (2\pi)^3 \delta^{(3)}\left(\vec{k} - \vec{k}_1 - \dots - \vec{k}_\ell\right). \tag{1.5.18}$$

Observe, for a given $\vec{k}$, the non-linearities of the potential $V(\phi)$ give rise to a driving force – the second line on the right hand side – due to the field itself but from superposing over a range of Fourier modes.

Hint: Let's work out the $\ell = 2$ contribution as an example. This is comes from the cubic p_3 term in the potential:

$$V'(\varphi) = p_3 \varphi(t, \vec{x})^2 + \dots. \tag{1.5.19}$$

The Fourier decomposition of φ^2 is

$$\widetilde{\varphi^2}(t, \vec{k}) = \int \mathrm{d}^3\vec{x} \varphi(x)^2 e^{-i\vec{k}\cdot\vec{x}} \tag{1.5.20}$$

$$= \int \mathrm{d}^3\vec{x} \int \frac{\mathrm{d}^3\vec{k}_1}{(2\pi)^3} \widetilde{\varphi}(t, \vec{k}_1) e^{i\vec{k}_1\cdot\vec{x}} \int \frac{\mathrm{d}^3\vec{k}_2}{(2\pi)^3} \widetilde{\varphi}(t, \vec{k}_2) e^{i\vec{k}_2\cdot\vec{x}} e^{-i\vec{k}\cdot\vec{x}} \tag{1.5.21}$$

$$= \int \frac{\mathrm{d}^3\vec{k}_1}{(2\pi)^3} \widetilde{\varphi}(t, \vec{k}_1) \int \frac{\mathrm{d}^3\vec{k}_2}{(2\pi)^3} \widetilde{\varphi}(t, \vec{k}_2) \int \mathrm{d}^3\vec{x} e^{i(\vec{k}_1+\vec{k}_2-\vec{k})\cdot\vec{x}} \tag{1.5.22}$$

$$= \int \frac{\mathrm{d}^3\vec{k}_1}{(2\pi)^3} \widetilde{\varphi}(t, \vec{k}_1) \int \frac{\mathrm{d}^3\vec{k}_2}{(2\pi)^3} \widetilde{\varphi}(t, \vec{k}_2) (2\pi)^3 \delta^{(3)}\left(\vec{k} - \vec{k}_1 - \vec{k}_2\right). \tag{1.5.23}$$

□

Problem 1.9. Dispersion relations Consider the *massive* Klein-Gordon equation in Minkowski spacetime:

$$\left(\partial^2 + m^2\right) \varphi(t, \vec{x}) = 0, \tag{1.5.24}$$

where φ is a real scalar field. Find the general solution for φ in terms of plane waves $\exp(-ik \cdot x)$ and obtain the dispersion relation:

$$k^2 = m^2 \qquad \Leftrightarrow \qquad E^2 = \vec{p}^2 + m^2, \tag{1.5.25}$$

$$E \equiv k^0, \qquad \vec{p} \equiv \vec{k}. \tag{1.5.26}$$

If each plane wave is associated with a particle of $d-$momentum k_μ, this states that it has mass m. The photon, which obeys $k^2 = 0$, has zero mass. □

1.6 ⋆Uniqueness of Lagrangians

In this section, let us ask the following question. Suppose two Lagrangian densities $\mathcal{L}(\varphi, \partial\varphi)$ and $\mathcal{L}'(\varphi, \partial\varphi)$, which we shall assume only depends on φ and its first derivatives, yield the same EoM:

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} = \frac{\partial \mathcal{L}'}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}'}{\partial \partial_\mu \varphi}. \tag{1.6.1}$$

What is the most general $\Delta \equiv \mathcal{L} - \mathcal{L}'$? In particular, this means Δ solves the Euler-Lagrange equation identically – namely, its form should not depend on the specific solution of φ.

$$0 = \frac{\partial \Delta}{\partial \varphi} - \partial_\mu \frac{\partial \Delta}{\partial \partial_\mu \varphi} \tag{1.6.2}$$

$$= \frac{\partial \Delta}{\partial \varphi} - \partial_\mu \varphi \frac{\partial^2 \Delta}{\partial \varphi \partial \partial_\mu \varphi} - \partial_\mu \partial_\gamma \varphi \frac{\partial^2 \Delta}{\partial \partial_\gamma \varphi \partial \partial_\mu \varphi} \tag{1.6.3}$$

Since this holds for any φ, the only way the second derivative terms vanish is

$$\frac{\partial^2 \Delta}{\partial \partial_\gamma \varphi \partial \partial_\mu \varphi} = 0, \tag{1.6.4}$$

i.e., Δ can only be linear in $\partial\varphi$.

$$\Delta = \Delta_{\mathrm{I}}(\varphi) + \partial_\alpha \varphi \Delta_{\mathrm{II}}^{\alpha}(\varphi) \tag{1.6.5}$$

Euler-Lagrange then reduces to

$$\Delta_{\mathrm{I}}'(\varphi) + \partial_\alpha \varphi \Delta_{\mathrm{II}}^{\alpha\prime}(\varphi) - \partial_\alpha \Delta_{\mathrm{II}}^{\alpha}(\varphi) = 0 \tag{1.6.6}$$
$$\Delta_{\mathrm{I}}'(\varphi) + \partial_\alpha \varphi \Delta_{\mathrm{II}}^{\alpha\prime}(\varphi) - \partial_\alpha \varphi \Delta_{\mathrm{II}}^{\alpha\prime}(\varphi) = 0 \tag{1.6.7}$$
$$\Delta_{\mathrm{I}}'(\varphi) = 0. \tag{1.6.8}$$

Hence, if we define

$$\Delta_0^{\alpha}(z) \equiv \int^z \Delta_{\mathrm{II}}^{\alpha}(z')\mathrm{d}z', \tag{1.6.9}$$

most general Δ is a total divergence (plus an irrelevant constant):

$$\Delta = \partial_\alpha \Delta_0^{\alpha}(\varphi) = \partial_\alpha \varphi \Delta_{\mathrm{II}}^{\alpha}(\varphi). \tag{1.6.10}$$

> If two Lagrangian densities depending on φ, $\partial\varphi$ but no higher derivatives yield the same Euler-Lagrange equations-of-motion, they must differ only up to a total divergence.

Actually, if the single scalar theory is Lorentz invariant, $\Delta_{\mathrm{II}}^{\alpha}$ would have to be proportional to $\partial^\alpha \varphi$, since there are no other vectors in the problem at hand. But $\Delta_{\mathrm{II}}^{\alpha}$ does not depend on the derivatives of φ. Hence, the result is likely stronger: a field theory involving a *single scalar* that has a Lorentz invariant Lagrangian density $\mathcal{L}(\varphi, \partial\varphi)$ must be unique up to an additive constant.

2 Quantum Field Theory of Massive Scalar Fields

We will quantize the scalar field in this chapter. As with all quantum systems – quantum mechanics or quantum field theories – we will continue to assume that Schrodinger's equation holds:

$$i\partial_t |\psi\rangle = H |\psi\rangle . \tag{2.0.1}$$

The key is to understand what the Hamiltonian of a quantum field theory is. We shall first do so for the non-interacting case.

2.1 Non-Interacting Case

2.1.1 Heisenberg Equations of Motion; Equal-Time Commutators

In d dimensional Minkowski spacetime, the theory of a free massive real/Hermitian scalar field is defined by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2 \tag{2.1.1}$$

$$= \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}\vec{\nabla}\varphi\cdot\vec{\nabla}\varphi - \frac{1}{2}m^2\varphi^2. \tag{2.1.2}$$

The momentum conjugate to φ is defined as

$$\Pi \equiv \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \dot{\varphi}. \tag{2.1.3}$$

Its Hamiltonian density is

$$\mathcal{H} \equiv \Pi\cdot\dot{\varphi} - \mathcal{L} = \Pi^2 - \left(\frac{1}{2}\Pi^2 - \frac{1}{2}\vec{\nabla}\varphi\cdot\vec{\nabla}\varphi - \frac{1}{2}m^2\varphi^2\right) \tag{2.1.4}$$

$$= \frac{1}{2}\Pi^2 + \frac{1}{2}\vec{\nabla}\varphi\cdot\vec{\nabla}\varphi + \frac{1}{2}m^2\varphi^2. \tag{2.1.5}$$

Quantization in "real" spacetime In position spacetime, and on a constant t surface, we shall identify the field at each spatial point $\vec{x}$ as an independent quantum mechanical "system". As such, in analogy with the quantum mechanical $[X,P]=i$ and $[X,X]=0=[P,P]$, we shall impose the equal-time commutation relations

$$[\varphi(t,\vec{x}),\Pi(t,\vec{x}')] = i\delta^{(d-1)}(\vec{x}-\vec{x}'), \tag{2.1.6}$$

$$[\varphi(t,\vec{x}),\varphi(t,\vec{x}')] = 0 = [\Pi(t,\vec{x}),\Pi(t,\vec{x}')]. \tag{2.1.7}$$

We reiterate, this holds when both fields are evaluated at the same time t; we will see below that $[\varphi(x),\varphi(y)]\neq 0$ in general. Moreover, note that the second line implies

$$[\partial_{x^i}\varphi(t,\vec{x}),\varphi(t,\vec{x}')] = 0. \tag{2.1.8}$$

As we shall witness, in Fourier space, this quantization condition is intimately related to the simple harmonic oscillator (SHO) algebra, which in turn leads us to the particle interpretation.

Problem 2.1. If we decompose the operators in Fourier space,

$$\varphi(t,\vec{x}) = \int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D}\widetilde{\varphi}(t,\vec{k})e^{i\vec{k}\cdot\vec{x}}, \tag{2.1.9}$$

$$\Pi(t,\vec{x}) = \int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D}\widetilde{\Pi}(t,\vec{k})e^{i\vec{k}\cdot\vec{x}}; \tag{2.1.10}$$

show that the commutation relations in eq. (2.1.6) and (2.1.7) translates to

$$\left[\widetilde{\varphi}(t,\vec{k}),\widetilde{\Pi}(t,\vec{k}')\right] = (2\pi)^D i\delta^{(D)}(\vec{k}+\vec{k}'), \tag{2.1.11}$$

$$\left[\widetilde{\varphi}(t,\vec{k}),\widetilde{\varphi}(t,\vec{k}')\right] = 0 = \left[\widetilde{\Pi}(t,\vec{k}),\widetilde{\Pi}(t,\vec{k}')\right]. \tag{2.1.12}$$

□

Since the scalar field depends on spacetime, it is necessarily time dependent. We shall postulate $\varphi(x)$ is a Heisenberg picture operator. It must therefore obey the equations of motion (EoM)

$$\dot{\varphi}(t,\vec{x}) = i\left[H_{\mathrm{H}}(t),\varphi(t,\vec{x})\right] \tag{2.1.13}$$

$$= i\int \mathrm{d}^{d-1}\vec{y}\left[\frac{1}{2}\Pi^2(t,\vec{y}) + \frac{1}{2}\vec{\nabla}\varphi(t,\vec{y})\cdot\vec{\nabla}\varphi(t,\vec{y}) + \frac{1}{2}m^2\varphi^2(t,\vec{y}),\varphi(x)\right] \tag{2.1.14}$$

$$= i\int \mathrm{d}^{d-1}\vec{y}\left[\frac{1}{2}\Pi^2(t,\vec{y}),\varphi(x)\right] \tag{2.1.15}$$

$$= \frac{i}{2}\int \mathrm{d}^{d-1}\vec{y}\left(\Pi(t,\vec{y})\left[\Pi(t,\vec{y}),\varphi(x)\right] + \left[\Pi(t,\vec{y}),\varphi(x)\right]\Pi(t,\vec{y})\right) \tag{2.1.16}$$

$$= \frac{i}{2}\int \mathrm{d}^{d-1}\vec{y}\left(\Pi(t,\vec{y})\left[\Pi(t,\vec{y}),\varphi(t,\vec{x})\right] + \left[\Pi(t,\vec{y}),\varphi(t,\vec{x})\right]\Pi(t,\vec{y})\right) \tag{2.1.17}$$

$$= i(-i)\int \mathrm{d}^{d-1}\vec{y}\Pi(t,\vec{y})\delta^{(d-1)}(\vec{y}-\vec{x}) = \Pi(t,\vec{x}). \tag{2.1.18}$$

Similarly,

$$\dot{\Pi}(t,\vec{x}) = i\left[H_{\mathrm{H}}(t),\Pi(t,\vec{x})\right] \tag{2.1.19}$$

$$= i\int \mathrm{d}^{d-1}\vec{y}\left[\frac{1}{2}\Pi^2(t,\vec{y}) + \frac{1}{2}\vec{\nabla}\varphi(t,\vec{y})\cdot\vec{\nabla}\varphi(t,\vec{y}) + \frac{1}{2}m^2\varphi^2(t,\vec{y}),\Pi(t,\vec{x})\right] \tag{2.1.20}$$

$$= \frac{i}{2}\int \mathrm{d}^{d-1}\vec{y}\left[\vec{\nabla}\varphi(t,\vec{y})\cdot\vec{\nabla}\varphi(t,\vec{y}) + m^2\varphi^2(t,\vec{y}),\Pi(t,\vec{x})\right] \tag{2.1.21}$$

$$= -\int \mathrm{d}^{d-1}\vec{y}\left(\vec{\nabla}\varphi(t,\vec{y})\cdot\vec{\nabla}_{\vec{y}}\delta^{(d-1)}(\vec{y}-\vec{x}) + m^2\varphi(t,\vec{y})\delta^{(d-1)}(\vec{y}-\vec{x})\right) \tag{2.1.22}$$

$$= \vec{\nabla}^2\varphi(t,\vec{x}) - m^2\varphi(t,\vec{x}). \tag{2.1.23}$$

Altogether, we have $\ddot{\varphi} = \dot{\Pi} = \vec{\nabla}^2\varphi - m^2\varphi$. Just like the quantum mechanical 1D SHO, the Heisenberg EoM for the φ is precisely its classical counterpart:

$$\left(\partial^2 + m^2\right)\varphi(x) = 0. \tag{2.1.24}$$

Problem 2.2. Covariant Alternative to Equal Time Commutation Relations An alternate means to canonical quantization that emphasizes the Lorentz covariance nature of quantum fields is to demand that their (anti-)commutators yield the difference between the classical retarded and advanced Green's functions. For a scalar field with mass $m > 0$,

$$\mathcal{L} = (1/2)(\partial\varphi)^2 - (m^2/2)\varphi^2, \tag{2.1.25}$$

we may postulate that

$$i\left[\varphi(x), \varphi(x')\right] = G^+(x - x') - G^-(x - x'), \tag{2.1.26}$$

$$G^\pm(z) = -\int_{-\infty\pm i0^+}^{+\infty\pm i0^+} \frac{\mathrm{d}\omega}{2\pi} \int_{\mathbb{R}^{d-1}} \frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{e^{-i\omega z^0} e^{i\vec{k}\cdot\vec{z}}}{\omega^2 - \vec{k}^2 - m^2}. \tag{2.1.27}$$

Note that:

$$(\partial^2 + m^2)G^\pm(x) = \delta^{(4)}(x). \tag{2.1.28}$$

Show that the covariant starting point in eq. (2.1.26) leads to the canonical equal time commutation relations

$$[\varphi(t, \vec{x}), \Pi(t, \vec{x}')] = [\varphi(t, \vec{x}), \dot{\varphi}(t, \vec{x}')] = i\delta^{(d-1)}\left(\vec{x} - \vec{x}'\right) \tag{2.1.29}$$

and

$$[\varphi(t, \vec{x}), \varphi(t, \vec{x}')] = 0 = [\Pi(t, \vec{x}), \Pi(t, \vec{x}')]\,. \tag{2.1.30}$$

Hint: You do not need to evaluate the entire Fourier integral, but you need to apply the residue theorem to the ω integral before computing the equal time commutator. You may also assume that $\Theta(x) + \Theta(-x) = 1$ for $x \in \mathbb{R}$. ☐

2.1.2 SHO Ladder Operators: Particles

Quantization in Fourier Space Let us now write the Heisenberg scalar field operator as a sum over its Fourier modes:

$$\varphi(t, \vec{x}) = \int_{\mathbb{R}^{d-1}} \frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}} \widetilde{\varphi}(t, \vec{k}) e^{i\vec{k}\cdot\vec{x}}. \tag{2.1.31}$$

The scalar EoM of eq. (2.1.24) now becomes

$$\int_{\mathbb{R}^{d-1}} \frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}} \left(\ddot{\widetilde{\varphi}} + E_{\vec{k}}^2 \widetilde{\varphi}\right) e^{i\vec{k}\cdot\vec{x}} = 0, \qquad E_{\vec{k}} \equiv \sqrt{\vec{k}^2 + m^2}. \tag{2.1.32}$$

This in turn means each Fourier mode of the scalar operator is a SHO, with frequency given by the energy $E_{\vec{k}}$:

$$\ddot{\widetilde{\varphi}}(t, \vec{k}) + E_{\vec{k}}^2 \widetilde{\varphi}(t, \vec{k}) = 0. \tag{2.1.33}$$

The solution is therefore a superposition of positive $e^{-iE_{\vec{k}}t}$ and negative $e^{+iE_{\vec{k}}t}$ modes, with the following manifestly Hermitian form:

$$\varphi(t,\vec{x}) = \int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D\sqrt{2E_{\vec{k}}}} \left(a_{\vec{k}} e^{-iE_{\vec{k}}t} e^{i\vec{k}\cdot\vec{x}} + a_{\vec{k}}^\dagger e^{+iE_{\vec{k}}t} e^{-i\vec{k}\cdot\vec{x}} \right). \tag{2.1.34}$$

(The $\sqrt{2E_{\vec{k}}}$ in the denominator is for convenience.) Let us now turn to work out the commutation relations of these $a_{\vec{k}}$ and $a_{\vec{k}}^\dagger$. For the canonical scalar, we have in fact already imposed their commutation relations in equations (2.1.6) and (2.1.7). Since $\Pi = \partial_t\varphi$,

$$\begin{aligned}
&i\delta^{(D)}(\vec{x}-\vec{x}') = i\int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} = [\varphi(t,\vec{x}), \Pi(t,\vec{x}')] \\
&= \int_{\vec{k},\vec{k}'} \left[a_{\vec{k}} e^{-iE_{\vec{k}}t} e^{i\vec{k}\cdot\vec{x}} + a_{\vec{k}}^\dagger e^{+iE_{\vec{k}}t} e^{-i\vec{k}\cdot\vec{x}}, -iE_{\vec{k}} \cdot a_{\vec{k}'} e^{-iE_{\vec{k}'}t} e^{i\vec{k}'\cdot\vec{x}'} + iE_{\vec{k}} \cdot a_{\vec{k}'}^\dagger e^{+iE_{\vec{k}'}t} e^{-i\vec{k}'\cdot\vec{x}'} \right] \qquad (2.1.35) \\
&= i\int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D} \int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}'}{(2\pi)^D} \frac{E_{\vec{k}}}{2\sqrt{E_{\vec{k}}E_{\vec{k}'}}} \\
&\qquad \times \Bigg\{ - \left[a_{\vec{k}}, a_{\vec{k}'}\right] e^{-iE_{\vec{k}}t} e^{i\vec{k}\cdot\vec{x}} e^{-iE_{\vec{k}'}t} e^{i\vec{k}'\cdot\vec{x}'} + \left[a_{\vec{k}}^\dagger, a_{\vec{k}'}^\dagger\right] e^{+iE_{\vec{k}}t} e^{-i\vec{k}\cdot\vec{x}} e^{+iE_{\vec{k}'}t} e^{-i\vec{k}'\cdot\vec{x}'} \\
&\qquad + \left[a_{\vec{k}}, a_{\vec{k}'}^\dagger\right] e^{-iE_{\vec{k}}t} e^{i\vec{k}\cdot\vec{x}} e^{+iE_{\vec{k}'}t} e^{-i\vec{k}'\cdot\vec{x}'} - \left[a_{\vec{k}}^\dagger, a_{\vec{k}'}\right] e^{+iE_{\vec{k}}t} e^{-i\vec{k}\cdot\vec{x}} e^{-iE_{\vec{k}'}t} e^{i\vec{k}'\cdot\vec{x}'} \Bigg\}; \qquad (2.1.36)
\end{aligned}$$

$$\begin{aligned}
&0 = [\varphi(t,\vec{x}), \varphi(t,\vec{x}')] \\
&= \int_{\vec{k},\vec{k}'} \left[a_{\vec{k}} e^{-iE_{\vec{k}}t} e^{i\vec{k}\cdot\vec{x}} + a_{\vec{k}}^\dagger e^{+iE_{\vec{k}}t} e^{-i\vec{k}\cdot\vec{x}}, a_{\vec{k}'} e^{-iE_{\vec{k}'}t} e^{i\vec{k}'\cdot\vec{x}'} + a_{\vec{k}'}^\dagger e^{+iE_{\vec{k}'}t} e^{-i\vec{k}'\cdot\vec{x}'} \right] \qquad (2.1.37) \\
&= \int_{\vec{k},\vec{k}'} \Bigg\{ \left[a_{\vec{k}}, a_{\vec{k}'}\right] e^{-iE_{\vec{k}}t} e^{i\vec{k}\cdot\vec{x}} e^{-iE_{\vec{k}'}t} e^{i\vec{k}'\cdot\vec{x}'} + \left[a_{\vec{k}}^\dagger, a_{\vec{k}'}^\dagger\right] e^{+iE_{\vec{k}}t} e^{-i\vec{k}\cdot\vec{x}} e^{+iE_{\vec{k}'}t} e^{-i\vec{k}'\cdot\vec{x}'} \\
&\qquad + \left[a_{\vec{k}}, a_{\vec{k}'}^\dagger\right] e^{-iE_{\vec{k}}t} e^{i\vec{k}\cdot\vec{x}} e^{+iE_{\vec{k}'}t} e^{-i\vec{k}'\cdot\vec{x}'} + \left[a_{\vec{k}}^\dagger, a_{\vec{k}'}\right] e^{+iE_{\vec{k}}t} e^{-i\vec{k}\cdot\vec{x}} e^{-iE_{\vec{k}'}t} e^{i\vec{k}'\cdot\vec{x}'} \Bigg\}; \qquad (2.1.38)
\end{aligned}$$

and

$$\begin{aligned}
&0 = [\Pi(t,\vec{x}), \Pi(t,\vec{x}')] \\
&= \int_{\vec{k},\vec{k}'} i^2 E_{\vec{k}} E_{\vec{k}'} \left[-a_{\vec{k}} e^{-iE_{\vec{k}}t} e^{i\vec{k}\cdot\vec{x}} + a_{\vec{k}}^\dagger e^{+iE_{\vec{k}}t} e^{-i\vec{k}\cdot\vec{x}}, -a_{\vec{k}'} e^{-iE_{\vec{k}'}t} e^{i\vec{k}'\cdot\vec{x}'} + a_{\vec{k}'}^\dagger e^{+iE_{\vec{k}'}t} e^{-i\vec{k}'\cdot\vec{x}'} \right] \qquad (2.1.39) \\
&= -\int_{\vec{k},\vec{k}'} E_{\vec{k}} E_{\vec{k}'} \Bigg\{ \left[a_{\vec{k}}, a_{\vec{k}'}\right] e^{-iE_{\vec{k}}t} e^{i\vec{k}\cdot\vec{x}} e^{-iE_{\vec{k}'}t} e^{i\vec{k}'\cdot\vec{x}'} + \left[a_{\vec{k}}^\dagger, a_{\vec{k}'}^\dagger\right] e^{+iE_{\vec{k}}t} e^{-i\vec{k}\cdot\vec{x}} e^{+iE_{\vec{k}'}t} e^{-i\vec{k}'\cdot\vec{x}'} \\
&\qquad - \left[a_{\vec{k}}, a_{\vec{k}'}^\dagger\right] e^{-iE_{\vec{k}}t} e^{i\vec{k}\cdot\vec{x}} e^{+iE_{\vec{k}'}t} e^{-i\vec{k}'\cdot\vec{x}'} - \left[a_{\vec{k}}^\dagger, a_{\vec{k}'}\right] e^{+iE_{\vec{k}}t} e^{-i\vec{k}\cdot\vec{x}} e^{-iE_{\vec{k}'}t} e^{i\vec{k}'\cdot\vec{x}'} \Bigg\}. \qquad (2.1.40)
\end{aligned}$$

Let's start with the $[\varphi, \Pi]$ commutation relation. The $[a_k, a_{k'}]e^{i\vec{k}\cdot\vec{x}+i\vec{k}'\cdot\vec{x}'}$ must be proportional to $\delta^{(D)}(\vec{k}+\vec{k}')e^{i\vec{k}(\vec{x}-\vec{x}')}$. The $[a_k, a^\dagger_{k'}]e^{i\vec{k}\cdot\vec{x}-i\vec{k}'\cdot\vec{x}'}$ term must be proportional to $\delta^{(D)}(\vec{k}-\vec{k}')e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$. The other two terms are related by hermitian conjugate. In detail,

$$\left[a_{\vec{k}}, a_{\vec{k}'}\right] = \chi_{\vec{k}} \cdot \delta^{(D)}(\vec{k}+\vec{k}'), \tag{2.1.41}$$

$$\left[a^\dagger_{\vec{k}}, a^\dagger_{\vec{k}'}\right] = -\chi^*_{\vec{k}} \cdot \delta^{(D)}(\vec{k}+\vec{k}'); \tag{2.1.42}$$

and

$$\left[a_{\vec{k}}, a^\dagger_{\vec{k}'}\right] = \gamma_{\vec{k}} \cdot \delta^{(D)}(\vec{k}-\vec{k}'), \tag{2.1.43}$$

$$\left[a^\dagger_{\vec{k}}, a_{\vec{k}'}\right] = -\gamma^*_{\vec{k}} \cdot \delta^{(D)}(\vec{k}-\vec{k}'). \tag{2.1.44}$$

At this point

$$\begin{aligned}
& i\delta^{(D)}(\vec{x}-\vec{x}') = i\int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} = [\varphi(t,\vec{x}), \Pi(t,\vec{x}')] \\
&= i\int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}'}{(2\pi)^D} \frac{E_{\vec{k}}}{2\sqrt{E_{\vec{k}}E_{\vec{k}'}}} \\
&\qquad \times \Bigg\{ -\chi_{\vec{k}} \cdot \delta^{(D)}(\vec{k}+\vec{k}') \cdot e^{-iE_{\vec{k}}t} e^{-iE_{\vec{k}'}t} - \chi^*_{\vec{k}} \cdot \delta^{(D)}(\vec{k}+\vec{k}') \cdot e^{+iE_{\vec{k}}t} e^{+iE_{\vec{k}'}t} \\
&\qquad + \gamma_{\vec{k}} \cdot \delta^{(D)}(\vec{k}-\vec{k}') \cdot e^{-iE_{\vec{k}}t} e^{+iE_{\vec{k}'}t} + \gamma^*_{\vec{k}} \cdot \delta^{(D)}(\vec{k}-\vec{k}') \cdot e^{+iE_{\vec{k}}t} e^{-iE_{\vec{k}'}t} \Bigg\}
\end{aligned} \tag{2.1.45}$$

$$\begin{aligned}
&= i\int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{2} \\
&\qquad \times \int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}'}{(2\pi)^D} \Bigg\{ -2\mathrm{Re}\left(\chi_{\vec{k}} \cdot e^{-2iE_{\vec{k}}t}\right) \delta^{(D)}(\vec{k}+\vec{k}') + 2\mathrm{Re}(\gamma_{\vec{k}}) \cdot \delta^{(D)}(\vec{k}-\vec{k}') \Bigg\};
\end{aligned} \tag{2.1.46}$$

$$\begin{aligned}
& 0 = [\varphi(t,\vec{x}), \varphi(t,\vec{x}')] \\
&= \int_{\vec{k},\vec{k}'} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \Bigg\{ \chi_{\vec{k}}\delta^{(D)}(\vec{k}+\vec{k}') \cdot e^{-iE_{\vec{k}}t} e^{-iE_{\vec{k}'}t} - \chi^*_{\vec{k}}\delta^{(D)}(\vec{k}+\vec{k}') \cdot e^{+iE_{\vec{k}}t} e^{+iE_{\vec{k}'}t} \\
&\qquad + \gamma_{\vec{k}}\delta^{(D)}(\vec{k}-\vec{k}') \cdot e^{-iE_{\vec{k}}t} e^{+iE_{\vec{k}'}t} - \gamma^*_{\vec{k}}\delta^{(D)}(\vec{k}-\vec{k}') \cdot e^{+iE_{\vec{k}}t} e^{-iE_{\vec{k}'}t} \Bigg\}
\end{aligned} \tag{2.1.47}$$

$$= \int_{\vec{k},\vec{k}'} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \Bigg\{ 2\mathrm{Im}\left(\chi_{\vec{k}} \cdot e^{-2iE_{\vec{k}}t}\right) \delta^{(D)}(\vec{k}+\vec{k}') + 2\mathrm{Im}\left(\gamma_{\vec{k}}\right) \delta^{(D)}(\vec{k}-\vec{k}') \Bigg\} \tag{2.1.48}$$

and

$$0 = [\Pi(t,\vec{x}), \Pi(t,\vec{x}')]$$

$$= -\int_{\vec{k},\vec{k}'} E_{\vec{k}} E_{\vec{k}'} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \Big\{ \chi_{\vec{k}} \delta^{(D)}(\vec{k}+\vec{k}') \cdot e^{-iE_{\vec{k}}t} e^{-iE_{\vec{k}'}t} - \chi^*_{\vec{k}} \delta^{(D)}(\vec{k}+\vec{k}') \cdot e^{+iE_{\vec{k}}t} e^{+iE_{\vec{k}'}t}$$
$$- \gamma_{\vec{k}} \delta^{(D)}(\vec{k}-\vec{k}') \cdot e^{-iE_{\vec{k}}t} e^{+iE_{\vec{k}'}t} + \gamma^*_{\vec{k}} \delta^{(D)}(\vec{k}-\vec{k}') \cdot e^{+iE_{\vec{k}}t} e^{-iE_{\vec{k}'}t} \Big\} \tag{2.1.49}$$
$$= -\int_{\vec{k},\vec{k}'} E_{\vec{k}} E_{\vec{k}'} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \Big\{ 2\mathrm{Im}\left(\chi_{\vec{k}} \cdot e^{-2iE_{\vec{k}}t}\right) \cdot \delta^{(D)}(\vec{k}+\vec{k}') - 2\mathrm{Im}(\gamma_{\vec{k}}) \delta^{(D)}(\vec{k}-\vec{k}') \Big\}. \tag{2.1.50}$$

To ensure time-independence of the commutation relations, we see that $\chi_{\vec{k}} = 0$. Next, Im $\gamma_{\vec{k}}$ must be real in order for the equal time $[\varphi, \varphi]$ and $[\Pi, \Pi]$ to be zero. In fact, $\gamma_{\vec{k}} = (2\pi)^D$ in order for the $\vec{k}'-$integration in the equal-time $[\varphi, \Pi]$ integral to yield 1. Altogether:

$$\left[a_{\vec{k}}, a_{\vec{k}'}\right] = 0 = \left[a^\dagger_{\vec{k}}, a^\dagger_{\vec{k}'}\right]; \tag{2.1.51}$$

and

$$\left[a_{\vec{k}}, a^\dagger_{\vec{k}'}\right] = (2\pi)^D \cdot \delta^{(D)}(\vec{k}-\vec{k}'). \tag{2.1.52}$$

These relations are the generalization of the SHO's ladder operators – i.e., every single $\vec{k}-$mode of the massive linear scalar field *is* a quantum SHO.

Problem 2.3. Show that the ladder operators may be extracted from the field operator φ at $t = 0$ as follows

$$a_{\vec{k}} = \int_{\mathbb{R}^D} \mathrm{d}^D \vec{x} e^{-i\vec{k}\cdot\vec{x}} \left(\sqrt{\frac{E_{\vec{k}}}{2}} \varphi[t=0,\vec{x}] + \frac{i}{\sqrt{2E_{\vec{k}}}} \partial_t \varphi[t=0,\vec{x}] \right), \tag{2.1.53}$$
$$a^\dagger_{\vec{k}} = \int_{\mathbb{R}^D} \mathrm{d}^D \vec{x} e^{i\vec{k}\cdot\vec{x}} \left(\sqrt{\frac{E_{\vec{k}}}{2}} \varphi[t=0,\vec{x}] - \frac{i}{\sqrt{2E_{\vec{k}}}} \partial_t \varphi[t=0,\vec{x}] \right). \tag{2.1.54}$$

□

Problem 2.4. Show that

$$\left[\varphi(t,\vec{x}), a^\dagger_{\vec{q}}\right] = \frac{1}{\sqrt{2E_{\vec{q}}}} e^{-iE_{\vec{q}}t} e^{+i\vec{q}\cdot\vec{x}}, \tag{2.1.55}$$
$$[\varphi(t,\vec{x}), a_{\vec{q}}] = -\frac{1}{\sqrt{2E_{\vec{q}}}} e^{+iE_{\vec{q}}t} e^{-i\vec{q}\cdot\vec{x}}. \tag{2.1.56}$$

Explain why the mode functions $\{\exp(\mp(iE_{\vec{q}} - i\vec{q}\cdot\vec{x}))\}$ are really the matrix element of the field operator with respect to the one particle state and the vacuum:

$$\langle \vec{q} \,|\varphi(t,\vec{x})|\, 0\rangle = e^{iE_{\vec{q}}t} e^{-i\vec{q}\cdot\vec{x}}, \tag{2.1.57}$$
$$\langle 0 \,|\varphi(t,\vec{x})|\, \vec{q}\rangle = e^{-iE_{\vec{q}}t} e^{+i\vec{q}\cdot\vec{x}}. \tag{2.1.58}$$

□

Hamiltonian: Fourier Decomposition We now compute the total Hamiltonian of the massive linear scalar field:

$$H(t) \equiv \int_{\mathbb{R}^D} \mathrm{d}^D \vec{x}' \left(\frac{1}{2}\Pi^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{m^2}{2}\varphi^2 \right) \tag{2.1.59}$$

$$= \int_{\mathbb{R}^D} \frac{\mathrm{d}^D \vec{k}}{(2\pi)^D} \frac{1}{2} \left(|\widetilde{\Pi}(t,\vec{k})|^2 + E_{\vec{k}}^2 |\widetilde{\varphi}(t,\vec{k})|^2 \right). \tag{2.1.60}$$

We have

$$\varphi(t,\vec{x}) = \int_{\mathbb{R}^D} \frac{\mathrm{d}^D \vec{k}}{(2\pi)^D} \frac{1}{\sqrt{2E_{\vec{k}}}} \left(a_{\vec{k}} e^{-iE_{\vec{k}}t} + a^\dagger_{-\vec{k}} e^{iE_{\vec{k}}t} \right) e^{i\vec{k}\cdot\vec{x}}, \tag{2.1.61}$$

$$\Pi(t,\vec{x}) = -i \int_{\mathbb{R}^D} \frac{\mathrm{d}^D \vec{k}}{(2\pi)^D} \sqrt{\frac{E_{\vec{k}}}{2}} \left(a_{\vec{k}} e^{-iE_{\vec{k}}t} - a^\dagger_{-\vec{k}} e^{iE_{\vec{k}}t} \right) e^{i\vec{k}\cdot\vec{x}}. \tag{2.1.62}$$

Therefore,

$$|\widetilde{\varphi}(t,\vec{k})|^2 = \widetilde{\varphi}(t,\vec{k})\widetilde{\varphi}(t,-\vec{k}) \tag{2.1.63}$$

$$= \frac{1}{2E_{\vec{k}}} \left(a_{\vec{k}} e^{-iE_{\vec{k}}t} + a^\dagger_{-\vec{k}} e^{iE_{\vec{k}}t} \right) \left(a_{-\vec{k}} e^{-iE_{\vec{k}}t} + a^\dagger_{\vec{k}} e^{iE_{\vec{k}}t} \right) \tag{2.1.64}$$

$$= \frac{1}{2E_{\vec{k}}} \left(a_{\vec{k}} a_{-\vec{k}} e^{-2iE_{\vec{k}}t} + a^\dagger_{-\vec{k}} a_{-\vec{k}} + a_{\vec{k}} a^\dagger_{\vec{k}} + a^\dagger_{-\vec{k}} a^\dagger_{\vec{k}} e^{2iE_{\vec{k}}t} \right) \tag{2.1.65}$$

$$E_{\vec{k}}^2 |\widetilde{\varphi}(t,\vec{k})|^2 = \frac{E_{\vec{k}}}{2} \left(a_{\vec{k}} a_{-\vec{k}} e^{-2iE_{\vec{k}}t} + a^\dagger_{-\vec{k}} a^\dagger_{\vec{k}} e^{2iE_{\vec{k}}t} + a^\dagger_{-\vec{k}} a_{-\vec{k}} + a^\dagger_{\vec{k}} a_{\vec{k}} + (2\pi)^D \delta^{(D)}(\vec{0}) \right), \tag{2.1.66}$$

where we have massaged $a_{\vec{k}} a^\dagger_{\vec{k}} = [a_{\vec{k}}, a^\dagger_{\vec{k}}] + a^\dagger_{\vec{k}} a_{\vec{k}}$. Similarly,

$$|\widetilde{\Pi}(t,\vec{k})|^2 = \widetilde{\Pi}(t,\vec{k})\widetilde{\Pi}(t,-\vec{k}) \tag{2.1.67}$$

$$= -\frac{E_{\vec{k}}}{2} \left(a_{\vec{k}} e^{-iE_{\vec{k}}t} - a^\dagger_{-\vec{k}} e^{iE_{\vec{k}}t} \right) \left(a_{-\vec{k}} e^{-iE_{\vec{k}}t} - a^\dagger_{\vec{k}} e^{iE_{\vec{k}}t} \right) \tag{2.1.68}$$

$$= -\frac{E_{\vec{k}}}{2} \left(a_{\vec{k}} a_{-\vec{k}} e^{-2iE_{\vec{k}}t} - a^\dagger_{-\vec{k}} a_{-\vec{k}} - a_{\vec{k}} a^\dagger_{\vec{k}} + a^\dagger_{-\vec{k}} a^\dagger_{\vec{k}} e^{2iE_{\vec{k}}t} \right) \tag{2.1.69}$$

$$= \frac{E_{\vec{k}}}{2} \left(-a_{\vec{k}} a_{-\vec{k}} e^{-2iE_{\vec{k}}t} - a^\dagger_{-\vec{k}} a^\dagger_{\vec{k}} e^{2iE_{\vec{k}}t} + a^\dagger_{-\vec{k}} a_{-\vec{k}} + a^\dagger_{\vec{k}} a_{\vec{k}} + (2\pi)^D \delta^{(D)}(\vec{0}) \right), \tag{2.1.70}$$

We collect

$$H = \int_{\mathbb{R}^D} \frac{\mathrm{d}^D \vec{k}}{(2\pi)^D} \frac{E_{\vec{k}}}{2} \left(a^\dagger_{-\vec{k}} a_{-\vec{k}} + a^\dagger_{\vec{k}} a_{\vec{k}} + (2\pi)^D \delta^{(D)}(\vec{0}) \right) \tag{2.1.71}$$

$$= \int_{\mathbb{R}^D} \frac{\mathrm{d}^D \vec{k}}{(2\pi)^D} E_{\vec{k}} \left(a^\dagger_{\vec{k}} a_{\vec{k}} + \frac{1}{2} \cdot (2\pi)^D \delta^{(D)}(\vec{0}) \right). \tag{2.1.72}$$

The momentum $\delta-$function may be interpreted as the total volume of space because

$$(2\pi)^D \delta^{(D)}(\vec{0}) = \lim_{\vec{k}\to\vec{0}} \int_{\mathbb{R}^D} \mathrm{d}^D \vec{x} e^{i\vec{k}\cdot\vec{x}} = \int_{\mathbb{R}^D} \mathrm{d}^D \vec{x}. \tag{2.1.73}$$

In the 1D QM SHO case, $H = \Omega(a^\dagger a + 1/2)$. We may therefore recognize the Hamiltonian for the massive linear scalar field to be simply that of the QM SHO, but summed over all $\vec{k}-$modes. On physical grounds, we expect that H may be appropriately regularized to become H':

$$H' = \int_{\mathbb{R}^D} \frac{\mathrm{d}^D \vec{k}}{(2\pi)^D} E_{\vec{k}} \left(a_{\vec{k}}^\dagger a_{\vec{k}} \right) + E_0, \tag{2.1.74}$$

for some well-defined ground state. As long as we ignore gravitational effects, also remember that physics is not sensitive to absolute energy scales; but only to *energy differences*. Hence, the choice of E_0 ought to be arbitrary.

Problem 2.5. Show that

$$[H, a_{\vec{q}}] = -E_{\vec{q}} a_{\vec{q}} \tag{2.1.75}$$

and

$$\left[H, a_{\vec{q}}^\dagger \right] = E_{\vec{q}} a_{\vec{q}}^\dagger. \tag{2.1.76}$$

These lead us to the particle interpretation. □

Particle Interpretation If $|\mathcal{E}\rangle$ is an eigenstate of H, consider

$$\begin{aligned} H(a_{\vec{q}} |\mathcal{E}\rangle) &= ([H, a_{\vec{q}}] + a_{\vec{q}} H) |\mathcal{E}\rangle & (2.1.77) \\ &= (\mathcal{E} - E_{\vec{q}}) a_{\vec{q}} |\mathcal{E}\rangle . & (2.1.78) \end{aligned}$$

Likewise, consider

$$\begin{aligned} H(a_{\vec{q}}^\dagger |\mathcal{E}\rangle) &= ([H, a_{\vec{q}}^\dagger] + a_{\vec{q}}^\dagger H) |\mathcal{E}\rangle & (2.1.79) \\ &= (\mathcal{E} + E_{\vec{q}}) a_{\vec{q}}^\dagger |\mathcal{E}\rangle . & (2.1.80) \end{aligned}$$

Therefore the $a_{\vec{q}}$ and $a_{\vec{q}}^\dagger$ are, respectively, the lowering and raising operators – i.e., they lower and raise energy by $E_{\vec{q}}$. Now, the $E_{\vec{k}} a_{\vec{k}}^\dagger a_{\vec{k}}$ portion of H yields a non-negative positive expectation value with respect to any state; hence, it appears reasonable to expect, there must be a minimum energy (aka ground) state with respect to an appropriately regularized H', with a well defined value E_0; where $H' |0\rangle = E_0 |0\rangle$. This in turn means $a_{\vec{q}} |0\rangle$ must be the zero vector, for otherwise there is now a non-trivial state with lower energy, violating the minimum energy assumption.

$$a_{\vec{q}} |0\rangle = |\text{zero}\rangle \tag{2.1.81}$$

Let us now use this ground state (aka vacuum state) to define the one particle state

$$\left| \vec{k} \right\rangle \equiv \sqrt{2E_{\vec{k}}} a_{\vec{k}}^\dagger |0\rangle . \tag{2.1.82}$$

The normalization is

$$\begin{aligned} \left\langle \vec{k} \middle| \vec{k}' \right\rangle &\equiv \sqrt{2E_{\vec{k}}} \sqrt{2E'_{\vec{k}}} \langle 0| a_{\vec{k}} a_{\vec{k}'}^\dagger |0\rangle & (2.1.83) \\ &= \sqrt{2E_{\vec{k}}} \sqrt{2E'_{\vec{k}}} \langle 0| \left([a_{\vec{k}}, a_{\vec{k}'}^\dagger] + a_{\vec{k}'}^\dagger a_{\vec{k}} \right) |0\rangle & (2.1.84) \\ &= (2E_{\vec{k}})(2\pi)^D \delta^{(D)}(\vec{k} - \vec{k}'). & (2.1.85) \end{aligned}$$

Despite appearances, this inner product is, in fact, invariant under Lorentz transformations. For, we may obtain the same result via the following manifestly Lorentz invariant positive energy integration measure:

$$\frac{\mathrm{d}^d k}{(2\pi)^d}\Theta(k_0)(2\pi)\delta\left(k_\alpha k^\alpha - m^2\right) = \frac{\mathrm{d}k_0 \mathrm{d}^D\vec{k}}{(2\pi)^D}\Theta(k_0)\left(\frac{\delta(k_0 - E_{\vec{k}})}{2E_{\vec{k}}} + \frac{\delta(k_0 + E_{\vec{k}})}{2E_{\vec{k}}}\right) \tag{2.1.86}$$

$$= \frac{\mathrm{d}k_0 \mathrm{d}^D\vec{k}}{(2\pi)^D}\Theta(k_0)\frac{\delta(k_0 - E_{\vec{k}})}{2E_{\vec{k}}}. \tag{2.1.87}$$

Hence we obtain the relativistically invariant result

$$\int_{\mathbb{R}^{D+1}} \frac{\mathrm{d}^d k}{(2\pi)^D}\Theta(k_0)\delta(k^2 - m^2)\left\langle \vec{k}' \middle| \vec{k} \right\rangle = \int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D(2E_{\vec{k}})}\left\langle \vec{k}' \middle| \vec{k} \right\rangle = 1. \tag{2.1.88}$$

Next, we turn to defining the $n-$particle state

$$\left|\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n\right\rangle \equiv \frac{1}{\sqrt{n!}}\sqrt{2E_{\vec{k}_1}}a^\dagger_{\vec{k}_1}\sqrt{2E_{\vec{k}_2}}a^\dagger_{\vec{k}_2}\dots\sqrt{2E_{\vec{k}_n}}a^\dagger_{\vec{k}_n}\left|0\right\rangle. \tag{2.1.89}$$

Its normalization is

$$\left\langle \vec{k}_1, \vec{k}_2, \dots, \vec{k}_n \middle| \vec{k}'_1, \vec{k}'_2, \dots, \vec{k}'_n \right\rangle = \frac{1}{n!}(2E_{\vec{k}_1})\dots(2E_{\vec{k}_n}) \tag{2.1.90}$$
$$\times \sum_{(\pi(1),\dots,\pi(n))\in\text{perm.}(1,2,\dots,n)} (2\pi)^D\delta^{(D)}\left(\vec{k}_1 - \vec{k}'_{\pi(1)}\right)\dots(2\pi)^D\delta^{(D)}\left(\vec{k}_n - \vec{k}'_{\pi(n)}\right).$$

This is relativistically invariant because

$$\prod_{\mathrm{J}=1}^{n}\left(\int_{\mathbb{R}^D}\frac{\mathrm{d}^D\vec{k}_{\mathrm{J}}}{(2\pi)^D(2E_{\vec{k}_{\mathrm{J}}})}\right)\left\langle \vec{k}_1, \vec{k}_2, \dots, \vec{k}_n \middle| \vec{k}'_1, \vec{k}'_2, \dots, \vec{k}'_n \right\rangle = 1. \tag{2.1.91}$$

The identity operator acting on one-particle states is

$$\mathbb{I}_{\text{1-particle}} = \int_{\mathbb{R}^D}\frac{\mathrm{d}^D\vec{k}}{(2\pi)^D(2E_{\vec{k}})}\left|\vec{k}\right\rangle\left\langle\vec{k}\right|. \tag{2.1.92}$$

As you will directly verify below, the set of $n-$particle momentum states diagonalizes the Hamiltonian H'. Hence, the collection of $|0\rangle$, $1-$, $2-$, and all $(n \geq 1)-$particle states span the Hilbert space upon each H' acts. This space is known as the *Fock space*.

Problem 2.6. Prove eq. (2.1.90). You may need to first prove

$$\left[a_{\vec{q}}, a^\dagger_{\vec{k}_1}a^\dagger_{\vec{k}_2}\dots a^\dagger_{\vec{k}_n}\right] = \sum_{\ell=1}^{n}(2\pi)^D\delta^{(D)}[\vec{q} - \vec{k}_\ell]a^\dagger_{\vec{k}_1}\dots a^\dagger_{\vec{k}_{\ell-1}}a^\dagger_{\vec{k}_{\ell+1}}\dots a^\dagger_{\vec{k}_n}, \tag{2.1.93}$$

where the product of $a^\dagger$s has the $\ell-$th term omitted; try mathematical induction. □

Problem 2.7. Show that, if H' is the some appropriately regularized H (see eq. (2.1.74)):

$$H' \left|\vec{k}_1, \dots, \vec{k}_n\right\rangle = \left(E_0 + E_{\vec{k}_1} + \dots + E_{\vec{k}_n}\right) \left|\vec{k}_1, \dots, \vec{k}_n\right\rangle, \tag{2.1.94}$$

where the $n-$particle state is defined in eq. (2.1.89). □

Just as the number operator for the SHO is $a^\dagger a$, the corresponding Hermitian number operator for the scalar field theory involves the sum over all momentum.

$$N \equiv \int \frac{\mathrm{d}^{d-1}\vec{q}}{(2\pi)^{d-1}} a_{\vec{q}}^\dagger a_{\vec{q}} \tag{2.1.95}$$

$$N |0\rangle = 0 \tag{2.1.96}$$

The commutation relation with $a_{\vec{k}}$ is

$$[N, a_{\vec{k}}] = \int \frac{\mathrm{d}^D\vec{q}}{(2\pi)^D} \left(\left[a_{\vec{q}}^\dagger, a_{\vec{k}}\right] a_{\vec{q}} + a_{\vec{q}}^\dagger \left[a_{\vec{q}}, a_{\vec{k}}\right]\right) \tag{2.1.97}$$

$$= -a_{\vec{k}}. \tag{2.1.98}$$

Taking the dagger on both sides,

$$\left[N, a_{\vec{k}}^\dagger\right] = a_{\vec{k}}^\dagger. \tag{2.1.99}$$

Therefore

$$N a_{\vec{k}}^\dagger |0\rangle = \left(\left[N, a_{\vec{k}}^\dagger\right] + a_{\vec{k}}^\dagger N\right) |0\rangle \tag{2.1.100}$$

$$= a_{\vec{k}}^\dagger |0\rangle. \tag{2.1.101}$$

Problem 2.8. Particle Number Verify that

$$N \left|\vec{k}_1, \dots, \vec{k}_n\right\rangle = n \left|\vec{k}_1, \dots, \vec{k}_n\right\rangle, \tag{2.1.102}$$

where the $n-$particle state is defined in eq. (2.1.89).

Explain why the $n-$particle momentum state is orthogonal to the $n'-$particle momentum state whenever $n \neq n'$. □

Problem 2.9. Conservation of particle number Explain why, for the linear massive scalar theory, particle number is conserved. (Hint: Show that $[H, N] = 0$.) This implies interactions – nonlinearities in the equations-of-motion – are needed for particle number to change. □

Problem 2.10. Completeness in Fock Space Explain why the identity operator acting on the full Hilbert space of $(n \geq 0)-$particle states is

$$\mathbb{I} = |0\rangle\langle 0| + \int \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D(2E_{\vec{k}})} \left|\vec{k}\right\rangle\left\langle\vec{k}\right| + \int \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D(2E_{\vec{k}})} \int \frac{\mathrm{d}^D\vec{k}'}{(2\pi)^D(2E_{\vec{k}'})} \left|\vec{k}, \vec{k}'\right\rangle\left\langle\vec{k}, \vec{k}'\right|$$
$$+ \int \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D(2E_{\vec{k}})} \int \frac{\mathrm{d}^D\vec{k}'}{(2\pi)^D(2E_{\vec{k}'})} \int \frac{\mathrm{d}^D\vec{k}''}{(2\pi)^D(2E_{\vec{k}''})} \left|\vec{k}, \vec{k}', \vec{k}''\right\rangle\left\langle\vec{k}, \vec{k}', \vec{k}''\right| + \dots \tag{2.1.103}$$

$$= |0\rangle\langle 0| + \sum_{\mathrm{J}=1}^{+\infty} \left(\prod_{\mathrm{I}=1}^{\mathrm{J}} \int \frac{\mathrm{d}^D\vec{k}_{\mathrm{I}}}{(2\pi)^D(2E_{\vec{k}_{\mathrm{I}}})}\right) \left|\vec{k}_1, \dots, \vec{k}_{\mathrm{J}}\right\rangle\left\langle\vec{k}_1, \dots, \vec{k}_{\mathrm{J}}\right|. \tag{2.1.104}$$

Hint: The orthogonality of the $n-$particle states is important here. □

Problem 2.11. Ladder Operators in Fock Space Explain why the ladder operators can be expressed as

$$a^\dagger_{\vec{q}} = \frac{1}{\sqrt{2E_{\vec{q}}}} \left(|\vec{q}\rangle \langle 0| + \sum_{\mathrm{J}=1}^{+\infty} \left(\prod_{\mathrm{I}=1}^{\mathrm{J}} \int \frac{\mathrm{d}^D \vec{k}_\mathrm{I}}{(2\pi)^D (2E_{\vec{k}_\mathrm{I}})} \right) \left| \vec{q}, \vec{k}_1, \dots, \vec{k}_\mathrm{J} \right\rangle \left\langle \vec{k}_1, \dots, \vec{k}_\mathrm{J} \right| \right) \tag{2.1.105}$$

and

$$a_{\vec{q}} = \frac{1}{\sqrt{2E_{\vec{q}}}} \left(|0\rangle \langle \vec{q}| + \sum_{\mathrm{J}=1}^{+\infty} \left(\prod_{\mathrm{I}=1}^{\mathrm{J}} \int \frac{\mathrm{d}^D \vec{k}_\mathrm{I}}{(2\pi)^D (2E_{\vec{k}_\mathrm{I}})} \right) \left| \vec{k}_1, \dots, \vec{k}_\mathrm{J} \right\rangle \left\langle \vec{q}, \vec{k}_1, \dots, \vec{k}_\mathrm{J} \right| \right). \tag{2.1.106}$$

Since the quantum scalar field φ is itself built out of a and $a^\dagger$, these relations provide an explicit demonstration that QFT *is* the quantum mechanics of variable $(n \geq 0)$−body physics. □

Problem 2.12. Define the momentum operator as

$$P^\mu \equiv \int \frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}} k^\mu a^\dagger_{\vec{k}} a_{\vec{k}}, \tag{2.1.107}$$

$$k^\mu = \left(\sqrt{\vec{k}^2 + m^2}, \vec{k} \right). \tag{2.1.108}$$

Explain why

$$P^\mu \left| \vec{k}_1 \dots \vec{k}_n \right\rangle = (k_1^\mu + \dots + k_n^\mu) \left| \vec{k}_1 \dots \vec{k}_n \right\rangle. \tag{2.1.109}$$

Demonstrate that P^μ is in fact the spatial integral of the zeroth Noether current ${T^\mu}_0$ derived in eq. (1.3.10), with $V = (m^2/2)\varphi^2$, up to an additive constant:

$$P^\mu \equiv \int \mathrm{d}^D \vec{x} T^{\mu 0}. \tag{2.1.110}$$

(This integral should be time-independent because of the conservation of the stress tensor – can you see why?) Finally, show that

$$i \left[P_\mu, \varphi(t, \vec{x}) \right] = \partial_\mu \varphi \tag{2.1.111}$$

and explain why that means

$$e^{+iP_\mu x'^\mu} \varphi(x) e^{-iP_\mu x'^\mu} = \varphi(x + x'). \tag{2.1.112}$$

Hint: This is related to spacetime translations. □

Wave Packets Now that we know how to construct the n particle momentum eigenstate, we may proceed to construct arbitrary wave packets. For instance, the general one particle wave packet is

$$|\psi\rangle \equiv \int \frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{\widetilde{\psi}(\vec{k})}{\sqrt{2E_{\vec{k}}}} \left| \vec{k} \right\rangle. \tag{2.1.113}$$

To ensure it is normalized to unity, we require

$$\langle\psi|\,\psi\rangle = 1 = \int \frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}} \int \frac{\mathrm{d}^{d-1}\vec{k}'}{(2\pi)^{d-1}} \frac{\widetilde{\psi}(\vec{k}')^*}{\sqrt{2E_{\vec{k}'}}} \frac{\widetilde{\psi}(\vec{k})}{\sqrt{2E_{\vec{k}}}} \left\langle \vec{k}' \middle| \vec{k} \right\rangle \tag{2.1.114}$$

$$= \int \frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}} |\widetilde{\psi}(\vec{k})|^2. \tag{2.1.115}$$

The general n particle wave packets state is

$$|\psi_1 \dots \psi_n\rangle \equiv \frac{1}{\sqrt{n!}} \int \frac{\mathrm{d}^{d-1}\vec{k}_1}{(2\pi)^{d-1}} \dots \int \frac{\mathrm{d}^{d-1}\vec{k}_n}{(2\pi)^{d-1}} \frac{\widetilde{\psi}_1(\vec{k}_1)}{\sqrt{2E_{\vec{k}_1}}} \dots \frac{\widetilde{\psi}_n(\vec{k}_n)}{\sqrt{2E_{\vec{k}_n}}} \left| \vec{k}_1, \dots, \vec{k}_n \right\rangle. \tag{2.1.116}$$

2.1.3 Time Evolution; Lorentz Covariance; Schrödinger Equation

Time Evolution Since the Hamiltonian for our massive linear scalar field is time independent, the time evolution operator is simply the usual

$$U(t) = \exp\left(-iH' \cdot t\right), \tag{2.1.117}$$

where we have assumed the initial conditions are specified at $t = 0$. For our wave packets above, its state in the future $t > 0$ is given by

$$|\psi_1 \dots \psi_n; t\rangle$$

$$= U(t)\,|\psi_1 \dots \psi_n\rangle \tag{2.1.118}$$

$$= \frac{1}{\sqrt{n!}} \int \frac{\mathrm{d}^D\vec{k}_1}{(2\pi)^D} \dots \int \frac{\mathrm{d}^D\vec{k}_n}{(2\pi)^D} \frac{\widetilde{\psi}_1(\vec{k}_1)}{\sqrt{2E_{\vec{k}_1}}} \dots \frac{\widetilde{\psi}_n(\vec{k}_n)}{\sqrt{2E_{\vec{k}_n}}} e^{-it\left(E_0 + E_{\vec{k}_1} + \dots + E_{\vec{k}_n}\right)} \left| \vec{k}_1, \dots, \vec{k}_n \right\rangle. \tag{2.1.119}$$

Furthermore, let us recognize from eq. (2.1.74) that

$$U(t) = \exp\left[-it\left(E_0 + \int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D} E_{\vec{k}} a^\dagger_{\vec{k}} a_{\vec{k}}\right)\right] \tag{2.1.120}$$

$$\sim e^{-itE_0} \prod_{\vec{k}} \exp\left[-it \cdot E_{\vec{k}} a^\dagger_{\vec{k}} a_{\vec{k}}\right]. \tag{2.1.121}$$

That is, the evolution operator acts on each momentum-mode separately. This also strongly suggests that the vacuum state may be constructed by multiplying the ground state of the SHO of each and every $\vec{k}-$mode.

Problem 2.13. N Body Non-Relativistic Schrödinger Equation Show that the $n-$particle momentum eigenstate

$$\left| \vec{k}_1, \dots, \vec{k}_n; t \right\rangle \equiv U(t) \left| \vec{k}_1, \dots, \vec{k}_n \right\rangle \tag{2.1.122}$$

obeys the $n-$particle non-relativistic Schrödinger equation

$$i\partial_t \left| \vec{k}_1, \dots, \vec{k}_n; t \right\rangle \approx \left(E_0 + n \cdot m + \sum_{\mathrm{J}=1}^{n} \frac{\vec{k}_{\mathrm{J}}^2}{2m}\right) \left| \vec{k}_1, \dots, \vec{k}_n; t \right\rangle \tag{2.1.123}$$

whenever $\vec{k}_{\mathrm{J}}^2/m^2 \ll 1$. This provides an explicit and simple example of how the Hamiltonian of a quantum *field* theory incorporates the dynamics of more than one body of the same 'species'. □

Bose Statistics Let us observe that relativistic QFT has led us to *Bose-Einstein* statistics for spin-0 particles through the $n-$particle state in eq. (2.1.89). Namely, because the $a^\dagger$s commute among themselves in eq. (2.1.89), we may swap the momenta in the n particle states, and arrive at the same state – i.e., these are n indistinguishable particles:

$$\left|\vec{k}_1 \dots \vec{k}_n\right\rangle = \left|\vec{k}_{\pi(1)} \dots \vec{k}_{\pi(n)}\right\rangle, \tag{2.1.124}$$

where $\pi(s)$, for $s = 1, 2, \dots, n$, simply denotes the permutation of the set $\{1, 2, \dots, n\}$. And since this indistinguishable n particle state holds for the momentum eigenstate, we may superpose them over arbitrary momenta weighted by the $\widetilde{\psi}_i$s. This tells us

$$|\psi_1 \dots \psi_n; t\rangle = \left|\psi_{\pi(1)} \dots \psi_{\pi(n)}; t\right\rangle. \tag{2.1.125}$$

Lorentz Boosts If Λ denotes a Lorentz transformation matrix, let us define the Lorentz transformed momentum vector as

$$k'_\alpha \equiv \Lambda_\alpha{}^\beta k_\beta. \tag{2.1.126}$$

Applying the operator

$$D_{1p}(\Lambda) \equiv \int \frac{\mathrm{d}^D \vec{q}}{(2\pi)^D (2E_{\vec{q}})} |\vec{q}'\rangle \langle\vec{q}| \tag{2.1.127}$$

to an arbitrary $1-$particle momentum state, we discover it boosts the eigenstate with momentum $\vec{k}$ to that with momentum $\vec{k}'$:

$$D_{1p}(\Lambda)\left|\vec{k}\right\rangle = \int \frac{\mathrm{d}^D \vec{q}}{(2\pi)^D (2E_{\vec{q}})} |\vec{q}'\rangle \left\langle\vec{q}\middle|\vec{k}\right\rangle \tag{2.1.128}$$

$$= \int \frac{\mathrm{d}^D \vec{q}}{(2\pi)^D (2E_{\vec{q}})} |\vec{q}'\rangle (2\pi)^D (2E_{\vec{q}}) \delta^{(D)}(\vec{q} - \vec{k}) \tag{2.1.129}$$

$$= \left|\vec{k}'\right\rangle. \tag{2.1.130}$$

Problem 2.14. Check that $D_{1p}(\Lambda)$ in eq. (2.1.127) is unitary. The set of $\{D_{1p}(\Lambda)\}$ forms an infinite dimensional unitary representation of the Lorentz group – can you explain why? □

Now, we consider

$$D(\Lambda)\left|\vec{k}\right\rangle = \left|\vec{k}'\right\rangle \tag{2.1.131}$$

$$\sqrt{2E_{\vec{k}}} D(\Lambda) a^\dagger_{\vec{k}} D(\Lambda)^\dagger D(\Lambda) |0\rangle = \sqrt{2E_{\vec{k}'}} a^\dagger_{\vec{k}'} |0\rangle. \tag{2.1.132}$$

Provided the vacuum is Lorentz invariant – i.e., the ground state$|0\rangle$ should be the same, regardless of the choice of inertial frame – we have $D(\Lambda)|0\rangle = |0\rangle$ and

$$D(\Lambda) a^\dagger_{\vec{k}} D(\Lambda)^\dagger = \frac{\sqrt{2E_{\vec{k}'}}}{\sqrt{2E_{\vec{k}}}} a^\dagger_{\vec{k}'}. \tag{2.1.133}$$

Taking the $\dagger$ on both sides,

$$D(\Lambda) a_{\vec{k}} D(\Lambda)^\dagger = \frac{\sqrt{2E_{\vec{k}'}}}{\sqrt{2E_{\vec{k}}}} a_{\vec{k}'}. \tag{2.1.134}$$

Problem 2.15. Lorentz Transformation on $n-$particle states Use eq. (2.1.133) to show that the $n-$particle state is Lorentz-transformed by $D(\Lambda)$; i.e.,

$$D(\Lambda)\left|\vec{k}_1,\dots,\vec{k}_n\right\rangle = \left|\vec{k}'_1,\dots,\vec{k}'_n\right\rangle; \tag{2.1.135}$$

where $(k'_i)_\alpha = \Lambda_\alpha{}^\beta(k_i)_\beta$ for $i = 1, 2, \dots, n$. □

Problem 2.16. Lorentz Transformation on Fock space What is the generalization of the Lorentz transformation in eq. (2.1.127), but acting on the entire Fock space? □

Problem 2.17. Lorentz Transformation of Scalar Field Show that

$$D(\Lambda_\alpha{}^\beta)\varphi(x^\mu)D(\Lambda_\alpha{}^\beta)^\dagger = \varphi(x'^\mu = \Lambda^\mu{}_\nu x^\mu). \tag{2.1.136}$$

The operators on the LHS implement Lorentz transformation on momenta via eq. (2.1.126); whereas, on the RHS, the inverse Lorentz transformation is applied to the Cartesian coordinates. Hence, note the placement of the indices on the Λ; those on the LHS are not the same as those on the RHS. This is the operator statement that the quantum scalar field does indeed transform as a scalar under Lorentz transformations. □

Field Basis The time dependent scalar field $\varphi(t,\vec{x})$ was taken to be the operator in the Heisenberg picture. If we choose, for convenience, $t = 0$ to be where the Schrödinger and Heisenberg picture coincide, then

$$\varphi_{\rm S}(\vec{x}) \equiv \varphi(t=0,\vec{x}) \tag{2.1.137}$$

is a Hermitian Schrödinger (time-independent) operator, whose eigenstates $\{|\Phi\rangle\}$ form a basis analagous to the position basis $\{|\vec{x}\rangle\}$ in quantum mechanics:

$$\varphi_{\rm S}(\vec{x})\,|\Phi\rangle = \Phi(\vec{x})\,|\Phi\rangle\,. \tag{2.1.138}$$

Moreover, we may write

$$\varphi_{\rm S}(\vec{x}) = \int_{\mathbb{R}^D}\frac{{\rm d}^D\vec{k}}{(2\pi)^D}\left(\frac{a_{\vec{k}}+a^\dagger_{-\vec{k}}}{\sqrt{2E_{\vec{k}}}}\right)e^{i\vec{k}\cdot\vec{x}} \equiv \int_{\mathbb{R}^D}\frac{{\rm d}^D\vec{k}}{(2\pi)^D}\widetilde{\varphi}_{\rm S}(\vec{k})e^{i\vec{k}\cdot\vec{x}} \tag{2.1.139}$$

where

$$\widetilde{\varphi}_{\rm S}(\vec{k})^\dagger = \widetilde{\varphi}_{\rm S}(-\vec{k}). \tag{2.1.140}$$

Likewise, the Schrödinger picture momentum operator is

$$\Pi_{\rm S}(\vec{x}) \equiv \dot{\varphi}(t=0,\vec{x}) = \int_{\mathbb{R}^D}\frac{{\rm d}^D\vec{k}}{(2\pi)^D}\sqrt{\frac{E_{\vec{k}}}{2}}\left(-ia_{\vec{k}}+ia^\dagger_{-\vec{k}}\right)e^{i\vec{k}\cdot\vec{x}} \tag{2.1.141}$$

$$\equiv \int_{\mathbb{R}^D}\frac{{\rm d}^D\vec{k}}{(2\pi)^D}\widetilde{\Pi}_{\rm S}(\vec{k})e^{i\vec{k}\cdot\vec{x}}, \tag{2.1.142}$$

where

$$\widetilde{\Pi}_{\rm S}(\vec{k})^\dagger = \widetilde{\Pi}_{\rm S}(-\vec{k}). \tag{2.1.143}$$

Problem 2.18. Explain why the position spacetime equal time commutation relation translates to

$$\left[\widetilde{\varphi}_{\text{S}}(\vec{k}), \widetilde{\Pi}_{\text{S}}(\vec{k}')^\dagger\right] = (2\pi)^D i\delta^{(D)}(\vec{k}-\vec{k}'), \tag{2.1.144}$$

$$\left[\widetilde{\varphi}_{\text{S}}(\vec{k})^\dagger, \widetilde{\Pi}_{\text{S}}(\vec{k}')\right] = (2\pi)^D i\delta^{(D)}(\vec{k}-\vec{k}'); \tag{2.1.145}$$

and

$$\left[\widetilde{\varphi}_{\text{S}}(\vec{k}), \widetilde{\varphi}_{\text{S}}(\vec{k}')^\dagger\right] = 0 = \left[\widetilde{\Pi}_{\text{S}}(\vec{k}), \widetilde{\Pi}_{\text{S}}(\vec{k}')^\dagger\right]. \tag{2.1.146}$$

This final relation implies both $\widetilde{\varphi}$ and $\widetilde{\Pi}$ are normal operators; and, hence, it must be possible to diagonalize either $\widetilde{\varphi}(\vec{k})$ or $\widetilde{\Pi}(\vec{k})$; though they are not simultaneously diagonalizable. □

These equal time commutation relations in Fourier space may be implemented in, say, the *Fourier field basis*, defined by the eigenstates $\{|\widetilde{\Psi}\rangle\}$ obeying

$$\widetilde{\varphi}_{\text{S}}(\vec{k})\left|\widetilde{\Psi}\right\rangle = \widetilde{\Psi}(\vec{k})\left|\widetilde{\Psi}\right\rangle, \tag{2.1.147}$$

$$\widetilde{\varphi}_{\text{S}}(\vec{k})^\dagger\left|\widetilde{\Psi}\right\rangle = \widetilde{\Psi}(\vec{k})^*\left|\widetilde{\Psi}\right\rangle. \tag{2.1.148}$$

These are functional eigensystem equations, where the eigenvalue is not a constant, but a function of momentum $\vec{k}$. We also have

$$\left\langle\widetilde{\Psi}\right|\widetilde{\varphi}_{\text{S}}(\vec{k})^\dagger = \left\langle\widetilde{\Psi}\right|\widetilde{\Psi}(\vec{k})^*, \tag{2.1.149}$$

$$\left\langle\widetilde{\Psi}\right|\widetilde{\varphi}_{\text{S}}(\vec{k}) = \left\langle\widetilde{\Psi}\right|\widetilde{\Psi}(\vec{k}). \tag{2.1.150}$$

Suppose we postulate the following functional derivative relations – for arbitrary state $|\Phi\rangle$ –

$$\left\langle\widetilde{\Psi}\left|\widetilde{\Pi}_{\text{S}}(\vec{k})^\dagger\right|\Phi\right\rangle = -i\frac{\delta}{\delta\widetilde{\Psi}(\vec{k})}\left\langle\widetilde{\Psi}\middle|\Phi\right\rangle, \tag{2.1.151}$$

$$\left\langle\widetilde{\Psi}\left|\widetilde{\Pi}_{\text{S}}(\vec{k})\right|\Phi\right\rangle = -i\frac{\delta}{\delta\widetilde{\Psi}(\vec{k})^*}\left\langle\widetilde{\Psi}\middle|\Phi\right\rangle. \tag{2.1.152}$$

Then we may verify,

$$\begin{aligned}\left\langle\widetilde{\Psi}\right|\left[\widetilde{\varphi}_{\text{S}}(\vec{k}), \widetilde{\Pi}_{\text{S}}(\vec{k}')^\dagger\right]|\Phi\rangle &= \left\langle\widetilde{\Psi}\right|\widetilde{\varphi}_{\text{S}}(\vec{k})\widetilde{\Pi}_{\text{S}}(\vec{k}')^\dagger - \widetilde{\Pi}_{\text{S}}(\vec{k}')^\dagger\widetilde{\varphi}_{\text{S}}(\vec{k})|\Phi\rangle \\ &= \widetilde{\Psi}(\vec{k})(-i)\frac{\delta}{\delta\widetilde{\Psi}(\vec{k}')}\left\langle\widetilde{\Psi}\middle|\Phi\right\rangle - (-i)\frac{\delta}{\delta\widetilde{\Psi}(\vec{k}')}\left(\widetilde{\Psi}(\vec{k})\left\langle\widetilde{\Psi}\middle|\Phi\right\rangle\right) \\ &= (2\pi)^D i\delta^{(D)}(\vec{k}-\vec{k}')\cdot\left\langle\widetilde{\Psi}\middle|\Phi\right\rangle,\end{aligned} \tag{2.1.153}$$

where we have employed the functional derivative

$$\frac{\delta\widetilde{\Psi}(\vec{k})^*}{\delta\widetilde{\Psi}(\vec{k}')^*} = (2\pi)^D\delta^{(D)}(\vec{k}-\vec{k}') = \frac{\delta\widetilde{\Psi}(\vec{k})}{\delta\widetilde{\Psi}(\vec{k}')}. \tag{2.1.154}$$

Likewise,

$$\left\langle\widetilde{\Psi}\right| \left[\widetilde{\varphi}_{\mathrm{S}}(\vec{k})^{\dagger}, \widetilde{\Pi}_{\mathrm{S}}(\vec{k}')\right] |\Phi\rangle \tag{2.1.155}$$

$$= \left\langle\widetilde{\Psi}\right| \widetilde{\varphi}_{\mathrm{S}}(\vec{k})^{\dagger}\widetilde{\Pi}_{\mathrm{S}}(\vec{k}') - \widetilde{\Pi}_{\mathrm{S}}(\vec{k}')\widetilde{\varphi}_{\mathrm{S}}(\vec{k})^{\dagger} |\Phi\rangle \tag{2.1.156}$$

$$= \widetilde{\Psi}(\vec{k})^{*}(-i)\frac{\delta}{\delta\widetilde{\Psi}(\vec{k}')^{*}} \left\langle\widetilde{\Psi}\middle| \Phi\right\rangle - (-i)\frac{\delta}{\delta\widetilde{\Psi}(\vec{k}')^{*}} \left(\widetilde{\Psi}(\vec{k})^{*} \left\langle\widetilde{\Psi}\middle| \Phi\right\rangle\right) \tag{2.1.157}$$

$$= (2\pi)^{D} i\delta^{(D)}(\vec{k}-\vec{k}') \cdot \left\langle\widetilde{\Psi}\middle| \Phi\right\rangle. \tag{2.1.158}$$

In this Schrödinger picture, any arbitrary but physical state must obey

$$\left\langle\widetilde{\Psi}\left|i\partial_t\right| \Phi(t)\right\rangle = \left\langle\widetilde{\Psi}\left|H\right| \Phi(t)\right\rangle; \tag{2.1.159}$$

leading us to the functional Schrodinger equation for the massive (real) scalar field

$$i\partial_t \left\langle\widetilde{\Psi}\middle| \Phi(t)\right\rangle = \int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D}\frac{1}{2}\left\langle\widetilde{\Psi}\left|\widetilde{\Pi}(\vec{k})\cdot\widetilde{\Pi}(\vec{k})^{\dagger} + E_{\vec{k}}^2\cdot\widetilde{\varphi}(\vec{k})\cdot\widetilde{\varphi}(\vec{k})^{\dagger}\right|\Phi(t)\right\rangle \tag{2.1.160}$$

$$= \int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D}\frac{1}{2}\left(-\frac{\delta}{\delta\widetilde{\Psi}(\vec{k})^{*}}\frac{\delta}{\delta\widetilde{\Psi}(\vec{k})} + E_{\vec{k}}^2|\widetilde{\Psi}(\vec{k})|^2\right)\left\langle\widetilde{\Psi}\middle| \Phi(t)\right\rangle. \tag{2.1.161}$$

Problem 2.19. Translation in Field Space Let $\widetilde{\Psi}$ be a classical field. Show that

$$\mathcal{T}(\widetilde{\Psi})^{\dagger}\widetilde{\varphi}_{\mathrm{S}}(\vec{k})\mathcal{T}(\widetilde{\Psi}) = \widetilde{\varphi}_{\mathrm{S}}(\vec{k}) + \widetilde{\Psi}(\vec{k}), \tag{2.1.162}$$

$$\mathcal{T}(\widetilde{\Psi}) \equiv \exp\left(-i\int_{\mathbb{R}^D}\frac{\mathrm{d}^D\vec{q}'}{(2\pi)^D}\widetilde{\Pi}_{\mathrm{S}}(-\vec{q}')\cdot\widetilde{\Psi}(\vec{q}')\right). \tag{2.1.163}$$

Explain why this implies $\widetilde{\Pi}_{\mathrm{S}}$ generates translations in field space.

Hints: You will need to use the commutation relations between $\widetilde{\Pi}$ and $\widetilde{\varphi}$; the Baker-Campbell-Hausdorff formula for $e^A B e^{-A}$ in terms of nested commutators. □

Problem 2.20. Vacuum Wavefunctional Argue that the *vacuum wavefunctional*, i.e., the ground state wavefunctional, of a massive scalar field theory is

$$\left\langle\widetilde{\Psi}\middle| 0\right\rangle = \mathcal{N}\exp\left(-\frac{1}{2}\int_{\mathbb{R}^D}\frac{\mathrm{d}^D\vec{k}}{(2\pi)^D}E_{\vec{k}}|\widetilde{\Psi}(\vec{k})|^2\right). \tag{2.1.164}$$

(Hint: One way is to use $0 = \langle\widetilde{\Psi}|a_{\vec{k}}^{\dagger}|0\rangle$ to derive a first order functional differential equation for $\langle\widetilde{\Psi}|0\rangle$.) You may wish to note that, since the massive scalar field is real,

$$\widetilde{\Psi}(\vec{k})^{*} = \widetilde{\Psi}(-\vec{k}) \tag{2.1.165}$$

and therefore,

$$\int_{\mathbb{R}^D}\frac{\mathrm{d}^D\vec{k}}{(2\pi)^D}E_{\vec{k}}|\widetilde{\Psi}(\vec{k})|^2 = \int_{\mathbb{R}^D}\frac{\mathrm{d}^D\vec{k}}{(2\pi)^D}\widetilde{\Psi}(\vec{k})E_{\vec{k}}\widetilde{\Psi}(-\vec{k}) \tag{2.1.166}$$

$$= \int_{\mathbb{R}^D}\frac{\mathrm{d}^D\vec{k}}{(2\pi)^D}\widetilde{\Psi}(-\vec{k})^{*}E_{\vec{k}}\widetilde{\Psi}(\vec{k})^{*}. \tag{2.1.167}$$

Can you figure out the correct normalization factor $\mathcal{N}$? Hint: First write down the ground state wave function of a single Fourier mode. □

2.1.4 Green's Functions; Gaussian Path Integrals

Feynman Green's Function At the heart of perturbative QFT is the Feynman Green's function. The Feynman Green's function $G_{\rm F}(x,x')$ is defined as the vacuum expectation value of the time-ordered product $\varphi(x)\varphi(x')$.

$$
\begin{aligned}
G_{\rm F}(x,x') &\equiv \langle 0|\mathsf{T}\left\{\varphi(x)\varphi(x')\right\}|0\rangle && (2.1.168)\\
&\equiv \langle 0|\Theta(t-t')\varphi(x)\varphi(x')+\Theta(t'-t)\varphi(x')\varphi(x)|0\rangle . && (2.1.169)
\end{aligned}
$$

From this definition, note the symmetry

$$
G_{\rm F}(x-x') = G_{\rm F}(x'-x). \qquad (2.1.170)
$$

The wave operator acting on $G_{\rm F}$ yields

$$
\begin{aligned}
&\left(\partial_x^2+m^2\right)G_{\rm F}[x,x'] \qquad (2.1.171)\\
&= \left\langle 0|\delta'(t-t')\varphi(x)\varphi(x')+2\delta(t-t')\partial_t\varphi(x)\varphi(x')+\Theta(t-t')(\partial_x^2+m^2)\varphi(x)\varphi(x')|0\right\rangle .\\
&\quad + \left\langle 0|\delta'(t'-t)\varphi(x')\varphi(x)-2\delta(t'-t)\varphi(x')\partial_t\varphi(x)+\Theta(t'-t)\varphi(x')(\partial_x^2+m^2)\varphi(x)|0\right\rangle .
\end{aligned}
$$

Use the Heisenberg EoM $(\partial^2+m^2)\varphi=0$ and the identity $z\delta'(z)=-\delta(z)$ to deduce

$$
\begin{aligned}
\left(\partial_x^2+m^2\right)G_{\rm F}[x,x'] &= \delta(t-t')\langle 0|\partial_t\varphi(x)\varphi(x')-\varphi(x')\partial_t\varphi(x)|0\rangle && (2.1.172)\\
&= -\delta(t-t')\langle 0|\left[\varphi(t,\vec{x}'),\Pi(t,\vec{x})\right]|0\rangle && (2.1.173)\\
&= -i\delta(t-t')\delta^{(d-1)}\left(\vec{x}-\vec{x}'\right). && (2.1.174)
\end{aligned}
$$

Similarly,

$$
\left(\partial_{x'}^2+m^2\right)G_{\rm F}[x,x'] = -i\delta(t-t')\delta^{(d-1)}\left(\vec{x}-\vec{x}'\right). \qquad (2.1.175)
$$

Let us record that, the Feynman Green's function for the masssive scalar field, $(\partial^2+m^2)\varphi=0$, has the following integral representation.

$$
G_{\rm F}(x-x') \equiv i\int\frac{{\rm d}^dk}{(2\pi)^d}\frac{e^{-ik\cdot(x-x')}}{k^2-m^2+i\epsilon}, \qquad \epsilon\equiv 0^+. \qquad (2.1.176)
$$

The small positive ϵ tells us the k_0 integral skirts the pole at $-E_{\vec{k}}$ by dipping into the negative imaginary plane; while skirting the pole at $+E_{\vec{k}}$ by going into the positive imaginary plane. Hence, if $t>t'$, we see that $\exp(-i(\pm i\infty)(t-t'))\to\exp((\pm\infty)(t-t'))$ tells us we need to close the contour in the lower half plane. Whereas, if $t<t'$, we see that $\exp(-i(\pm i\infty)(t-t'))\to\exp((\mp\infty)|t-t'|)$ tells us we need to close the contour in the upper half plane. Noting $k^2-m^2=(k_0-E_{\vec{k}})(k_0+E_{\vec{k}})$,

$$
\begin{aligned}
G_{\rm F}(x-x') &= \int\frac{{\rm d}^{d-1}\vec{k}}{(2\pi)^{d-1}}\frac{1}{2E_{\vec{k}}}\left(\frac{i(-i)}{+1}\Theta(t-t')e^{-iE_{\vec{k}}(t-t')}+\frac{i(+i)}{-1}\Theta(t'-t)e^{+iE_{\vec{k}}(t-t')}\right)e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\\
&= \int\frac{{\rm d}^{d-1}\vec{k}}{(2\pi)^{d-1}}\frac{1}{2E_{\vec{k}}}\left(\Theta(t-t')e^{-ik\cdot(x-x')}+\Theta(t'-t)e^{+ik\cdot(x-x')}\right), \qquad k^2=m^2.
\end{aligned}
$$

Within the canonical formalism, we may do a direct calculation of the following Wightman function

$$\langle 0 |\varphi(x)\varphi(x')| 0\rangle = \int \frac{\mathrm{d}^{d-1}\vec{k}_1}{(2\pi)^d}\frac{1}{\sqrt{2E_1}}\int \frac{\mathrm{d}^{d-1}\vec{k}_2}{(2\pi)^d}\frac{1}{\sqrt{2E_2}}\left\langle 0 \left| a_{\vec{k}_1} a^\dagger_{\vec{k}_2}\right| 0\right\rangle e^{-ik_1\cdot x}e^{ik_2\cdot x'} \tag{2.1.177}$$

$$= \int \frac{\mathrm{d}^{d-1}\vec{k}_1}{(2\pi)^d}\frac{1}{\sqrt{2E_1}}\int \frac{\mathrm{d}^{d-1}\vec{k}_2}{(2\pi)^d}\frac{1}{\sqrt{2E_2}}\left\langle 0 \left| \left[a_{\vec{k}_1}, a^\dagger_{\vec{k}_2}\right] + a^\dagger_{\vec{k}_2} a_{\vec{k}_1}\right| 0\right\rangle e^{-ik_1\cdot x}e^{ik_2\cdot x'}$$

$$= \int \frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^d}\frac{1}{2E_k}e^{-ik\cdot(x-x')} \tag{2.1.178}$$

Hence, the time ordered product itself is the Feynman Green's function

$$\langle 0 |\mathsf{T}\ \varphi(x)\varphi(x')| 0\rangle = \int \frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^d}\frac{1}{2E_{\vec{k}}}\left(\Theta(t-t')e^{-ik\cdot(x-x')} + \Theta(t'-t)e^{+ik\cdot(x-x')}\right) \tag{2.1.179}$$

$$= G_{\mathrm{F}}(x-x'). \tag{2.1.180}$$

Powers of $\hbar$s In d dimensional flat spacetime $\int \mathrm{d}^d x(\partial\varphi)^2$ must be of dimensions $\hbar$; which means

$$[\varphi^2]L^{d-2} = \hbar. \tag{2.1.181}$$

Since G_F arises from the $\langle 0| T\varphi\varphi |0\rangle$, this immediately implies

$$[G_{\mathrm{F}}] = \hbar/L^{d-2}. \tag{2.1.182}$$

Now, the *classical* Green's functions obeys $(\partial^2 + m^2)G = \delta^{(d)}$, which means $[G] = 1/L^{d-2}$. Moreover if m is mass then $[m^2/\hbar^2] = 1/L^2$ if $c = 1$; hence, the Green's function equation really reads $(\partial^2 + (m/\hbar)^2)G = \delta^{(d)}$. Therefore, we may identify the vacuum expectation value of the time-ordered product of two φs to be $\hbar$ times a classical Green's function:

$$\langle 0|\mathsf{T}\left\{\varphi(x)\varphi(x')\right\}|0\rangle = i\hbar\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{e^{-ik\cdot(x-x')}}{k^2-(m/\hbar)^2+i\epsilon}. \tag{2.1.183}$$

Retarded/Advanced Green's Functions, Commutators & Micro-causality We will now argue that the retarded Green's function G^+ of the massive scalar field is given by the commutator multiplied by the causal step function, namely

$$G^+[x,x'] \equiv i\Theta[t-t']\left[\varphi[x],\varphi[x']\right]. \tag{2.1.184}$$

Let us apply the wave operator:

$$\left(\partial_x^2 + m^2\right)G^+[x,x'] = i\delta'(t-t')\left[\varphi[x],\varphi[x']\right] + 2i\delta(t-t')\left[\partial_t\varphi[x],\varphi[x']\right]$$
$$+ i\Theta(t-t')\left[\left(\partial^2+m^2\right)\varphi(x),\varphi(x')\right] \tag{2.1.185}$$

Using the identity $z\delta'(z) = -\delta(z)$ and using the Heisenberg equations of motion $(\partial^2+m^2)\varphi = 0$, we gather

$$\left(\partial_x^2 + m^2\right)G^+[x,x'] = i\delta'(t-t')\left[\partial_t\varphi[t,\vec{x}],\varphi[t,\vec{x}']\right]$$
$$= -i\delta'(t-t')\left[\varphi[t,\vec{x}'],\Pi[t,\vec{x}]\right] \tag{2.1.186}$$
$$= \delta(t-t')\delta^{(d-1)}\left(\vec{x}-\vec{x}'\right). \tag{2.1.187}$$

By construction, G^+ is zero whenever $t < t'$. Furthermore it satisfies the Green's function equation. Hence, it is the retarded Green's function. A similar argument would reveal, the advanced Green's function G^- is

$$G^-[x,x'] \equiv -i\Theta[t'-t]\,[\varphi[x],\varphi[x']]\,. \tag{2.1.188}$$

We may then subtract the retarded G^+ and advanced G^- Green's functions, recognize $\Theta(t-t')+\Theta(t'-t)=1$, and deduce

$$i\,[\varphi[x],\varphi[x']] = G^+[x,x'] - G^-[x,x']. \tag{2.1.189}$$

Whenever the commutator of the fields $\varphi(x)$ and $\varphi(x')$ vanish outside the light cone, i.e.,

$$[\varphi[x],\varphi[x']] = 0 \qquad \text{whenever} \qquad (x-x')^2 < 0, \tag{2.1.190}$$

micro-causality is said to be respected. In fact, it can be shown that the retarded and advanced Green's functions of wave operators (in generic curved spacetimess) do in fact vanish outside the light cone. Hence, the micro-causality of quantum fields is tied to the causal propagation of classical fields.

Gaussian Path Integrals and 2 Point Functions We will turn to obtaining the Feynman Green's function, i.e., the two point function, via the path integral

$$\langle\widetilde{\varphi}(k_1)\widetilde{\varphi}(k_2)\rangle = \int \mathrm{d}^d x_1 \int \mathrm{d}^d x_2 e^{ik_1\cdot x_1} e^{ik_2\cdot x_2} \frac{\int \mathcal{D}\varphi\varphi(x_1)\varphi(x_2)\exp(iS)}{\int \mathcal{D}\varphi\exp(iS)} \tag{2.1.191}$$

$$S \equiv \int \mathrm{d}^d x \left(\frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2\right). \tag{2.1.192}$$

The two point function $\langle\widetilde{\varphi}(k_1)\widetilde{\varphi}(k_2)\rangle$, which is simply the Feynman Green's function in Fourier spacetime, reads

$$\langle\widetilde{\varphi}(k_1)\widetilde{\varphi}(k_2)\rangle = \frac{i}{k_1^2 - m^2 + i\epsilon}(2\pi)^d\delta^{(d)}(k_1+k_2). \tag{2.1.193}$$

Check:

$$\langle 0|\mathsf{T}\ \varphi(x_1)\varphi(x_2)|0\rangle = \int \frac{\mathrm{d}^d k_1}{(2\pi)^d}\int \frac{\mathrm{d}^d k_2}{(2\pi)^d}\frac{i}{k_1^2-m^2+i\epsilon}e^{-ik_1\cdot x_1}e^{-ik_2\cdot x_2}(2\pi)^d\delta^{(d)}(k_1+k_2) \tag{2.1.194}$$

$$= \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{i}{k^2-m^2+i\epsilon}e^{-ik\cdot(x_1-x_2)} \tag{2.1.195}$$

Let us see that the path integral would recover this result. First, convert the action into Fourier spacetime

$$S \equiv \int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{1}{2}\left(q^2-m^2\right)|\widetilde{\varphi}(q)|^2. \tag{2.1.196}$$

The path integral measure $\int\mathcal{D}\varphi \equiv \prod_x \int \mathrm{d}\varphi(x)$ may be converted to Fourier spacetime.

$$\prod_x \mathrm{d}\varphi(x) \leftrightarrow \prod_k \mathrm{d}\widetilde{\varphi}_{\mathrm{R}}(k)\mathrm{d}\widetilde{\varphi}_{\mathrm{I}}(k) \tag{2.1.197}$$

Hence,

$$\int \mathrm{d}^d x_1 \int \mathrm{d}^d x_2 e^{ik_1\cdot x_1} e^{ik_2\cdot x_2} \frac{\int \mathcal{D}\varphi \varphi(x_1)\varphi(x_2) \exp(iS)}{\int \mathcal{D}\varphi \exp(iS)}$$
$$= \frac{\prod_q \int \mathrm{d}\widetilde{\varphi}_{\mathrm{R}}(q) \int \mathrm{d}\widetilde{\varphi}_{\mathrm{I}}(q) \widetilde{\varphi}(k_1)\widetilde{\varphi}(k_2) \exp\left(\frac{i}{2}\left(q^2 - m^2 + i\epsilon\right)\left(\widetilde{\varphi}_{\mathrm{R}}(q')^2 + \widetilde{\varphi}_{\mathrm{I}}(q')^2\right)\right)}{\prod_{q'} \int \mathrm{d}\widetilde{\varphi}_{\mathrm{R}}(q') \int \mathrm{d}\widetilde{\varphi}_{\mathrm{I}}(q') \exp\left(\frac{i}{2}\left(q'^2 - m^2 + i\epsilon\right)\left(\widetilde{\varphi}_{\mathrm{R}}(q')^2 + \widetilde{\varphi}_{\mathrm{I}}(q')^2\right)\right)}. \tag{2.1.198}$$

For the integration over the real and imaginary portions of the Fourier spacetime $\widetilde{\varphi}$ to converge, we shall shift the $q^2 - m^2$ by $q^2 - m^2 + i\epsilon$. In more detail, each momentum slice integral goes as

$$\int_{\mathbb{R}} \exp\left((i/2)\alpha z^2\right) \ldots \mathrm{d}z, \qquad \alpha \in \mathbb{R}. \tag{2.1.199}$$

The gaussian fluctuates wildly as $|z| \to \infty$ unless we shift $\alpha \to \alpha + i\epsilon$. Moreover, each momentum slice in the numerator and denominator cancels out except for the slices corresponding to $q = k_1$ and $q = k_2$. Furthermore, if $k_1 \neq \pm k_2$, then the integral over the $q = k_1$ slice and that over the $q = k_2$ slice are both odd integrands.

$$\int \mathrm{d}\widetilde{\varphi}_{\mathrm{R}} \int \mathrm{d}\widetilde{\varphi}_{\mathrm{I}} \left(\widetilde{\varphi}_{\mathrm{R}} + i\widetilde{\varphi}_{\mathrm{I}}\right) \exp\left(\frac{i}{2}\left(k_1^2 - m^2 + i\epsilon\right)\left(\widetilde{\varphi}_{\mathrm{R}}^2 + \widetilde{\varphi}_{\mathrm{I}}^2\right)\right) = 0 \tag{2.1.200}$$

Hence, the entire path integral is zero unless $k_1 = k_2$. When $k_1 = k_2$, the path integral becomes

$$\int \mathrm{d}^d x_1 \int \mathrm{d}^d x_2 e^{ik_1\cdot x_1} e^{ik_2\cdot x_2} \frac{\int \mathcal{D}\varphi \varphi(x_1)\varphi(x_2) \exp(iS)}{\int \mathcal{D}\varphi \exp(iS)}$$
$$= \frac{\int \mathrm{d}\widetilde{\varphi}_{\mathrm{R}} \int \mathrm{d}\widetilde{\varphi}_{\mathrm{I}} \left(\widetilde{\varphi}_{\mathrm{R}}^2 + i^2\widetilde{\varphi}_{\mathrm{I}}^2\right) \exp\left(\frac{i}{2}\left(k_1^2 - m^2 + i\epsilon\right)\left(\widetilde{\varphi}_{\mathrm{R}}^2 + \widetilde{\varphi}_{\mathrm{I}}^2\right)\right)}{\int \mathrm{d}\widetilde{\varphi}_{\mathrm{R}} \int \mathrm{d}\widetilde{\varphi}_{\mathrm{I}} \exp\left(\frac{i}{2}\left(k_1^2 - m^2 + i\epsilon\right)\left(\widetilde{\varphi}_{\mathrm{R}}^2 + \widetilde{\varphi}_{\mathrm{I}}^2\right)\right)}, \tag{2.1.201}$$

where we have thrown out those terms linear in $\widetilde{\varphi}_{\mathrm{R}}$ and in $\widetilde{\varphi}_{\mathrm{I}}$. But this integral is zero because $i^2 = -1$. Hence, the remaining possibility is when $k_1 = -k_2$. In such a situation, $\widetilde{\varphi}(-k_1) = \widetilde{\varphi}(k_1)^*$, $\widetilde{\varphi}(k_1)\widetilde{\varphi}(-k_1) = |\widetilde{\varphi}(k_1)|^2$, and the path integral becomes

$$\int \mathrm{d}^d x_1 \int \mathrm{d}^d x_2 e^{ik_1\cdot x_1} e^{ik_2\cdot x_2} \frac{\int \mathcal{D}\varphi \varphi(x_1)\varphi(x_2) \exp(iS)}{\int \mathcal{D}\varphi \exp(iS)}$$
$$= \frac{\int \mathrm{d}\widetilde{\varphi}_{\mathrm{R}} \int \mathrm{d}\widetilde{\varphi}_{\mathrm{I}} \left(\widetilde{\varphi}_{\mathrm{R}}^2 + \widetilde{\varphi}_{\mathrm{I}}^2\right) \exp\left(\frac{i}{2}\left(k_1^2 - m^2 + i\epsilon\right)\left(\widetilde{\varphi}_{\mathrm{R}}^2 + \widetilde{\varphi}_{\mathrm{I}}^2\right)\right)}{\int \mathrm{d}\widetilde{\varphi}_{\mathrm{R}} \int \mathrm{d}\widetilde{\varphi}_{\mathrm{I}} \exp\left(\frac{i}{2}\left(k_1^2 - m^2 + i\epsilon\right)\left(\widetilde{\varphi}_{\mathrm{R}}^2 + \widetilde{\varphi}_{\mathrm{I}}^2\right)\right)} \tag{2.1.202}$$
$$= 2\frac{\int \mathrm{d}z z^2 \exp\left(\frac{i}{2}\left(k_1^2 - m^2 + i\epsilon\right) z^2\right)}{\int \mathrm{d}z \exp\left(\frac{i}{2}\left(k_1^2 - m^2 + i\epsilon\right) z^2\right)} = \frac{2i}{k_1^2 - m^2 + i\epsilon}. \tag{2.1.203}$$

Peskin & Schroeder found some way to argue away the 1/2 in the exponential; this would yield $i/(k^2 - m^2 + i\epsilon)$. Notice, too, this is why free field theories are oftentimes dubbed 'gaussian' – the exponential is in fact gaussian, because the action is quadratic in the fields.

In any case, let us arrive at this answer by completing the square instead. To this end we follow Schwinger and first introduce a linear coupling to the scalar field via

$$Z[J] \equiv \frac{\int \mathcal{D}\varphi \exp\left[i \int \mathrm{d}^d x \left(\frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2 + J(x)\varphi(x)\right)\right]}{\int \mathcal{D}\varphi \exp\left[i \int \mathrm{d}^d x \left(\frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2\right)\right]}, \tag{2.1.204}$$

where $J(x)$ is real. Let us define

$$\widetilde{\varphi}(k) \equiv \widetilde{\phi}(k) + i\widetilde{G}(k)\widetilde{J}(k). \tag{2.1.205}$$

The Lagrangian in the numerator in Fourier spacetime now reads

$$\mathcal{L}_k[J] \equiv \frac{1}{2}\widetilde{\varphi}(k)(k^2 - m^2 + i\epsilon)\widetilde{\varphi}(-k) + \frac{1}{2}\widetilde{J}(k)\widetilde{\varphi}(-k) + \frac{1}{2}\widetilde{J}(-k)\widetilde{\varphi}(k) \tag{2.1.206}$$

$$= \frac{1}{2}(k^2 - m^2 + i\epsilon)\left(|\widetilde{\phi}|^2 + \widetilde{G}(k)\widetilde{G}(-k)|\widetilde{J}|^2 + i\widetilde{G}(k)\widetilde{J}(k)\widetilde{\phi}(-k) + \widetilde{\phi}(k)i\widetilde{G}(-k)\widetilde{J}(-k)\right)$$

$$+ \frac{1}{2}\widetilde{J}(k)\widetilde{\phi}(-k) + \frac{i}{2}\widetilde{J}(k)\widetilde{G}(-k)\widetilde{J}(-k) + \frac{1}{2}\widetilde{J}(-k)\widetilde{\phi}(k) + \frac{i}{2}\widetilde{J}(-k)\widetilde{G}(-k)\widetilde{J}(-k). \tag{2.1.207}$$

Focusing on the terms linear in $\widetilde{\phi}$,

$$\mathcal{L}_k[J] = \frac{1}{2}(k^2 - m^2 + i\epsilon)\Bigg(\widetilde{\varphi}(k)i\left(\widetilde{G}(-k) - \frac{i}{k^2 - m^2 + i\epsilon}\right)\widetilde{J}(-k) \tag{2.1.208}$$

$$+ \widetilde{J}(k)i\left(\widetilde{G}(k) - \frac{i}{k^2 - m^2 + i\epsilon}\right)\widetilde{\phi}(-k)\Bigg) + \dots \tag{2.1.209}$$

If we choose

$$\widetilde{G}(k) = \frac{i}{k^2 - m^2 + i\epsilon} = \widetilde{G}(-k) \tag{2.1.210}$$

all the linear-in-$\widetilde{\phi}$ terms will vanish and we obtain

$$\mathcal{L}_k[J] = \frac{1}{2}(k^2 - m^2 + i\epsilon)|\widetilde{\phi}|^2$$

$$+ \frac{1}{2}\widetilde{J}(k)\frac{1}{k^2 - m^2 - i\epsilon}\widetilde{J}(-k) - \frac{1}{2}\widetilde{J}(k)\frac{1}{k^2 - m^2 - i\epsilon}\widetilde{J}(-k) + \frac{i}{2}\widetilde{J}(-k)\widetilde{G}(k)\widetilde{J}(k) \tag{2.1.211}$$

$$= \frac{1}{2}\widetilde{\phi}(k)(k^2 - m^2 + i\epsilon)\widetilde{\phi}(-k) + \frac{1}{2}(i\widetilde{J}(k))(k^2 - m^2 + i\epsilon)^{-1}(i\widetilde{J}(-k)) \tag{2.1.212}$$

The $\widetilde{J}$ term does not take part in the functional integral, while the $\widetilde{\phi}$ terms simply correspond to the functional integral without $\widetilde{J}$s. At this point, the functional integrals in the numerator and denominator cancel, and we are left with

$$Z = \exp\left[\frac{1}{2}\int_k i\widetilde{J}(k)\frac{i}{k^2 - m^2 + i\epsilon}i\widetilde{J}(-k)\right] \tag{2.1.213}$$

$$= \exp\left[\frac{1}{2}\int_{x,x'} iJ(x)G_{\mathrm{F}}(x - x')iJ(x')\right] \tag{2.1.214}$$

This tells us

$$\left.\frac{\delta^2 Z}{\delta(iJ(z))\delta(iJ(z'))}\right|_{J=0} = G_{\mathrm{F}}(x - x'). \tag{2.1.215}$$

In Fourier spacetime,

$$\left.\frac{\delta^2 Z}{\delta(i\widetilde{J}(k))\delta(i\widetilde{J}(k'))}\right|_{J=0} = \frac{i}{k^2 - m^2 + i\epsilon}(2\pi)^d \delta^{(d)}(k + k'). \tag{2.1.216}$$

On the other hand, by directly differentiating the ratio of path integrals,

$$\left.\frac{\delta^2 Z}{\delta(iJ(z))\delta(iJ(z'))}\right|_{J=0} = \frac{\int \mathcal{D}\varphi \varphi(z)\varphi(z') \exp\left[i \int \mathrm{d}^d x \left(\frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2\right)\right]}{\int \mathcal{D}\varphi \exp\left[i \int \mathrm{d}^d x \left(\frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2\right)\right]} \tag{2.1.217}$$

$$= G_{\mathrm{F}}(x - x'). \tag{2.1.218}$$

Hence, we have shown that ratios of the path integrals in Z' does indeed generate the two point function.

Problem 2.21. Non-locality Explain why the seemingly local-in-momentum space object becomes a non-local one in spacetime:

$$\int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\widetilde{J}(k) i \widetilde{J}(-k)}{k^2 - m^2 + i\epsilon} = \int_{\mathbb{R}^d} \mathrm{d}^d x \int_{\mathbb{R}^d} \mathrm{d}^d x' J(x) G_{\mathrm{F}}(x - x') J(x'). \tag{2.1.219}$$

Remember $J(x)$ is real. □

Odd Point Functions Vanish Let us also note that all odd point functions vanish.

$$\frac{\int \mathcal{D}\varphi \varphi(z_1)\dots\varphi(z_{2n+1}) \exp\left[i \int \mathrm{d}^d x \left(\frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2\right)\right]}{\int \mathcal{D}\varphi \exp\left[i \int \mathrm{d}^d x \left(\frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2\right)\right]} = 0. \tag{2.1.220}$$

We may see this by directly differentiating Z'.

$$\sum_{s=0}^{\infty} \frac{(1/2)^s}{s!} \left.\frac{\delta^{2n+1}}{\delta(iJ(z_1))\delta(iJ(z_2))\dots\delta(iJ(z_{2n+1}))}\right|_{J=0} iJ_{x_1} G_{\mathrm{F}}^{x_1,x_1'} iJ_{x_1'} \cdots iJ_{x_s} G_{\mathrm{F}}^{x_s,x_s'} iJ_{x_s'} \tag{2.1.221}$$

But each term in the sum has even number of Js. So there must be at least one J leftover from differentiation; which will render the entire expression trivial upon setting $J = 0$.

Even Point Functions & Wick's Theorem For even point function, the result may be expressed through Wick's theorem:

> The $2n$ point function is simply the sum over all fully contracted fields, where which Wick contracted pair of fields is replaced with the corresponding Feynman Green's function.

(I forgot how to do Wick contractions in LaTeX.) Simply differentiate.

$$\sum_{s=0}^{\infty} \frac{(1/2)^s}{s!} \left.\frac{\delta^{2n}}{\delta(iJ(z_1))\delta(iJ(z_2))\dots\delta(iJ(z_{2n}))}\right|_{J=0} iJ_{x_1} G_{\mathrm{F}}^{x_1,x_1'} iJ_{x_1'} \cdots iJ_{x_s} G_{\mathrm{F}}^{x_s,x_s'} iJ_{x_s'} \tag{2.1.222}$$

Only the $s=n$ term is relevant, because there are exactly $2n$ Js there. For every

$$\frac{\delta iJ(x_k)}{\delta iJ(z_i)} G_{\mathrm{F}}^{x_k,x'_k} \frac{\delta iJ(x'_k)}{\delta iJ(z'_i)} \tag{2.1.223}$$

there is also a

$$\frac{\delta iJ(x_k)}{\delta iJ(z'_i)} G_{\mathrm{F}}^{x_k,x'_k} \frac{\delta iJ(x'_k)}{\delta iJ(z_i)}. \tag{2.1.224}$$

Because there are s pairs of Js, the $(1/2)^s$ will cancel out. Moreover, since there are precisely $s!$ ways to re-arrange s Green's functions of fixed arguments, the $s!$ in the denominator will cancel out too. For example, we have the following 4 point function.

$$\langle 0|\mathsf{T}\varphi_1\varphi_2\varphi_3\varphi_4|0\rangle = G_{\mathrm{F}}^{1,2}G_{\mathrm{F}}^{3,4} + G_{\mathrm{F}}^{1,3}G_{\mathrm{F}}^{2,4} + G_{\mathrm{F}}^{1,4}G_{\mathrm{F}}^{2,3} \tag{2.1.225}$$

To sum: for the free massive scalar theory, the n point function is completely fixed (up to combinatorics) once the two point function is known. This is analogous to the situation in statistics: when the underlying probability distribution is Gaussian, all the $n \geq 1$ moments are completely determined by the variance.

Problem 2.22. Write down the 6−point function. □

2.1.5 Misc. Problems

Problem 2.23. Quantization Around A Classical Background Starting from the following action

$$S \equiv \int_{\mathbb{R}^{D,1}} \mathrm{d}t\mathrm{d}^D\vec{x} \left(\frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi + \varphi\cdot J\right), \tag{2.1.226}$$

Show that the Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{m^2}{2}\varphi^2 - \varphi\cdot J, \tag{2.1.227}$$

and that the resulting Hamilton's equations yield

$$\left(\partial^2 + m^2\right)\varphi(t,\vec{x}) = J(t,\vec{x}). \tag{2.1.228}$$

Let us attempt to quantize the fluctuations on top of this classical background. That is, starting from the action **YZ: Incomplete.** □

Problem 2.24. QFT in a "Square Well" & Casimir Effect Consider the massless $m=0$ free scalar field in (1+1)-dimensional Minkowski, but within a finite spatial domain of length L:

$$x^\mu \equiv (t\in\mathbb{R}, 0\leq x\leq L). \tag{2.1.229}$$

Demand that the field vanishes at $x = 0, L$. Compute the vacuum expectation value of the energy density ρ, defined as:

$$\rho \equiv \left\langle \text{vac} \left| T^{00} \right| \text{vac} \right\rangle, \tag{2.1.230}$$

$$T^{00} \equiv \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\partial_x \varphi)^2. \tag{2.1.231}$$

What is the total energy E inside the square well, as a function of L? What is $\partial E/\partial L$? The latter can be interpreted as an effective force experienced by the walls of the square well as a result of quantum fluctuations. (Bonus: What are the pressure and momentum densities?) □

Problem 2.25. Cosmology We live in a universe that is well described, at large scales, by the metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a(t)^2 \mathrm{d}\vec{x} \cdot \mathrm{d}\vec{x}; \tag{2.1.232}$$

i.e., with non-zero components $g_{00} = 1$ and $g_{ij} = -a^2 \delta_{ij}$. The associated wave operator can be constructed out of the absolute value of its determinant $|g| = a^6$ and inverse $g^{\mu\nu}$:

$$\Box \varphi = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu \varphi \right) = \frac{1}{a^3} \partial_t \left(a^3 \dot{\varphi} \right) - a^{-2} \vec{\nabla}^2 \varphi. \tag{2.1.233}$$

The Lagrangian density of the massive scalar field is now

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{m^2}{2} \varphi^2 = \frac{1}{2} \dot{\varphi}^2 - \frac{a^{-2}}{2} \vec{\nabla} \varphi \cdot \vec{\nabla} \varphi - \frac{m^2}{2} \varphi^2. \tag{2.1.234}$$

Work out the conjugate momentum Π. Show that the equal time commutation relations

$$[\varphi(t, \vec{x}), \Pi(t, \vec{x}')] = i \frac{\delta^{(3)}(\vec{x} - \vec{x}')}{a(t)^3} \tag{2.1.235}$$

$$[\varphi(t, \vec{x}), \varphi(t, \vec{x}')] = 0 = [\Pi(t, \vec{x}), \Pi(t, \vec{x}')]; \tag{2.1.236}$$

leads to the Heisenberg EoM

$$\left(\Box + m^2 \right) \varphi = 0. \tag{2.1.237}$$

That is, just like the Minkowski case, we recover the massive scalar wave equation. □

Problem 2.26. de Sitter Power Spectrum de Sitter spacetime in 'flat slicing' coordinates is described by the metric

$$\mathrm{d}s^2 = a(\eta)^2 \left(\mathrm{d}\eta^2 - \mathrm{d}\vec{x}^2 \right), \tag{2.1.238}$$

$$a[\eta] \equiv -(H\eta)^{-1}, \qquad \eta \in (-\infty, 0). \tag{2.1.239}$$

In this problem, compute in this geometry the one point function

$$\left\langle \text{vac} \left| \widetilde{\varphi}(\eta, \vec{k}) \right| \text{vac} \right\rangle \tag{2.1.240}$$

and the power spectrum of a massless scalar field

$$\left\langle \text{vac} \left| \widetilde{\varphi}(\eta, \vec{k}) \widetilde{\varphi}(\eta, \vec{k}') \right| \text{vac} \right\rangle . \tag{2.1.241}$$

Hints: If $\sqrt{|g|}$ is the square root of the determinant of the metric $g_{\mu\nu} = \text{diag}[1, -1, -1, -1]/(H\eta)^2$ and $g^{\mu\nu} = \text{diag}[1, -1, -1, -1](H\eta)^2$ is its inverse, the scalar field obeys the wave equation

$$\Box\varphi = \frac{\partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu \varphi \right)}{\sqrt{|g|}} = 0. \tag{2.1.242}$$

The spatial translation invariance means we may Fourier decompose our scalar field in plane waves. Specifically, if we first re-scale

$$\varphi(\eta, \vec{x}) = \frac{\phi(\eta, \vec{x})}{a(\eta)}, \tag{2.1.243}$$

followed by examining a single Fourier mode

$$\phi(\eta, \vec{x}) = f(\xi) e^{i\vec{k}\cdot\vec{x}}, \tag{2.1.244}$$

where $\xi \equiv k\eta$ and $k \equiv |\vec{k}|$; show that f obeys

$$f''(\xi) + \left(1 - \frac{2}{\xi^2} \right) f(\xi) = 0. \tag{2.1.245}$$

You should find the two linearly independent solutions to be

$$f_\pm(\xi) \equiv \frac{e^{\pm i\xi}}{\sqrt{2}} \left(1 \pm \frac{i}{\xi} \right). \tag{2.1.246}$$

Choose your mode functions for φ such that, as $\eta \to -\infty$, the positive energy solutions approach those of Minkowski spacetime. Moreover, to quantize such a system, you should be able to verify its associated Lagrangian is

$$L_f \equiv \frac{1}{2} f'(\xi)^2 + \frac{f(\xi) f'(\xi)}{\xi} + \frac{1}{2} f(\xi)^2 \left(\frac{1}{\xi^2} - 1 \right). \tag{2.1.247}$$

These vacuum fluctuations – if inflationary cosmologists are right – may be responsible for generating inhomogeneities in the very early universe, from which all of cosmic structure (galaxy clusters, etc.) were produced. In particular, comment on how the calculations apply to primordial gravitational waves. □

Problem 2.27. Massless Spin-1 Photons and Spin-2 Gravitons Consider a scalar-vector decomposition of the photon vector potential A_i and electric current J_i:

$$A_i = \partial_i \alpha + \alpha_i, \qquad \partial_i \alpha_i = 0, \tag{2.1.248}$$

$$J_i = \partial_i \Gamma + \Gamma_i, \qquad \partial_i \Gamma_i = 0. \tag{2.1.249}$$

Explain why the conservation of current $\partial^\mu J_\mu = 0$ implies

$$\dot{J}^0 = \partial_i \partial_i \Gamma. \tag{2.1.250}$$

Next, recall the gauge-invariant vector potential variables

$$\Phi \equiv A_0 - \dot{\alpha} \qquad \text{and} \qquad A_i^{\mathrm{T}} \equiv \alpha_i. \tag{2.1.251}$$

Show that the electromagnetic action

$$S_{\mathrm{EM}} \equiv \int \mathrm{d}^d x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu \right) \tag{2.1.252}$$

can be written in the manifestly gauge-invariant form

$$S_{\mathrm{EM}} \equiv \int \mathrm{d}^d x \left(\frac{1}{2} \partial_\alpha \alpha_j \partial^\alpha \alpha_j + \alpha_i \Gamma_i + \frac{1}{2} \partial_i \Phi \partial_i \Phi - \Phi J^0 \right). \tag{2.1.253}$$

(Assume it is alright to integrate-by-parts freely.) Now, let us quantize the dynamical massless spin-1 photon α_i in the (3+1)D vacuum, i.e., where $J_\mu = 0$.

$$\alpha_i(x) = \int \frac{\mathrm{d}^3 \vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2k}} \sum_{s=1}^{2} \left\{ a_{\vec{k}}^s \epsilon_i^s e^{-ik\cdot x} + (a_{\vec{k}}^s)^\dagger (\epsilon_i^s)^* e^{+ik\cdot x} \right\}, \qquad k \equiv |\vec{k}|. \tag{2.1.254}$$

The $\{\epsilon_i^s | s = 1, 2\}$ are the two transverse orthonormal polarization vectors of the photon: $k^i \epsilon_i^s = 0$. Whereas $k_\mu k^\mu = 0$ and the ladder operators obey the simple harmonic algebra

$$\left[a_{\vec{k}}^r, (a_{\vec{k}'}^s)^\dagger \right] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{k} - \vec{k}'). \tag{2.1.255}$$

Show that, for $x^\mu \equiv (t, \vec{x})$ and $x'^\mu \equiv (t', \vec{x}')$,

$$i \left[\alpha_i(x), \alpha_j(x') \right] = \mathrm{sgn}(t - t') \int \frac{\mathrm{d}^3 \vec{k}}{(2\pi)^3} P_{ij}(\vec{k}) \frac{\sin(k(t - t'))}{k} e^{i\vec{k}\cdot(\vec{x} - \vec{x}')}, \tag{2.1.256}$$

$$P_{ij}(\vec{k}) \equiv \delta_{ij} - \frac{k_i k_j}{\vec{k}^2}. \tag{2.1.257}$$

Next, we turn to the massless spin-2 graviton h_{ij}^{TT},[2] which is transverse and traceless:

$$\partial_i h_{ij}^{\mathrm{TT}} = 0 = \delta^{ij} h_{ij}^{\mathrm{TT}}. \tag{2.1.258}$$

Its quantum operator admits the expansion, in (3+1)D,

$$h_{ij}^{\mathrm{TT}}(x) = \int \frac{\mathrm{d}^3 \vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2k}} \sum_{s=1}^{2} \left\{ a_{\vec{k}}^s \epsilon_{ij}^s e^{-ik\cdot x} + (a_{\vec{k}}^s)^\dagger (\epsilon_{ij}^s)^* e^{+ik\cdot x} \right\}, \qquad k \equiv |\vec{k}|. \tag{2.1.259}$$

[2] In a weakly curved spacetime $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $|h_{\mu\nu}| \ll 1$, the h_{ij}^{TT} is the transverse-traceless part of $h_{\mu\nu}$.

The transverse traceless conditions in Fourier space reads

$$k^i \epsilon^s_{ij} = 0 = \delta^{ij} \epsilon^s_{ij}. \tag{2.1.260}$$

Imposing the simple harmonic algebra

$$\left[a^r_{\vec{k}}, (a^s_{\vec{k}'})^\dagger\right] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{k} - \vec{k}'). \tag{2.1.261}$$

demonstrate that

$$i\left[h^{\rm TT}_{ij}(x), h^{\rm TT}_{mn}(x')\right] = \text{sgn}(t - t') \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} P_{ijmn}(\vec{k}) \frac{\sin(k(t-t'))}{k} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}, \tag{2.1.262}$$

$$P_{ijmn}(\vec{k}) \equiv \frac{1}{2}\left(P_{im}(\vec{k})P_{jn}(\vec{k}) + P_{in}(\vec{k})P_{jm}(\vec{k}) - P_{ij}(\vec{k})P_{mn}(\vec{k})\right). \tag{2.1.263}$$

Hints: Recall that the (Fourier space) transverse polarization vector can be obtained by projection:

$$\varepsilon^{\rm T}_i = P_{ij}\varepsilon_j; \tag{2.1.264}$$

and likewise for the (Fourier space) TT gravitation polarization tensor:

$$\varepsilon^{\rm TT}_{ij} = \frac{1}{2}\left(P_{im}(\vec{k})P_{jn}(\vec{k}) + P_{in}(\vec{k})P_{jm}(\vec{k}) - P_{ij}(\vec{k})P_{mn}(\vec{k})\right)\varepsilon_{mn}. \tag{2.1.265}$$

This implies, when one encounters a sum over orthonormal polarization vectors such as

$$\sum_s \epsilon^{(s)\rm T}_i (\varepsilon^{(s)\rm T}_j)^* = P_{ia}P_{jb} \sum_s \epsilon^{(s)}_a (\varepsilon^{(s)}_b)^*; \tag{2.1.266}$$

even though the sum only runs over the two orthonormal polarizations perpendicular to k_i, it can be extended to include $\epsilon^{(3)}_i = \widehat{k}_i$ since it is annihilated by P_{ij} anyway. This observation may then be invoked to then apply the completeness relation. Similar remarks apply for the graviton case.

2.2 Interacting Fields

2.2.1 In- and Out-Vacuum (Heisenberg Picture)

We will define the in-vacuum to be the Heisenberg picture ground state of the full Hamiltonian H, evaluated at the infinite past

$$|\text{in-vac}\rangle \equiv |\Omega, t \to -\infty\rangle_{\rm H}.$$

Similarly, the out-vacuum is that evaluated at the infinite future

$$|\text{out-vac}\rangle \equiv |\Omega, t \to +\infty\rangle_{\rm H}.$$

In Minkowski spacetime with no external time dependent fields, the Hamiltonian of a given QFT is time independent. We have, of course,

$$H|\text{in-vac}\rangle = \Omega|\text{in-vac}\rangle,$$
$$H|\text{out-vac}\rangle = \Omega|\text{out-vac}\rangle,$$

where Ω is the ground state energy, a real number. This means $|\text{in-vac}\rangle$ and $|\text{out-vac}\rangle$ can only differ by a (possibly ambiguous) phase

$$|\text{in-vac}\rangle = e^{i\Sigma}|\text{out-vac}\rangle, \qquad e^{i\Sigma}\langle\text{in-vac}| = \langle\text{out-vac}|$$
$$e^{i\Sigma} \equiv \lim_{T\to\infty} e^{-i(2T)\Omega}.$$

Therefore, the vacuum expectation value of the time-ordered product of operators

$$\mathcal{O}_1[t_1], \mathcal{O}_2[t_2], \dots, \mathcal{O}_n[t_n], \tag{2.2.1}$$

can be expressed in various ways. Firstly, it does not matter whether the expectation value is taken with respect with either the in-vacuum or out-vacuum, since they differ only by a phase.

$$\langle\text{in-vac}|\mathcal{T}\left\{\mathcal{O}_1[t_1]\mathcal{O}_2[t_2]\dots\mathcal{O}_n[t_n]\right\}|\text{in-vac}\rangle = \langle\text{out-vac}|\mathcal{T}\left\{\mathcal{O}_1[t_1]\mathcal{O}_2[t_2]\dots\mathcal{O}_n[t_n]\right\}|\text{out-vac}\rangle.$$

Secondly, the vacuum expectation value can also be expressed as an in-out transition amplitude,

$$\langle\Omega'|\mathcal{T}\left\{\mathcal{O}_1[t_1]\mathcal{O}_2[t_2]\dots\mathcal{O}_n[t_n]\right\}|\Omega'\rangle = \frac{\langle\text{out-vac}|\mathcal{T}\left\{\mathcal{O}_1[t_1]\mathcal{O}_2[t_2]\dots\mathcal{O}_n[t_n]\right\}|\text{in-vac}\rangle}{\langle\text{out-vac}|\text{in-vac}\rangle}, \tag{2.2.2}$$

where $\Omega' = \text{in-vac}$ or $\Omega' = \text{out-vac}$. To see this, we begin from the RHS.

$$\begin{aligned}\frac{\langle\text{out-vac}|\mathcal{T}\left\{\mathcal{O}_1[t_1]\mathcal{O}_2[t_2]\dots\mathcal{O}_n[t_n]\right\}|\text{in-vac}\rangle}{\langle\text{out-vac}|\text{in-vac}\rangle} &= \frac{\langle\text{out-vac}|\mathcal{T}\left\{\mathcal{O}_1[t_1]\mathcal{O}_2[t_2]\dots\mathcal{O}_n[t_n]\right\}e^{i\Sigma}|\text{out-vac}\rangle}{e^{i\Sigma}\langle\text{out-vac}|\text{out-vac}\rangle} \\ &= \frac{e^{i\Sigma}\langle\text{in-vac}|\mathcal{T}\left\{\mathcal{O}_1[t_1]\mathcal{O}_2[t_2]\dots\mathcal{O}_n[t_n]\right\}|\text{in-vac}\rangle}{e^{i\Sigma}\langle\text{in-vac}|\text{in-vac}\rangle},\end{aligned}$$

which yields the LHS of eq. (2.2.2) as long as the vacuum state is normalized to unity. (Actually, the $|\Omega'\rangle$ can be the vacuum at *any* time – can you see why?) This discussion is important for the upcoming discussion in the next section, because the right hand side of eq. (2.2.2) can be expressed as ratios of path integrals, allowing us to use path integrals to compute expectation values – at least when H is time independent. Up to an overall constant $\mathcal{A}$, we have

$$\langle\text{out-vac}|\mathsf{T}\left\{\mathcal{O}_1[t_1]\mathcal{O}_2[t_2]\dots\mathcal{O}_n[t_n]\right\}|\text{in-vac}\rangle = \mathcal{A}\int\mathcal{D}\varphi\exp\left[iS[\varphi]\right]\mathcal{O}_1[t_1], \mathcal{O}_2[t_2], \dots, \mathcal{O}_n[t_n],$$
$$\langle\text{out-vac}|\text{in-vac}\rangle = \mathcal{A}\int\mathcal{D}\varphi\exp\left[iS[\varphi]\right]. \tag{2.2.3}$$

Schwinger-Keldysh/"in-in" When H is *not* time independent, then energy eigenstates are no longer stationary states – in general we may only speak about the instantaneous eigenstates of $H[t]$ at a given time. In particular, the $|\text{in-vac}\rangle$ is no longer some phase multiplied by the $|\text{out-vac}\rangle$ and the argument above, relating the ratio of the path integrals to the in-vacuum expectation value, will not go through because time evolution is now considerably more complicated. In terms of the now highly non-trivial $U = \mathsf{T}\exp(-i\int_{-\infty}^{+\infty} H[s]\mathrm{d}s)$,

$$\begin{aligned}
|\text{in-vac}\rangle &= U\,|\text{out-vac}\rangle \qquad (2.2.4)\\
\langle\text{out-vac}|\mathsf{T}\left\{\mathcal{O}_1[t_1]\mathcal{O}_2[t_2]\dots\mathcal{O}_n[t_n]\right\}|\text{in-vac}\rangle &= \langle\text{in-vac}|U\mathsf{T}\left\{\mathcal{O}_1[t_1]\mathcal{O}_2[t_2]\dots\mathcal{O}_n[t_n]\right\}|\text{in-vac}\rangle\\
\frac{\langle\text{out-vac}|\mathsf{T}\left\{\mathcal{O}_1[t_1]\mathcal{O}_2[t_2]\dots\mathcal{O}_n[t_n]\right\}|\text{in-vac}\rangle}{\langle\text{out-vac}|\,\text{in-vac}\rangle} &= \frac{\langle\text{in-vac}|U\mathsf{T}\left\{\mathcal{O}_1[t_1]\mathcal{O}_2[t_2]\dots\mathcal{O}_n[t_n]\right\}|\text{in-vac}\rangle}{\langle\text{in-vac}\,|U|\,\text{in-vac}\rangle}\\
&\neq \frac{\langle\text{in-vac}|\mathsf{T}\left\{\mathcal{O}_1[t_1]\mathcal{O}_2[t_2]\dots\mathcal{O}_n[t_n]\right\}|\text{in-vac}\rangle}{\langle\text{in-vac}|\,\text{in-vac}\rangle}.
\end{aligned}$$

On the other hand, the more restricted problem of computing the vacuum expectation value of some operator $O[t]$ at some time t, namely

$$\langle\text{in-vac}\,|O[t]|\,\text{in-vac}\rangle\,, \qquad (2.2.5)$$

may still be tackled in terms of path integrals – two copies of them in fact – but the ensuing formalism is different from the above "in-out" one. It is instead known as the Schwinger-Keldysh or "in-in" formalism, named after Julian Schwinger and Leonid Keldysh; though if the path integral formalism is used it should really be known as the Feynman-Vernon method.

2.2.2 Correlations, Wick's Theorem & Path Integrals

One of the key objects that occur in QFT is the $n \geq 1$ point correlation function

$$\langle\Omega\,|\mathsf{T}\;\varphi(x_1)\dots\varphi(x_n)|\,\Omega\rangle\,. \qquad (2.2.6)$$

We will find it easier to work in Fourier spacetime.

$$\langle\widetilde{\varphi}(k_1)\dots\widetilde{\varphi}(k_n)\rangle \equiv \int \mathrm{d}^d x_1 \cdots \int \mathrm{d}^d x_n\,\langle\Omega\,|\mathsf{T}\;\varphi(x_1)\dots\varphi(x_n)|\,\Omega\rangle\, e^{ik_1\cdot x_1}\cdots e^{ik_n\cdot x_n}. \qquad (2.2.7)$$

To begin, let us remind ourselves of the following fact from QM:

$$\begin{aligned}
\int_{\vec{x}'}^{\vec{x}}\mathcal{D}\vec{q}\int\mathcal{D}\vec{p}\exp\left(i\int_{t'}^{t}\left\{\vec{p}\cdot\dot{\vec{q}} - H[\vec{q},\vec{p}]\right\}\mathrm{d}\tau\right)A(s)B(s')C(s'')\dots \qquad (2.2.8)\\
= {}_{\mathrm{H}}\langle\vec{x},t\,|\mathsf{T}\left\{A_{\mathrm{H}}(s)B_{\mathrm{H}}(s')C_{\mathrm{H}}(s'')\dots\right\}|\,\vec{x}',t'\rangle_{\mathrm{H}}\,; \qquad (2.2.9)
\end{aligned}$$

where on the right hand side A_{H}, B_{H}, C_{H}, ... are arbitrary Heisenberg-picture operators diagonal in the position representation; and A, B, C, ... on the left-hand-side correspond to their complex-number counterparts; and $t > (s, s', s'', \dots) > t'$. Whereas, T means the operators within the curly brackets are arranged such that the ones with later times stand on the right. For instance, if $s_2 > s_1$, then

$$\mathsf{T}\left\{A_{\mathrm{H}}(s_1)B_{\mathrm{H}}(s_2)\right\} = B_{\mathrm{H}}(s_2)A_{\mathrm{H}}(s_1); \qquad (2.2.10)$$

or, if $s'' > s > s'$, then

$$\mathsf{T}\left\{A_{\mathrm{H}}(s)B_{\mathrm{H}}(s')C_{\mathrm{H}}(s'')\right\} = C_{\mathrm{H}}(s'')A_{\mathrm{H}}(s)B_{\mathrm{H}}(s'). \qquad (2.2.11)$$

Nonlinearities We may now proceed to construct perturbation theory. One simply breaks up the action into quadratic-in-fields piece S_2 plus the higher-than-quadratic-in-fields $S_{n\geq 3}$, followed by Taylor expanding the exponential in powers of the latter.

The general n point function with respect to the *full interacting vacuum* is now

$$\langle\Omega|\mathsf{T}\varphi(z_1)\dots\varphi(z_n)|\Omega\rangle = \frac{\int\mathcal{D}\varphi\varphi(z_1)\dots\varphi(z_n)\exp\left[i\int\mathrm{d}^dx\left(\frac{1}{2}(\partial\varphi)^2-\frac{m^2}{2}\varphi^2\right)+i\sum_{n\geq 3}S_n[\varphi]\right]}{\int\mathcal{D}\varphi\exp\left[i\int\mathrm{d}^dx\left(\frac{1}{2}(\partial\varphi)^2-\frac{m^2}{2}\varphi^2\right)+i\sum_{n\geq 3}S_n[\varphi]\right]}. \tag{2.2.12}$$

We will again employ Schwinger's source trick, by considering the following generating functional.

$$Z[J]\equiv\frac{\int\mathcal{D}\varphi\exp\left[i\int\mathrm{d}^dx\left(\frac{1}{2}(\partial\varphi)^2-\frac{m^2}{2}\varphi^2+J\cdot\varphi\right)+i\sum_{n\geq 3}S_n[\varphi]\right]}{\int\mathcal{D}\varphi\exp\left[i\int\mathrm{d}^dx\left(\frac{1}{2}(\partial\varphi)^2-\frac{m^2}{2}\varphi^2\right)+i\sum_{n\geq 3}S_n[\varphi]\right]} \tag{2.2.13}$$

The general n point function may again be obtained from differentiation.

$$\langle\Omega|\mathsf{T}\varphi(z_1)\dots\varphi(z_n)|\Omega\rangle = \left.\frac{\delta^n Z}{\delta(iJ(z_1))\delta(iJ(z_2))\dots\delta(iJ(z_n)))}\right|_{J=0} \tag{2.2.14}$$

Another way to see this is

$$Z[J]=\sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{z_1\dots z_n}J(z_1)\dots J(z_n)\left\langle\Omega\right|\mathsf{T}\varphi(z_1)\dots\varphi(z_n)\left|\Omega\right\rangle. \tag{2.2.15}$$

This $n-$point function will turn out to be

$$\langle\Omega|\mathsf{T}\varphi(z_1)\dots\varphi(z_n)|\Omega\rangle = \sum(\text{Fully Contracted Feynman Diagrams involving the }\{z_1,\dots,z_n\}). \tag{2.2.16}$$

Now, the general path integral with external sources may be shown to be the functional determinant $\mathcal{N}$ of the kinetic term of the action times the exponential of the 'vacuum bubbles' (related to vacuum energy) times the exponential of the sum of fully connected Feynman diagrams involving the external sources. For e.g.,

$$\int\mathcal{D}\varphi\exp\left[iS[\varphi]+i\int J\cdot\varphi\mathrm{d}^dx\right]=\mathcal{N}\exp\left[\sum(\text{vacuum bubbles})\right] \tag{2.2.17}$$
$$\times\exp\left[\sum(\text{Fully connected Feynman diagrams involving }J)\right]$$

Therefore, when taking the ratio of path integrals, only the fully connected diagrams remain.

$$Z[J]=\exp\left[\sum(\text{Fully connected Feynman diagrams involving }J)\right] \tag{2.2.18}$$

A slightly different way to arrive at this result is to simply split

$$Z[J]\equiv\frac{\int\mathcal{D}\varphi\exp\left[i\int\mathrm{d}^dx\left(\frac{1}{2}(\partial\varphi)^2-\frac{m^2}{2}\varphi^2+J\cdot\varphi\right)+i\sum_{n\geq 3}S_n[\varphi]\right]}{\int\mathcal{D}\varphi\exp\left[i\int\mathrm{d}^dx\left(\frac{1}{2}(\partial\varphi)^2-\frac{m^2}{2}\varphi^2\right)\right]} \tag{2.2.19}$$

$$\times\frac{\int\mathcal{D}\varphi\exp\left[i\int\mathrm{d}^dx\left(\frac{1}{2}(\partial\varphi)^2-\frac{m^2}{2}\varphi^2\right)\right]}{\int\mathcal{D}\varphi\exp\left[i\int\mathrm{d}^dx\left(\frac{1}{2}(\partial\varphi)^2-\frac{m^2}{2}\varphi^2\right)+i\sum_{n\geq 3}S_n[\varphi]\right]}. \tag{2.2.20}$$

This will translate into

$$Z[J] = \frac{\exp\left[\sum(\text{Vacuum Bubbles})\right]\exp\left[\sum(\text{Full Connected Feynman diagrams involving } J)\right]}{\exp\left[\sum(\text{Vacuum Bubbles})\right]}$$
$$= \exp\left[\sum(\text{Full Connected Feynman diagrams involving } J)\right]. \tag{2.2.21}$$

For example, for the free massive scalar theory, the only Feynman diagram was the one involving

$$\langle 0|\mathsf{T}\frac{1}{2!}\int iJ_1\varphi_1\int iJ_2\varphi_2|0\rangle = \frac{1}{2!}\int_{1,2} iJ_1 G_{\mathrm{F}}^{1,2} iJ_2. \tag{2.2.22}$$

Example: φ^4 Theory A common textbook example involves φ^4 theory, defined by the Lagrangian

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4, \tag{2.2.23}$$
$$\mathcal{L}_2 = \frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2, \tag{2.2.24}$$
$$\mathcal{L}_4 = -\frac{\lambda}{4!}\varphi^4. \tag{2.2.25}$$

In 4D the Lagrangian density should be of mass dimensions 4, so that it then yields a dimensionless quantity when it is integrated over 4D spacetime (i.e., $\mathrm{d}^4x\mathcal{L}$ is dimensionless; or, of dimensions $\hbar$). Hence $[\varphi] = m$ and therefore λ is dimensionless.

Let us first observe that the odd point function vanishes:

$$\langle\Omega|\,\mathsf{T}\varphi_1\varphi_2\dots\varphi_{2n+1}\,|\Omega\rangle \propto \int D\varphi\varphi_1\dots\varphi_{2n+1}e^{i\int\mathcal{L}_2}\sum_\ell(-i\lambda/4!)^\ell\left(\int\varphi^4\right)^\ell = 0 \tag{2.2.26}$$

because the integrand always involves the Wick contractions of odd powers of φ.

Next, we turn to the two point function up to first order in λ:

$$\langle\Omega|\,\mathsf{T}\varphi_1\varphi_2\,|\Omega\rangle \tag{2.2.27}$$
$$= \frac{\int D\varphi\varphi_1\varphi_2 e^{i\int\mathcal{L}_2}(1 - i\lambda/4!\int_z\varphi_z^4 + \mathcal{O}[\lambda^2])}{\int D\varphi e^{i\int\mathcal{L}_2}(1 - i\lambda/4!\int_z\varphi_z^4 + \mathcal{O}[\lambda^2])} \tag{2.2.28}$$
$$= \frac{G_F^{x_1,x_2}(1 - i(\lambda/4!)3\int_z G_F^{z,z}G_F^{z,z}) - i(\lambda/4!)4\cdot 3\int_z G_F^{x_1,z}G_F^{x_2,z}G_F^{z,z}}{1 - i(\lambda/4!)3\int_z G_F^{z,z}G_F^{z,z}} + \mathcal{O}[\lambda^2] \tag{2.2.29}$$
$$= G_F^{x_1,x_2} - i\frac{\lambda}{2}\int_z G_F^{x_1,z}G_F^{x_2,z}G_F^{z,z} + \mathcal{O}[\lambda^2] \tag{2.2.30}$$

These are fully connected diagrams. An example of a vacuum bubble is the figure 8 term

$$-(i\lambda/4!)3\int_z G_F^{z,z}G_F^{z,z}. \tag{2.2.31}$$

Notice the vacuum bubbles really do cancel out at first order in λ.

The four point function up to λ:

$$\langle\Omega|\,\mathsf{T}\varphi_1\varphi_2\varphi_3\varphi_4\,|\Omega\rangle \tag{2.2.32}$$

$$= \frac{\int D\varphi\varphi_1\varphi_2\varphi_3\varphi_4 e^{i\int\mathcal{L}_2}(1-i\lambda/4!\int_z\varphi_z^4+\mathcal{O}[\lambda^2])}{\int D\varphi e^{i\int\mathcal{L}_2}(1-i\lambda/4!\int_z\varphi_z^4+\mathcal{O}[\lambda^2])} \tag{2.2.33}$$

$$= \Bigg((G_F^{x_1,x_2}G_F^{x_3,x_4}+G_F^{x_1,x_3}G_F^{x_2,x_4}+G_F^{x_1,x_4}G_F^{x_2,x_3})(1-i(\lambda/4!)3\int_z G_F^{z,z}G_F^{z,z})$$

$$- i(\lambda/4!)4!\int_z G_F^{x_1,z}G_F^{x_2,z}G_F^{x_3,z}G_F^{x_4,z} \tag{2.2.34}$$

$$- i(\lambda/4!)G_F^{x_1,x_2}4\cdot 3\int_z G_F^{x_3,z}G_F^{x_4,z}G_F^{z,z} \tag{2.2.35}$$

$$- i(\lambda/4!)G_F^{x_1,x_3}4\cdot 3\int_z G_F^{x_2,z}G_F^{x_4,z}G_F^{z,z} - i(\lambda/4!)G_F^{x_1,x_4}4\cdot 3\int_z G_F^{x_2,z}G_F^{x_3,z}G_F^{z,z} \tag{2.2.36}$$

$$- i(\lambda/4!)G_F^{x_2,x_3}4\cdot 3\int_z G_F^{x_1,z}G_F^{x_4,z}G_F^{z,z} - i(\lambda/4!)G_F^{x_2,x_4}4\cdot 3\int_z G_F^{x_1,z}G_F^{x_3,z}G_F^{z,z} \tag{2.2.37}$$

$$- i(\lambda/4!)G_F^{x_3,x_4}4\cdot 3\int_z G_F^{x_1,z}G_F^{x_2,z}G_F^{z,z}\Bigg)\left(1-i(\lambda/4!)3\int_z G_F^{z,z}G_F^{z,z}\right)^{-1}+\mathcal{O}[\lambda^2] \tag{2.2.38}$$

$$= G_F^{x_1,x_2}G_F^{x_3,x_4}+G_F^{x_1,x_3}G_F^{x_2,x_4}+G_F^{x_1,x_4}G_F^{x_2,x_3}-i\lambda\int_z G_F^{x_1,z}G_F^{x_2,z}G_F^{x_3,z}G_F^{x_4,z} \tag{2.2.39}$$

$$- i\frac{\lambda}{2}\Bigg(G_F^{x_1,x_2}\int_z G_F^{x_3,z}G_F^{x_4,z}G_F^{z,z}+G_F^{x_1,x_3}\int_z G_F^{x_2,z}G_F^{x_4,z}G_F^{z,z} \tag{2.2.40}$$

$$+ G_F^{x_1,x_4}\int_z G_F^{x_2,z}G_F^{x_3,z}G_F^{z,z}+G_F^{x_2,x_3}\int_z G_F^{x_1,z}G_F^{x_4,z}G_F^{z,z} \tag{2.2.41}$$

$$+ G_F^{x_2,x_4}\int_z G_F^{x_1,z}G_F^{x_3,z}G_F^{z,z}+G_F^{x_3,x_4}\int_z G_F^{x_1,z}G_F^{x_2,z}G_F^{z,z}\Bigg)+\mathcal{O}[\lambda^2]. \tag{2.2.42}$$

The second term involving $-i\lambda$ is fully connected; the rest are not. At first order in λ, it is not possible to form a fully connected diagram at 6−points, because the $\lambda\varphi^4$ term is a 4−point interaction.

We may gather

$$Z[J]-1 = \frac{i^2}{2!}\int_{x_1,x_2}J(x_1)J(x_2)\left(G_F^{x_1,x_2}-i\frac{\lambda}{2}\int_z G_F^{x_1,z}G_F^{x_2,z}G_F^{z,z}\right) \tag{2.2.43}$$

$$+\frac{i^4}{4!}\int_{x_1,x_2,x_3,x_4}J(x_1)J(x_2)J(x_3)J(x_4)\Bigg(-i\lambda\int_z G_F^{x_1,z}G_F^{x_2,z}G_F^{x_3,z}G_F^{x_4,z} \tag{2.2.44}$$

$$3G_F^{x_1,x_2}G_F^{x_3,x_4}-i\frac{\lambda}{2}6G_F^{x_1,x_2}\int_z G_F^{x_3,z}G_F^{x_4,z}G_F^{z,z}\Bigg)+\mathcal{O}[J^4,\lambda^2] \tag{2.2.45}$$

$$= \frac{1}{2}\int_{x_1,x_2}iJ(x_1)G_F^{x_1,x_2}iJ(x_2)-i\frac{\lambda}{4}\int_{x_1,x_2}iJ(x_1)iJ(x_2)\int_z G_F^{x_1,z}G_F^{x_2,z}G_F^{z,z} \tag{2.2.46}$$

$$+\int_{x_1,x_2,x_3,x_4} iJ(x_1)iJ(x_2)iJ(x_3)iJ(x_4)\Bigg(-i\frac{\lambda}{4!}\int_z G_F^{x_1,z}G_F^{x_2,z}G_F^{x_3,z}G_F^{x_4,z} \tag{2.2.47}$$

$$+\frac{1}{8}G_F^{x_1,x_2}G_F^{x_3,x_4} - i\frac{\lambda}{8}G_F^{x_1,x_2}\int_z G_F^{x_3,z}G_F^{x_4,z}G_F^{z,z}\Bigg) + \mathcal{O}[J^4,\lambda^2] \tag{2.2.48}$$

Note that, up to order J^4 and λ^1,

$$\frac{1}{2!}\Bigg(\int_{x_1,x_2} iJ(x_1)iJ(x_2)\left(\frac{1}{2}G_F^{x_1,x_2} - i\frac{\lambda}{4}\int_z G_F^{x_1,z}G_F^{x_2,z}G_F^{z,z}\right)$$

$$-\frac{i\lambda}{4!}\int_{x_1,x_2,x_3,x_4} iJ(x_1)iJ(x_2)iJ(x_3)iJ(x_4)\int_z G_F^{x_1,z}G_F^{x_2,z}G_F^{x_3,z}G_F^{x_4,z} + \mathcal{O}[J^4,\lambda^2]\Bigg)^2 \tag{2.2.49}$$

$$= \frac{1}{8}\left(\int_{x_1,x_2} iJ(x_1)iJ(x_2)G_F^{x_1,x_2}\right)^2 - i\frac{\lambda}{8}\int_{x_1,x_2} iJ(x_1)iJ(x_2)G_F^{x_1,x_2}\int_{x_3,x_4} iJ(x_3)iJ(x_4)\int_z G_F^{x_3,z}G_F^{x_4,z}G_F^{z,z}.$$

We have therefore verified that – up to order J^4 and λ^1 –

$$Z[J] = \exp\Bigg[\int_{x_1,x_2} iJ(x_1)iJ(x_2)\left(\frac{1}{2}G_F^{x_1,x_2} - i\frac{\lambda}{4}\int_z G_F^{x_1,z}G_F^{x_2,z}G_F^{z,z}\right)$$

$$-\frac{i\lambda}{4!}\int_{x_1,x_2,x_3,x_4} iJ(x_1)iJ(x_2)iJ(x_3)iJ(x_4)\int_z G_F^{x_1,z}G_F^{x_2,z}G_F^{x_3,z}G_F^{x_4,z} + \mathcal{O}[J^4,\lambda^2]\Bigg]. \tag{2.2.50}$$

Summary To sum: the $n-$point function may be computed perturbatively as the sum of all Feynman diagrams without any disconnected vacuum bubbles:

$$\langle\Omega\,|\mathsf{T}\{\varphi(x_1)\dots\varphi(x_n)\}|\,\Omega\rangle$$

$$= \sum_{\ell=0}^{\infty}\frac{i^\ell}{\ell!}\int_{z_1\dots z_\ell}\langle 0\,|\mathsf{T}\{\varphi(x_1)\dots\varphi(x_n)\mathcal{L}_{\mathrm{I}}[z_1]\dots\mathcal{L}_{\mathrm{I}}[z_n]\}|\,0\rangle_{\text{No disconnected vacuum bubbles}} \tag{2.2.51}$$

and

$$Z[J] = \exp\left[\sum(\text{Fully connected diagrams involving } J)\right]. \tag{2.2.52}$$

Note that:

- Generically, Feynman diagrams diverge at loop level. Hence, a consistent regularization and renormalization scheme is part of the *definition* of a QFT, in order to ensure finite physical observables can be extracted.
- Even after regularization and renormalization, perturbation theory – e.g., in powers of λ in the $\varphi-4$ theory above, or α in QED – the series is asymptotic, with zero radius of convergence. For some fixed (Nature-given) coupling constant λ, α, etc., the series needs to be truncated beyond some perturbative order; else the answer is no longer a good approximation.

Problem 2.28. 1D Euclidean φ^4 QFT Compute the 2 point Euclidean correlator:

$$\langle\varphi(x_1)\varphi(x_2)\rangle \equiv \frac{\int \mathcal{D}\varphi\varphi(x_1)\varphi(x_2)\exp\left(-S_{\rm E}\right)}{\int \mathcal{D}\varphi\exp\left(-S_{\rm E}\right)}, \tag{2.2.53}$$

$$S_{\rm E} \equiv \int_{\mathbb{R}} \mathrm{d}x \left(\frac{1}{2}\varphi'(x)^2 + \frac{m^2}{2}\varphi(x)^2 + \frac{\lambda}{4!}\varphi(x)^4\right); \tag{2.2.54}$$

up to $\mathcal{O}(\lambda)$. You should find

$$\langle\varphi(x_1)\varphi(x_2)\rangle = \frac{e^{-m|x_1-x_2|}}{2m}\left(1 - \frac{\lambda}{8m^3}\left(1+m|x_1-x_2|\right) + \mathcal{O}(\lambda^2)\right). \tag{2.2.55}$$

Note that, even though this is a Euclidean QFT, the perturbation theory is similar to that in Lorentzian QFT. Hint: First solve for the *non-interacting* two point function – it is $D_{\rm E}(x) = \int_{\mathbb{R}}(\mathrm{d}\omega/(2\pi))e^{-i\omega z}/(\omega^2+m^2) = e^{-m|x|}/(2m)$. □

$\hbar$ Power Counting Generally, we would attribute higher powers of $\hbar$ to indicate a higher degree of "quantum-ness". Let us therefore count the powers of $\hbar$ in the $n-$point correlations within scalar field theory by studying the generating function

$$Z[J] = \frac{\int D\varphi \exp[(i/\hbar)\int(1/2)((\partial\varphi)^2 - m^2\varphi^2) + (i/\hbar)\sum_{n\geq 3} S_n + i\int J\varphi]}{\int D\varphi \exp[(i/\hbar)\int(1/2)((\partial\varphi)^2 - m^2\varphi^2) + (i/\hbar)\sum_{n\geq 3} S_n]}, \tag{2.2.56}$$

where we have restored the $1/\hbar$ in front of each term of the action; though we *do not* put a $1/\hbar$ for the $J\varphi$ term since our main goal is to read off the correlator as the coefficient of the Js, and carrying extra factors of $\hbar$s does not affect the result and is in fact merely a distraction – i.e., compare $(i/n!)(J_1/\hbar)\dots(J_n/\hbar)\langle\Omega|\mathsf{T}\varphi_1\dots\varphi_n|\Omega\rangle$ with $(i/n!)(J_1\dots J_n)\langle\Omega|\mathsf{T}\varphi_1\dots\varphi_n|\Omega\rangle$. Now, let us note that each and every φ occurring within the beyond-quadratic action can be obtained by a functional derivative acting on the exponential $e^{i\int J\varphi}$:

$$S_{n\geq 3}[\varphi]e^{i\int J\varphi} = S_n\left[\frac{\delta}{\delta(iJ)}\right]e^{i\int J\varphi}. \tag{2.2.57}$$

For example, the φ^4 example from above would read

$$\int_z \left(\frac{\delta}{\delta(iJ_z)}\right)^4 e^{i\int_x J_x\varphi_x} \tag{2.2.58}$$

$$= \int_z \int_{x_1} \delta_{x_1-z}\varphi_{x_1} \int_{x_2} \delta_{x_2-z}\varphi_{x_2} \int_{x_3} \delta_{x_3-z}\varphi_{x_3} \int_{x_4} \delta_{x_4-z}\varphi_{x_4} e^{i\int_x J_x\varphi_x} \tag{2.2.59}$$

$$= \int_z \varphi_z^4 e^{i\int_x J_x\varphi_x}. \tag{2.2.60}$$

Hence,

$$Z[J] = \frac{\exp[\frac{i}{\hbar}\sum_{n\geq 3} S_n[\frac{\delta}{\delta(iJ)}]]\int D\varphi\exp[\frac{i}{\hbar}\int\frac{1}{2}((\partial\varphi)^2 - m^2\varphi^2) + i\int J\varphi]}{(\exp[\frac{i}{\hbar}\sum_{n\geq 3} S_n[\frac{\delta}{\delta(iJ)}]]\int D\varphi\exp[\frac{i}{\hbar}\int\frac{1}{2}((\partial\varphi)^2 - m^2\varphi^2) + i\int J\varphi])_{J=0}}, \tag{2.2.61}$$

where we have now introduced a J in the denominator as well – we just need to set it to zero at the end of the calculation. Next, we complete the square, but keep in mind that each factor of the Feynman Green's function is multiplied by an $\hbar$, namely $\langle 0| \mathsf{T}\varphi_x\varphi_{x'} |0\rangle = \hbar G_{\mathrm{F}}[x-x']$:

$$Z[J] = \frac{\exp[\frac{i}{\hbar}\sum_{n\geq 3} S_n[\frac{\delta}{\delta(iJ)}]] \exp\left[\frac{\hbar}{2}\int_{x,x'} iJ(x) G_{\mathrm{F}}(x-x') iJ(x')\right]}{\left(\exp[\frac{i}{\hbar}\sum_{n\geq 3} S_n[\frac{\delta}{\delta(iJ)}]] \exp\left[\frac{\hbar}{2}\int_{x,x'} iJ(x) G_{\mathrm{F}}(x-x') iJ(x')\right]\right)_{J=0}}. \tag{2.2.62}$$

The $\hbar$ power counting is now manifest: Each $n-$vertex carries a $1/\hbar$ whereas each internal line (i.e., a propagator) contains a $\hbar$. At this point, we invoke the following formula. If L is the number of loops, P is the number of *internal* propagator lines, and V is the number of vertices in each Feynman diagram, they are related by

$$L = P - V + 1. \tag{2.2.63}$$

For example, the $n=2$ point function in our φ^4 theory at one-loop order contains $P=1$, $V=1$; the $n=4$ point function in the same theory at one loop order also has $P=1$ and $V=1$. A Feynman digram contributing to a given correlator would therefore contain the product of coupling constants associated with each and every vertex times

$$\hbar^{\#\text{ external propagators}} \cdot \hbar^P \cdot \hbar^{-V} = \hbar^{\#\text{ external propagators}} \cdot \hbar^{L-1}. \tag{2.2.64}$$

This is why the loop expansion is oftentimes associated with an expansion in powers of $\hbar$s. Notice, though, that the coupling constants associated with each vertex could in principle contain $\hbar$s. Moreover, the massive propagator itself does not scale homogeneously with $\hbar$ too: $\hbar\widetilde{G}_{\mathrm{F}}(k) = i\hbar/(k^2 - (m/\hbar)^2)$.

Problem 2.29. Two-Point Function of φ^4 theory Verify directly that the two point function of φ^2 theory up to one loop order goes as $\hbar G_{\mathrm{F}} + \hbar^2\lambda(\dots)$, where $\dots$ should be spelted out explicitly. □

N Point Functions in Fourier Spacetime To prepare for the following section on the scattering matrix, we wish to now re-express the n point function in Fourier space. We see that

$$\langle\widetilde{\varphi}(k_1)\dots\widetilde{\varphi}(k_n)\rangle = \frac{\int D\varphi \widetilde{\varphi}(k_1)\cdots\widetilde{\varphi}(k_n)\exp(iS)}{\int D\varphi \exp(iS)} \tag{2.2.65}$$

$$= \frac{\int D\varphi \frac{\delta}{\delta(i\widetilde{J}(-k_1))}\cdots\frac{\delta}{\delta(i\widetilde{J}(-k_n))}\exp\left(iS + i\int_{k'}\widetilde{\varphi}(k')\widetilde{J}(-k')\right)\Big|_{J=0}}{\int D\varphi \exp(iS)}. \tag{2.2.66}$$

Each interaction vertex can first be re-expressed in momentum spacetime. For example, the φ^4 self-interaction becomes

$$\int_z \varphi_z^4 e^{i\int_z J_z\varphi_z} \tag{2.2.67}$$

$$= \int_{k_1,k_2,k_3,k_4} \widetilde{\varphi}(k_1)\widetilde{\varphi}(k_2)\widetilde{\varphi}(k_3)\widetilde{\varphi}(k_4)(2\pi)^d\delta^{(d)}[k_1+k_2+k_3+k_4]e^{i\int_{k'}\widetilde{\varphi}(k')\widetilde{J}(-k')} \tag{2.2.68}$$

$$= \int_{k_1,k_2,k_3,k_4} (2\pi)^d\delta^{(d)}[k_1+k_2+k_3+k_4]\frac{\delta}{\delta(i\widetilde{J}(-k_1))}\cdots\frac{\delta}{\delta(i\widetilde{J}(-k_4))}e^{i\int_{k'}\widetilde{\varphi}(k')\widetilde{J}(-k')}. \tag{2.2.69}$$

We may therefore write the generating function as

$$Z[J] = \exp\left[i\sum_{n\geq 3}\int_{k_1,\dots,k_n}\delta_{k_1+\dots+k_n}\widetilde{\mathcal{L}}\left[\frac{\delta}{\delta(i\widetilde{J}(-k_1))},\dots,\frac{\delta}{\delta(i\widetilde{J}(-k_n))}\right]\right] \times \exp\left[\frac{1}{2}\int_{q,q'}(i\widetilde{J}(-q))\frac{i\delta_{q+q'}}{q^2-m^2+i\epsilon}(i\widetilde{J}(-q'))\right] \tag{2.2.70}$$

$$\times\left(\exp\left[i\sum_{n\geq 3}\int_{k_1,\dots,k_n}\delta_{k_1+\dots+k_n}\widetilde{\mathcal{L}}\left[\frac{\delta}{\delta(i\widetilde{J}(-k_1))},\dots,\frac{\delta}{\delta(i\widetilde{J}(-k_n))}\right]\right] \times \exp\left[\frac{1}{2}\int_{q,q'}(i\widetilde{J}(-q))\frac{i\delta_{q+q'}}{q^2-m^2+i\epsilon}(i\widetilde{J}(-q'))\right]\right)^{-1}_{J=0}, \tag{2.2.71}$$

where δ_k is shorthand for $(2\pi)^d\delta^{(d)}(k)$. Note that

$$\frac{\delta\widetilde{J}(q)}{\delta\widetilde{J}(-k)} = \delta_{q+k} \qquad \text{and} \qquad \frac{\delta\widetilde{J}(q)}{\delta\widetilde{J}(k)} = \delta_{q-k}. \tag{2.2.72}$$

We deduce: Each line in the Feynman diagram is now $i/(p^2-m^2+i\epsilon)$ times a $\delta-$function; whereas each $n-$vertex comes with a momentum conserving $\delta-$function and associated $n-$integrals.

For instance, in φ^4 theory, the 2 point function with external momentum q_1 and q_2 up to order λ therefore

$$\langle\widetilde{\varphi}_{q_1}\widetilde{\varphi}_{q_2}\rangle \tag{2.2.73}$$

$$= \frac{i\delta_{q_1+q_2}}{q_1^2-m^2+i\epsilon} - \frac{i\lambda}{2}\int_{k_1,\dots,k_4}\delta_{k_1+k_2+k_3+k_4}\frac{i\delta_{q_1+k_1}}{q_1^2-m^2+i\epsilon}\frac{i\delta_{k_3+k_4}}{k_3^2-m^2+i\epsilon}\frac{i\delta_{q_2+k_2}}{q_2^2-m^2+i\epsilon}+\dots \tag{2.2.74}$$

$$= \frac{i\delta_{q_1+q_2}}{q_1^2-m^2+i\epsilon} - \frac{i\lambda}{2}\frac{i}{q_1^2-m^2+i\epsilon}\frac{i\delta_{q_1+q_2}}{q_2^2-m^2+i\epsilon}\int_{k''}\frac{i}{k''^2-m^2+i\epsilon}+\dots \tag{2.2.75}$$

In turn, the $4-$point function reads

$$\langle\widetilde{\varphi}_{k_1}\widetilde{\varphi}_{k_2}\widetilde{\varphi}_{k_3}\widetilde{\varphi}_{k_4}\rangle = \frac{i\delta_{k_1+k_2}}{k_1^2-m^2+i\epsilon}\frac{i\delta_{k_3+k_4}}{k_3^2-m^2+i\epsilon} + \text{distinct perms of } \{k_1,\dots,k_4\} \tag{2.2.76}$$

$$- i\lambda\delta_{k_1+\dots+k_4}\left(\prod_{\ell=1}^{4}\frac{i}{k_\ell^2-m^2+i\epsilon}\right) \tag{2.2.77}$$

We may interpret the first row as simply 'nothing happens'; whereas the first order λ term tells us four φ particles can scatter off each other.

2.2.3 LSZ Reduction Formula

See Chapters 4, 5 and 7 of Peskin and Schroeder. Chapter 4.5 of Peskin and Schroeder (P&S) relates the $S-$matrix to cross sections; whereas 4.6 argues how the $S-$matrix can be computed from Feynman diagrams. Chapter 7.2 relates correlators to the $S-$matrix. This section is only a very rough sketch. The key object is the transition amplitude

$${}_{\text{out}}\langle p_1', s_1'; p_2', s_2'; \dots | \, p_1, s_1; p_2, s_2; \dots\rangle_{\text{in}}\,, \tag{2.2.78}$$

where the ket is the $n-$particle incoming momentum eigenstate and the bra is the $n'-$particle outgoing momentum eigenstate; the s's are the spins. 'In' here simply means the Heisenberg picture eigenstate in the infinite past; while 'out' means the eigenstate in the infinite future. We define

$$_{\text{out}}\langle p_1', s_1'; p_2', s_2'; \dots | \, p_1, s_1; p_2, s_2; \dots \rangle_{\text{in}} , \equiv \, _{\text{s}}\langle p_1', s_1'; p_2', s_2'; \dots | \, S \, | p_1, s_1; p_2, s_2; \dots \rangle_{\text{s}} ; \tag{2.2.79}$$

$$S \equiv \mathbb{I} + iT; \tag{2.2.80}$$

The subscript s indicates "Schrödinger" picture; and at tree level these momentum eigenstates are simply the appropriate raising operators acting upon the non-interacting vacuum $|0\rangle$. The S is known as the $S-$matrix. By momentum conservation, we have

$$\begin{aligned} &_{\text{s}}\langle p_1', s_1'; p_2', s_2'; \dots | \, iT \, | p_1, s_1; p_2, s_2; \dots \rangle_{\text{s}} \\ &= (2\pi)^d \delta^{(d)}[p_1' + p_2' + \dots - (p_1 + p_2 + \dots)] i\mathcal{M}[p_1, s_1; p_2, s_2; \dots \to p_1', s_1'; p_2', s_2'; \dots], \end{aligned} \tag{2.2.81}$$

where $\mathcal{M}$ is usually dubbed the *scattering amplitude*. We may now generalize P&S eq. 4.90 to relate the $T-$matrix elements to Feynman diagrams; up to renormalization $Z-$factors:

$$_{\text{s}}\langle p_1', s_1'; p_2', s_2'; \dots | \, S \, | p_1, s_1; p_2, s_2; \dots \rangle_{\text{s}} \tag{2.2.82}$$

$$= \, _{\text{I}}\langle p_1', s_1'; p_2', s_2'; \dots | \, \mathsf{T} \exp \left(-i \int_{\mathbb{R}} H_{\text{I}}(t') \text{d}t' \right) | p_1, s_1; p_2, s_2; \dots \rangle_{\text{I}} \tag{2.2.83}$$

$$= \, _{\text{I}}\langle p_1', s_1'; p_2', s_2'; \dots | \, \mathbb{I} - \mathsf{T} \sum_{\ell} \frac{1}{\ell!} \left(-i \int_{\mathbb{R}} H_{\text{I}}(t') \text{d}t' \right)^{\ell} | p_1, s_1; p_2, s_2; \dots \rangle_{\text{I}} ; \tag{2.2.84}$$

where the kets and bras with subscript I indicates they are now eigenstates of the quadratic (i.e., non-interacting) Hamiltonian. P&S summarizes the $T-$matrix (S with the identity subtracted out) as

$$_{\text{s}}\langle p_1', s_1'; p_2', s_2'; \dots | \, iT \, | p_1, s_1; p_2, s_2; \dots \rangle_{\text{s}} \tag{2.2.85}$$

$$= \left(_{\text{I}}\langle p_1', s_1'; p_2', s_2'; \dots | \, \mathsf{T} \exp \left(-i \int_{\mathbb{R}} H_{\text{I}}(t') \text{d}t' \right) | p_1, s_1; p_2, s_2; \dots \rangle_{\text{I}} \, .. \right)_{\text{Fully connected, amputated}} .$$

Connected means, any part of the Feynman diagram is connected to any other portion; while amputated means the quantum-corrected Green's functions directly connected to the external lines are discarded. More precisely, P&S tells us

> *. . . we define amputation in the following way. Starting from the tip of each external leg, find the last point at which the diagram can be cut by removing a single propagator, such that this operation separates the leg from the rest of the diagram. Cut there.*

One of the key applications of the $n-$point function involves its relation to the scattering matrix (S-matrix). This is provided by the LSZ reduction formula. From Peskin and Schroeder eq. (7.42), the transition amplitude $\langle p_1, \dots, p_n \, | S | \, k_1, \dots, k_m \rangle$, describing the process of m incoming particles $(k_1, \ldots, k_m)$ transforming to n outgoing ones $(p_1, \ldots, p_n)$, can be read off from

$$\begin{aligned}&\langle \widetilde{\varphi}(p_1)\ldots\widetilde{\varphi}(p_n)\widetilde{\varphi}(-k_1)\ldots\widetilde{\varphi}(-k_m)\rangle \\ &\sim \left(\prod_{\ell}\frac{i\sqrt{Z}}{p_\ell^2-m^2+i\epsilon}\right)\left(\prod_{\ell'}\frac{i\sqrt{Z}}{k_{\ell'}^2-m^2+i\epsilon}\right)\langle p_1,\ldots,p_n\,|S|\,k_1,\ldots,k_m\rangle\,. \end{aligned} \tag{2.2.86}$$

The Z factors arise from renormalization; while the $\sim$ means the expressions are to be taken near the mass shell $p^2 = m^2$. The prescription for the S-matrix for this $m \to n$ particle process is therefore as follows: Compute the $m+n$ point function with the momenta $(p_1, \ldots, p_n, -k_1, \ldots, -k_m)$. Then throw out the 'outermost' layer of quantum-corrected Green's functions; i.e., throw out the $i\sqrt{Z}/(p^2 - m^2 + i\epsilon)$ factors.

The $T-$matrix, or scattering amplitude, is the 'off-diagonal' portion of the above transition amplitude. It is the non-trivial part of scattering that is sensitive to the interactions:

$$S = \mathbb{I} + iT. \tag{2.2.87}$$

From the above expression, we see that it is the sum of fully connected amputated Feynman diagrams involving the appropriate external momentum lines.

$$\begin{aligned}&{}_{\text{s}}\left\langle \{p'_j\}\,|iT|\,\{p_i\}\right\rangle_{\text{s}} \\ &= \sum_{\ell=1}^{\infty}\frac{i^\ell}{\ell!}\int_{z_1\ldots z_\ell}\left({}_{\text{s}}\left\langle \{p'_j\}\,|\mathsf{T}\{\mathcal{L}[z_1]\ldots\mathcal{L}[z_\ell]\}|\,\{p_i\}\right\rangle_{\text{s}}\right)_{\text{Fully connected, Amputated}}\end{aligned} \tag{2.2.88}$$

This formula holds provided we define Wick contractions to also include that with the external momentum eigenstates. For scalars, contraction with external momentum states simply yields unity.

For $\varphi-4$ theory these considerations allow us to summarize the computation of the $T-$matrix via the following Feynman rules (see Peskin and Schroeder Appendix A.1)

- Scalar propagator (line): $i/(p^2 - m^2 + i\epsilon)$.
- φ^4 vertex: $-i\lambda$.
- External scalar: 1.

3 Quantum Field Theory of (3+1)D Spinor Fields

3.1 Quantum Mechanics

The Dirac equation reads

$$(i\partial\!\!\!/ - m)\psi = 0 \tag{3.1.1}$$

where $\partial\!\!\!/ \equiv \gamma^\mu \partial_\mu$, and in the chiral basis

$$\gamma^\mu \equiv \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}, \tag{3.1.2}$$

$$\sigma^\mu \equiv \left(\mathbb{I}_{2\times 2}, \sigma^i\right), \qquad \bar{\sigma}^\mu \equiv \left(\mathbb{I}_{2\times 2}, -\sigma^i\right). \tag{3.1.3}$$

Note that these 4×4 matrices $\{\gamma^\mu\}$ satisfy the following anti-commutation relations, otherwise known as the *Clifford Algebra*:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{3.1.4}$$

We note in passing: in arbitrary $d \geq 2$ spacetime dimensions, whenever a set of matrices $\{\gamma^\mu\}$ are found such they satisfy the Clifford algebra, then $(i/2)[\gamma^\mu, \gamma^\nu]$ may be used to generate the Lorentz transformations of spinors.

Let us note that the Dirac Lagrangian density is

$$\mathcal{L}_\psi \equiv \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi. \tag{3.1.5}$$

Here, we have defined

$$\bar{\psi} \equiv \psi^\dagger \gamma^0. \tag{3.1.6}$$

which yields

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi^\dagger} = 0 \tag{3.1.7}$$

$$\frac{\partial \mathcal{L}}{\partial \psi^\dagger} = i\gamma^0\gamma^\mu\partial_\mu\psi - m\gamma^0\psi; \tag{3.1.8}$$

and

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} = i\partial_\mu\psi^\dagger\gamma^0\gamma^\mu \tag{3.1.9}$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = -m\psi^\dagger\gamma^0; \tag{3.1.10}$$

i.e., we have

$$i\gamma^0\gamma^\mu\partial_\mu\psi - m\gamma^0\psi = 0 \tag{3.1.11}$$

$$(i\gamma^\mu\partial_\mu - m)\psi = 0. \tag{3.1.12}$$

Problem 3.1. Hermitian Lagrangian Explain why

$$(\gamma^0\gamma^\mu)^\dagger = (\gamma^\mu)^\dagger\gamma^0 = \gamma^0\gamma^\mu. \tag{3.1.13}$$

Show that the following Lagrangian

$$\mathcal{L}'_\psi = \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{i}{2}\partial_\mu\bar{\psi}\gamma^\mu\psi - m\bar{\psi}\psi \tag{3.1.14}$$

is Hermitian and also yields the Dirac equation in eq. (3.1.1). □

Problem 3.2. Stress-Energy Tensor Show that the Noether current of spacetime translation symmetry generated from eq. (3.1.5) is, up to an additive divergence-free term,

$$T^\mu{}_\nu = i\bar{\psi}\gamma^\mu\partial_\nu\psi. \tag{3.1.15}$$

Next show that the Noether current of spacetime translation symmetry generated from eq. (3.1.14) is, up to an additive divergence-free term,

$$T'^\mu{}_\nu = \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\nu\psi - \frac{i}{2}\partial_\nu\bar{\psi}\gamma^\mu\psi. \tag{3.1.16}$$

In both cases, assume the Dirac equation $(i\partial\!\!\!/ - m)\psi = 0$ is satisfied. Comment on the physical differences between the two results. □

Problem 3.3. Global U$_1$ Symmetry Verify that the Dirac Lagrangian in eq. (3.1.5) is invariant under the global U_1 transformation

$$\psi \to e^{i\vartheta}\psi, \tag{3.1.17}$$

where by global we mean θ does not depend on spacetime. Next, compute that the associated Noether current and show that is (up to overall constant factors)

$$J^\mu = \bar{\psi}\gamma^\mu\psi. \tag{3.1.18}$$

□

3.1.1 Invariants and Covariants

Left- and Right-Handed Spinors

Tensors from Spinors in (3+1)D we may identify the following scalar/tensor objects, built out of fermion bilinears.

$$\bar{\psi}\psi \text{ (Scalar)}, \qquad \bar{\psi}\gamma^\mu\psi \text{ (Vector)}, \qquad \bar{\psi}\gamma^{[\mu}\gamma^{\nu]}\psi \text{ (Rank-2 tensor)}, \tag{3.1.19}$$

$$\bar{\psi}\gamma^\mu\gamma^5\psi \text{ (Pseudo-vector)}, \qquad \bar{\psi}\gamma^5\psi \text{ (Pseudo-Scalar)} \tag{3.1.20}$$

The 'pseudo-' here means, under parity $\bar{\psi}\gamma^i\gamma^5\psi$ goes into itself; and $\bar{\psi}\gamma^5\psi$ goes into $-\bar{\psi}\gamma^5\psi$. Note that $\mathbb{I}$, γ^5, the four $\{\gamma^\mu\}$, the four $\{\gamma^\mu\gamma^5\}$, and $(4^2-4)/2 = 6$ matrices $\{\gamma^{[\mu}\gamma^{\nu]}\}$ form a basis for the 4×4 matrices. There is no need to ever consider symmetrized products of the $\gamma-$matrices because they obey the Clifford Algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}_{4\times 4}. \tag{3.1.21}$$

We also record

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3, \qquad \{\gamma^5, \gamma^\mu\} = 0. \tag{3.1.22}$$

The commonly used Weyl (aka chiral) basis reads

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}, \qquad \gamma^5 = \begin{bmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{bmatrix} \tag{3.1.23}$$

The $\{\sigma^i\}$ are the Pauli matrices; while $\sigma^0 = \mathbb{I}_{2\times 2}$. Observe that the left- and right-handed Weyl spinors can be extracted via

$$\left(\frac{\mathbb{I} - \gamma^5}{2}\right)\psi = \begin{bmatrix} \psi_L \\ 0 \end{bmatrix}, \tag{3.1.24}$$

$$\left(\frac{\mathbb{I} + \gamma^5}{2}\right)\psi = \begin{bmatrix} 0 \\ \psi_R \end{bmatrix}. \tag{3.1.25}$$

Problem 3.4. Left and Right Spinors Express the fermion bilinears in equations (3.1.19) and (3.1.20) in terms of left- and right-handed spinors. For example, $\bar{\psi}\psi = \psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L$. $\square$

Below, we use the notation

$$\not{p} \equiv \gamma^\mu p_\mu. \tag{3.1.26}$$

This allows us to recognize, the mass dimension of ψ is 3/2. Moreover, $\bar{\psi}\psi$ itself has mass dimension 3 and therefore its coefficient is precisely that of mass – i.e., the massive Dirac equation comes from extremizing

$$S = \int \mathrm{d}^4x\bar{\psi}(i\not{\partial} - m)\psi \tag{3.1.27}$$

$$\delta_{\psi^\dagger}S = 0 \qquad \Rightarrow \qquad (i\not{\partial} - m)\psi = 0. \tag{3.1.28}$$

If we wish to allow the fermion to be electrically charged, we must write down an interaction of the form $A_\mu J^\mu$, where J^μ must be conserved on-shell (i.e., when the EoM is satisfied) and transforms as a four vector. Indeed, we may identify the electromagnetic current

$$J^\mu = -e\bar{\psi}\gamma^\mu\psi, \tag{3.1.29}$$

$$e = -|e|. \tag{3.1.30}$$

[3]Introducing a 'covariant derivative'

$$D_\mu \equiv \partial_\mu + ieA_\mu, \tag{3.1.31}$$

we may now write the fermionic part of the QED action as

$$S_\psi = \int \mathrm{d}^4x\bar{\psi}(i\not{D} - m)\psi, \tag{3.1.32}$$

$$\delta_{\psi^\dagger}S = 0 \qquad \Rightarrow \qquad (i\not{D} - m)\psi = 0. \tag{3.1.33}$$

[3]I follow Peskin and Schroeder, where $e = -|e|$ the electric charge is negative.

3.2 Canonical Quantization and Fermi-Dirac Statistics

Treating ψ as the Heisenberg picture quantum field, we may obtain the mode expansion solution to the Dirac equation:

$$\psi(x) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3\sqrt{2E_{\vec{k}}}} \sum_{s=1}^{2} \left(a_k^s u_k^s e^{-ik\cdot x} + (b_k^s)^\dagger v_k^s e^{+ik\cdot x} \right), \qquad k_\mu \equiv \left(E_{\vec{k}}, k_i \right). \tag{3.2.1}$$

Remember there are two sets of solutions, one with positive and the other with negative energy:

$$u_p^s = \begin{bmatrix} \sqrt{\sigma\cdot p}\zeta^s \\ \sqrt{\bar{\sigma}\cdot p}\zeta^s \end{bmatrix} \qquad \text{and} \qquad v_p^s = \begin{bmatrix} \sqrt{\sigma\cdot p}\zeta^s \\ -\sqrt{\bar{\sigma}\cdot p}\zeta^s \end{bmatrix}; \tag{3.2.2}$$

obeying

$$\left(\not{p} - m\right) u_p^s = 0, \tag{3.2.3}$$
$$\left(\not{p} + m\right) v_p^s = 0. \tag{3.2.4}$$

We may also note

$$\bar{u}_p^r \left(\not{p} - m\right) = 0 = \bar{v}_p^r(\not{p} + m). \tag{3.2.5}$$

The s refers to the spin state – i.e., 'up' or 'down' – for a fixed inertial frame, it is possible to choose

$$(\widehat{p}_i\sigma^i)u_p^s = (-)^s u_p^s, \tag{3.2.6}$$
$$(\widehat{p}_i\sigma^i)v_p^s = (-)^{s+1} v_p^s. \tag{3.2.7}$$

The conjugate momentum is

$$\Pi \equiv \frac{\partial\mathcal{L}_\psi}{\partial\dot{\psi}} = i\bar{\psi}\gamma^0 = i\psi^\dagger. \tag{3.2.8}$$

And, the Dirac Hamiltonian is therefore

$$\mathcal{H} = \Pi\cdot\dot{\psi} - \psi^\dagger\gamma^0 i\gamma^0\partial_0\psi - \psi^\dagger\gamma^0\left(i\gamma^i\partial_i - m\right)\psi \tag{3.2.9}$$
$$= -\psi^\dagger\gamma^0\left(i\gamma^i\partial_i - m\right)\psi \tag{3.2.10}$$
$$= i\Pi\gamma^0\left(i\gamma^i\partial_i - m\right)\psi. \tag{3.2.11}$$

Problem 3.5. Dirac Equation from Hamilton's Equations Let $\{\cdot,\cdot\}_{\mathrm{PB}}$ denote the Poisson bracket. Show that

$$\dot{\psi}(t,\vec{x}) = \{\psi(t,\vec{x}), H(t)\}_{\mathrm{PB}} \tag{3.2.12}$$
$$\text{and} \qquad \dot{\Pi}(t,\vec{x}) = \{\Pi(t,\vec{x}), H(t)\}_{\mathrm{PB}} \tag{3.2.13}$$

are equivalent to the Dirac equation $(i\not{\partial} - m)\psi = 0$. Hint: See Problem (1.6). □

Canonical Commutation Relations? Let us attempt to quantize the Dirac field to obey the equal time commutation relations

$$[\psi(t,\vec{x}),\Pi(t,\vec{x}')] = i\mathbb{I}_{4\times 4}\cdot\delta^{(3)}(\vec{x}-\vec{x}'), \tag{3.2.14}$$

$$\left[\psi(t,\vec{x}),\psi^\dagger(t,\vec{x}')\right] = \mathbb{I}_{4\times 4}\cdot\delta^{(3)}(\vec{x}-\vec{x}'). \tag{3.2.15}$$

We have

$$\begin{aligned}
&\left[\psi(t,\vec{x}),\psi^\dagger(t,\vec{x}')\right] \\
&= \int_{\vec{k},\vec{k}'} \frac{1}{\sqrt{2E_{\vec{k}}}\sqrt{2E'_{\vec{k}}}} \sum_{s,s'} \Big[a_k^s u_k^s e^{-iE_k t + i\vec{k}\cdot\vec{x}} + (b_k^s)^\dagger v_k^s e^{+iE_k t - i\vec{k}\cdot\vec{x}}, \\
&\qquad + (a_{k'}^{s'})^\dagger (u_{k'}^{s'})^\dagger e^{+iE_{k'}t - i\vec{k}'\cdot\vec{x}'} + b_{k'}^{s'} (v_{k'}^{s'})^\dagger e^{-iE_{k'}t + i\vec{k}'\cdot\vec{x}'}\Big] \qquad (3.2.16)\\
&= \int_{\vec{k},\vec{k}'} \frac{1}{\sqrt{2E_{\vec{k}}}\sqrt{2E'_{\vec{k}}}} \sum_{s,s'} \\
&\times \Big\{ \left[a_k^s,(a_{k'}^{s'})^\dagger\right] u_k^s (u_{k'}^{s'})^\dagger e^{-iE_k t + i\vec{k}\cdot\vec{x}} e^{+iE_{k'}t - i\vec{k}'\cdot\vec{x}'} + \left[a_k^s, b_{k'}^{s'}\right] u_k^s (v_{k'}^{s'})^\dagger e^{-iE_k t + i\vec{k}\cdot\vec{x}} e^{-iE_{k'}t + i\vec{k}'\cdot\vec{x}'} \\
&+ \left[(b_k^s)^\dagger,(a_{k'}^{s'})^\dagger\right] v_k^s (u_{k'}^{s'})^\dagger e^{+iE_k t - i\vec{k}\cdot\vec{x}} e^{+iE_{k'}t - i\vec{k}'\cdot\vec{x}'} + \left[(b_k^s)^\dagger, b_{k'}^{s'}\right] v_k^s (v_{k'}^{s'})^\dagger e^{+iE_k t - i\vec{k}\cdot\vec{x}} e^{-iE_{k'}t + i\vec{k}'\cdot\vec{x}'} \Big\}.
\end{aligned}$$

To ensure we obtain a $t-$independent result, we see that $[a,b]=0=[a^\dagger,b^\dagger]$. We then impose

$$\left[a_k^s,(a_{k'}^{s'})^\dagger\right] = \alpha_{k,k'}\delta_{s'}^s (2\pi)^3\delta^{(3)}[\vec{k}-\vec{k}'], \tag{3.2.17}$$

$$\left[(b_k^s)^\dagger, b_{k'}^{s'}\right] = -\beta_{k,k'}^*\delta_{s'}^s (2\pi)^3\delta^{(3)}[\vec{k}-\vec{k}']. \tag{3.2.18}$$

This yields

$$\left[\psi(t,\vec{x}),\psi^\dagger(t,\vec{x}')\right] = \int_{\vec{k}} \frac{1}{2E_{\vec{k}}} \sum_s \Big\{\alpha_{k,k'} u_k^s (u_k^s)^\dagger e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} - \beta_{k,k'}^* v_k^s (v_k^s)^\dagger e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}\Big\}. \tag{3.2.19}$$

At this point we may compute

$$\sum_s u_p^s (u_p^s)^\dagger = \begin{bmatrix} \sqrt{\sigma\cdot p}\,\zeta^s \\ \sqrt{\bar{\sigma}\cdot p}\,\zeta^s \end{bmatrix} \begin{bmatrix} (\zeta^s)^\dagger\sqrt{\sigma\cdot p} & (\zeta^s)^\dagger\sqrt{\bar{\sigma}\cdot p} \end{bmatrix} \tag{3.2.20}$$

$$= \begin{bmatrix} \sqrt{\sigma\cdot p}\,\zeta^s(\zeta^s)^\dagger\sqrt{\sigma\cdot p} & \sqrt{\sigma\cdot p}\,\zeta^s(\zeta^s)^\dagger\sqrt{\bar{\sigma}\cdot p} \\ \sqrt{\bar{\sigma}\cdot p}\,\zeta^s(\zeta^s)^\dagger\sqrt{\sigma\cdot p} & \sqrt{\bar{\sigma}\cdot p}\,\zeta^s(\zeta^s)^\dagger\sqrt{\bar{\sigma}\cdot p} \end{bmatrix}. \tag{3.2.21}$$

Assuming the basis spinors $\{\zeta^s\}$ are normalized to unit length, they obey the completeness relation

$$\sum_s \zeta^s(\zeta^s)^\dagger = \mathbb{I}_{2\times 2}. \tag{3.2.22}$$

This leads us to

$$\sum_s u_p^s (u_p^s)^\dagger = \begin{bmatrix} \sigma\cdot p & m \\ m & \bar{\sigma}\cdot p \end{bmatrix} \tag{3.2.23}$$

$$= \left(\not{p} + m\right)\gamma^0. \tag{3.2.24}$$

Problem 3.6. Show that

$$\sum_s v_p^s (v_p^s)^\dagger = \left(\not{p} - m\right) \gamma^0. \tag{3.2.25}$$

□

At this point we may gather

$$\begin{aligned}
&\left[\psi(t,\vec{x}), \psi^\dagger(t,\vec{x}')\right] &(3.2.26)\\
&= \int_{\vec{k}} \frac{1}{2E_{\vec{k}}} \left\{ \alpha_{k=k'} (\not{k} + m)\gamma^0 e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} - \beta^*_{k=k'} (\not{k} - m)\gamma^0 e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')} \right\} &(3.2.27)\\
&= \int_{\vec{k}} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{2E_{\vec{k}}} \left\{ \alpha_k (\gamma^0 E_k + \gamma^i k_i + m)\gamma^0 - \beta^*_{-k} (\gamma^0 E_k - \gamma^i k_i - m)\gamma^0 \right\}. &(3.2.28)
\end{aligned}$$

If we choose

$$\alpha = (2\pi)^3 = -\beta \tag{3.2.29}$$

$$\begin{aligned}
&\left[\psi(t,\vec{x}), \psi^\dagger(t,\vec{x}')\right] &(3.2.30)\\
&= \int_{\vec{k}} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{2E_{\vec{k}}} \left\{ (E_k + (\gamma^i k_i + m)\gamma^0) + (E_k - (\gamma^i k_i + m)\gamma^0) \right\} &(3.2.31)\\
&= \delta^{(3)}[\vec{x} - \vec{x}']. &(3.2.32)
\end{aligned}$$

Because $\beta = -(2\pi)^3$, which means we may re-define $b_k^s \equiv (c_k^s)^\dagger$ and $(b_k^s)^\dagger \equiv c_k^s$. Then

$$\left[c_k^s, (c_{k'}^{s'})^\dagger\right] = \delta^s_{s'} (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}'). \tag{3.2.33}$$

All seems well! But let us now compute the Hamiltonian.

$$\begin{aligned}
H(t) &\equiv \int_{\mathbb{R}^3} \mathcal{H}(t,\vec{x}) \mathrm{d}^3\vec{x} &(3.2.34)\\
&= -\int_{\mathbb{R}^3} \bar{\psi} \left(i\gamma^0\partial_0 + i\vec{\gamma}\cdot\vec{\nabla} - m - i\gamma^0\partial_0 \right) \psi = i \int_{\mathbb{R}^3} \psi^\dagger \partial_0 \psi \mathrm{d}^3\vec{x} &(3.2.35)\\
&= \int_{\vec{x},\vec{k},\vec{k}'} \sum_{s,s'} \frac{1}{\sqrt{2E_k}\sqrt{2E_{k'}}} \left((a_k^s)^\dagger (u_k^s)^\dagger e^{+ik\cdot x} + (v_k^s)^\dagger (c_k^s)^\dagger e^{-ik\cdot x} \right) \\
&\qquad \times E_{k'} \left(a_{k'}^{s'} u_{k'}^{s'} e^{-ik'\cdot x} - v_{k'}^{s'} c_{k'}^{s'} e^{ik'\cdot x} \right). &(3.2.36)
\end{aligned}$$

Let us note that

$$\begin{aligned}
(u_{\vec{p}}^s)^\dagger v_{-\vec{p}}^r &= (\zeta^s)^\dagger \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}}\zeta^r + (\zeta^s)^\dagger \sqrt{p\cdot\bar{\sigma}}(-)\sqrt{p\cdot\sigma}\zeta^r &(3.2.37)\\
&= (\zeta^s)^\dagger \zeta^r (m - m) = 0. &(3.2.38)
\end{aligned}$$

Likewise

$$(v^s_{\vec{p}})^\dagger u^r_{-\vec{p}} = (\eta^s)^\dagger \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}}\zeta^r - (\eta^s)^\dagger \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma}\zeta^r \tag{3.2.39}$$

$$= (m-m)(\eta^s)^\dagger\zeta^r = 0. \tag{3.2.40}$$

Therefore,

$$H(t) = \frac{1}{2}\int_{\vec{k}}\sum_{s,s'}\left((a^s_k)^\dagger a^{s'}_k (u^s_k)^\dagger u^{s'}_k - (v^s_k)^\dagger v^{s'}_k (c^s_k)^\dagger (c^{s'}_k)\right). \tag{3.2.41}$$

We have

$$(u^r_p)^\dagger u^s_p = (\zeta^r)^\dagger p\cdot\sigma\zeta^s + (\zeta^r)^\dagger p\cdot\bar{\sigma}\zeta^s \tag{3.2.42}$$

$$= 2E_p\delta^r_s \tag{3.2.43}$$

and

$$(v^r_p)^\dagger v^s_p = (\zeta^r)^\dagger p\cdot\sigma\eta^s + (\zeta^r)^\dagger p\cdot\bar{\sigma}\eta^s \tag{3.2.44}$$

$$= 2E_p\delta^r_s. \tag{3.2.45}$$

We have arrived at

$$H = \int\frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\sum_s E_k\left((a^s_k)^\dagger a^s_k - (c^s_k)^\dagger c^s_k\right). \tag{3.2.46}$$

The Hamiltonian has no lower bound because of the $-$ sign in front of the term on the right!

Problem 3.7. Verify the following results.

$$\bar{u}^r_p u^s_p = 2m\delta^r_s \tag{3.2.47}$$

and

$$\bar{v}^r_p v^s_p = -2m\delta^r_s. \tag{3.2.48}$$

□

Anti-Commutation Relations We now solve this unbounded-from-below problem with the Hamiltonian by imposing instead the anti-commutation relations[4]

$$\{\psi(t,\vec{x}),\Pi(t,\vec{x}')\} = i\delta^{(3)}(\vec{x}-\vec{x}') \tag{3.2.49}$$

$$\left\{\psi(t,\vec{x}),\psi(t,\vec{x}')^\dagger\right\} = \delta^{(3)}(\vec{x}-\vec{x}'). \tag{3.2.50}$$

(Note: $\{A,B\}\equiv AB+BA=\{B,A\}$.) We will assume ψ and $\psi^\dagger$ anti-commutes with itself

$$\{\psi(t,\vec{x}),\psi(t,\vec{x}')\} = 0 = \left\{\psi(t,\vec{x})^\dagger,\psi(t,\vec{x}')^\dagger\right\}. \tag{3.2.51}$$

[4] Peskin and Schroeder tries to justify these further, by appealing to microcausality arguments.

As we will soon witness, these amount to imposing anti-commutation relations for the ladder operators

$$\left\{a_k^r, (a_{k'}^s)^\dagger\right\} = \delta_s^r (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') = \left\{b_k^r, (b_{k'}^s)^\dagger\right\} \tag{3.2.52}$$

$$\left\{a_k^r, a_{k'}^s\right\} = 0 = \left\{b_k^r, b_{k'}^s\right\}. \tag{3.2.53}$$

Starting from the LHS,

$$\begin{aligned}
&\left\{\psi(t,\vec{x}), \psi^\dagger(t,\vec{x}')\right\} \\
&= \int_{\vec{k},\vec{k}'} \sum_{s,s'} \frac{1}{\sqrt{2E_k}\sqrt{2E_{k'}}} \left\{u_k^s a_k^s e^{-ik\cdot x} + v_k^s (b_k^s)^\dagger e^{ik\cdot x}, (u_{k'}^{s'})^\dagger (a_{k'}^{s'})^\dagger e^{ik'\cdot x'} + (v_{k'}^{s'})^\dagger (b_{k'}^{s'}) e^{-ik'\cdot x'}\right\} \\
&= \int_{\vec{k},\vec{k}'} \sum_{s,s'} \frac{1}{\sqrt{2E_k}\sqrt{2E_{k'}}} \left(u_k^s (\bar{u}_{k'}^{s'})^\dagger e^{-ik\cdot x + ik'\cdot x'} \left\{a_k^s, (a_{k'}^{s'})^\dagger\right\} + v_k^s (\bar{v}_{k'}^{s'}) e^{ik\cdot x - ik'\cdot x'} \left\{b_{k'}^{s'}, (b_k^s)^\dagger\right\}\right) \gamma^0 \\
&= \int_{\vec{k}} \frac{1}{2E_k} \left((\not{k} + m) e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} + (\not{k} - m) e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}\right) \gamma^0 && (3.2.54) \\
&= \frac{1}{2} \int_{\vec{k}} \frac{1}{E_k} \left((\gamma^0 E_k + \gamma^j k_j + m) e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} + (\gamma^0 E_p - \gamma^j k_j - m) e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\right) \gamma^0 && (3.2.55) \\
&= \delta^{(3)}(\vec{x} - \vec{x}'). && (3.2.56)
\end{aligned}$$

Next, let us compute the Hamiltonian. Everything goes through as before.

$$\begin{aligned}
H(t) &\equiv \int_{\mathbb{R}^3} \mathcal{H}(t,\vec{x}) \mathrm{d}^3\vec{x} && (3.2.57) \\
&= -\int_{\mathbb{R}^3} \bar{\psi} \left(i\gamma^0\partial_0 + i\vec{\gamma}\cdot\vec{\nabla} - m - i\gamma^0\partial_0\right) \psi = i \int_{\mathbb{R}^3} \psi^\dagger \partial_0 \psi \mathrm{d}^3\vec{x} && (3.2.58) \\
&= \int_{\vec{x},\vec{k},\vec{k}'} \sum_{s,s'} \frac{1}{\sqrt{2E_k}\sqrt{2E_{k'}}} \left((a_k^s)^\dagger (u_k^s)^\dagger e^{+ik\cdot x} + (v_k^s)^\dagger (b_k^s) e^{-ik\cdot x}\right) \\
&\qquad \times E_{k'} \left(a_{k'}^{s'} u_{k'}^{s'} e^{-ik'\cdot x} - v_{k'}^{s'} (b_{k'}^{s'})^\dagger e^{ik'\cdot x}\right). && (3.2.59)
\end{aligned}$$

Hence,

$$\begin{aligned}
H &= \frac{1}{2} \int_{\vec{k}} \sum_{s,s'} \left((a_k^s)^\dagger a_k^{s'} (u_k^s)^\dagger u_k^{s'} - (v_k^s)^\dagger v_k^{s'} (b_k^s)(b_k^{s'})^\dagger\right) && (3.2.60) \\
&= \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \sum_s E_k \left((a_k^s)^\dagger a_k^s - (b_k^s)(b_k^s)^\dagger\right). && (3.2.61)
\end{aligned}$$

But now, we may say $(b_k^s)(b_k^s)^\dagger = \{(b_k^s), (b_k^s)^\dagger\} - (b_k^s)^\dagger(b_k^s)$ and

$$H = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \sum_s E_k \left((a_k^s)^\dagger a_k^s + (b_k^s)^\dagger (b_k^s)\right) + \text{constant}. \tag{3.2.62}$$

Problem 3.8. Number Operators If we define

$$N_\psi \equiv \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \widetilde{N}_\psi(\vec{k}) \equiv \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \sum_s (a_k^s)^\dagger a_k^s \tag{3.2.63}$$

and

$$N_{\bar{\psi}} \equiv \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \widetilde{N}_{\bar{\psi}}(\vec{k}) \equiv \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \sum_s (b_k^s)^\dagger b_k^s; \tag{3.2.64}$$

Show that

$$\left[N_\psi, (a_p^r)^\dagger\right] = (a_p^r)^\dagger \tag{3.2.65}$$

$$\left[N_\psi, a_p^r\right] = -a_p^r. \tag{3.2.66}$$

and

$$\left[N_{\bar{\psi}}, (b_p^r)^\dagger\right] = (b_p^r)^\dagger \tag{3.2.67}$$

$$\left[N_{\bar{\psi}}, b_p^r\right] = -b_p^r. \tag{3.2.68}$$

Finally, if $\mathcal{N}^\mu \equiv \bar{\psi}\gamma^\mu\psi$ is viewed as the Noether current due to global U_1 symmetry – recall Problem (3.3) – show that

$$N_{\text{total}} \equiv \int \mathrm{d}^3\vec{x}\mathcal{N}^0 = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \left(\widetilde{N}_\psi - \widetilde{N}_{\bar{\psi}}\right). \tag{3.2.69}$$

Note that these hold because of the *anti-commutation* relations. □

Fermion Number Conservation We may view the number current $\mathcal{N}^\mu = \bar{\psi}\gamma^\mu\psi$ as the conserved Noether current arising from the global symmetry transformation $\psi \to e^{i\vartheta}\psi$, for ϑ real and constant. Thus, N_{total} in eq. (3.2.69) tells us, for every fermion created (destroyed) there must be an anti-fermion created (destroyed); so that the total number of fermions plus anti-fermions never changes: $\Delta N_{\text{total}} = 0 \Leftrightarrow \Delta N_\psi = \Delta N_{\bar{\psi}}$.

Ladder Operators If we define the vacuum to be such that

$$a_p^s |0\rangle = 0 = b_p^s |0\rangle, \tag{3.2.70}$$

then we see that

$$N_\psi (a_p^s)^\dagger |0\rangle = \left(\left[N_\psi, (a_p^s)^\dagger\right] + (a_p^s)^\dagger N_\psi\right)|0\rangle \tag{3.2.71}$$

$$= (a_p^s)^\dagger |0\rangle \tag{3.2.72}$$

and

$$N_{\bar{\psi}} (b_p^s)^\dagger |0\rangle = \left(\left[N_{\bar{\psi}}, (b_p^s)^\dagger\right] + (b_p^s)^\dagger N_{\bar{\psi}}\right)|0\rangle \tag{3.2.73}$$

$$= (b_p^s)^\dagger |0\rangle. \tag{3.2.74}$$

In words: $N_\psi = \int_{\vec{k}} \sum_s (a_k^s)^\dagger a_k^s$ is the number operator for *fermions* created by $(a_p^r)^\dagger$ and destroyed by a_p^r; while $N_{\bar{\psi}} = \int_{\vec{k}} \sum_s (b_k^s)^\dagger b_k^s$ is the number operator for *anti-fermions* created by $(b_p^r)^\dagger$ and destroyed by b_p^r.

Furthermore, up to an additive constant, we may now write the Hamiltonian as

$$H = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} E_k \left(\widetilde{N}_\psi(\vec{k}) + \widetilde{N}_{\bar{\psi}}(\vec{k})\right). \tag{3.2.75}$$

It is now bounded from below!

Fermi-Dirac Statistics An important consequence of the anti-commutation relations is that of Fermi-Dirac statistics: no more than one fermion or anti-fermion can occupy the same state, since

$$(a_p^s)^\dagger (a_p^s)^\dagger |0\rangle = \frac{1}{2}\left\{(a_p^s)^\dagger, (a_p^s)^\dagger\right\}|0\rangle = 0 \tag{3.2.76}$$

$$(b_p^s)^\dagger (b_p^s)^\dagger |0\rangle = \frac{1}{2}\left\{(b_p^s)^\dagger, (b_p^s)^\dagger\right\}|0\rangle = 0. \tag{3.2.77}$$

The n particle fermion and n' particle anti-fermion state is now

$$\begin{aligned}&|p_1, s_1; p_2, s_2; \dots; p_n, s_n; \bar{p}_1, \bar{s}_1; \dots; \bar{p}_n, \bar{s}_n\rangle \\ &\equiv \left(\prod_{i=1}^{n} \sqrt{2E_{p_i}}(a_{p_i}^{s_i})^\dagger\right)\left(\prod_{i'=1}^{n'} \sqrt{2E_{\bar{p}_{i'}}}(b_{\bar{p}_{i'}}^{\bar{s}_{i'}})^\dagger\right)|0\rangle\,;\end{aligned} \tag{3.2.78}$$

with the implication that it is fully anti-symmetric under the swap of any pair

$$(k_a, \sigma_a) \leftrightarrow (k_b, \sigma_b) \tag{3.2.79}$$

because of the anti-commuting character of the $a^\dagger$s and $b^\dagger$s.

Problem 3.9. Particle Number Apply N_{total} in eq. (3.2.69) to the (n, n') particle state in eq. (3.2.78) to verify that

$$\begin{aligned}&N_{\text{total}}\,|p_1, s_1; p_2, s_2; \dots; p_n, s_n; \bar{p}_1, \bar{s}_1; \dots; \bar{p}_n, \bar{s}_n\rangle \\ &= (n - n')\,|p_1, s_1; p_2, s_2; \dots; p_n, s_n; \bar{p}_1, \bar{s}_1; \dots; \bar{p}_n, \bar{s}_n\rangle\,.\end{aligned} \tag{3.2.80}$$

□

3.3 Fermionic Path Integrals

Feynman Green's Function Because of the anti-commuting character of fermionic fields, the non-interacting vacuum expectation value of the time ordered products is now

$$\left\langle 0\left|\mathsf{T}\psi(x)\bar{\psi}(y)\right|0\right\rangle = \left\langle 0\left|\psi(x)\bar{\psi}(y)\right|0\right\rangle \qquad \text{when } x^0 > y^0 \tag{3.3.1}$$

$$= -\left\langle 0\left|\bar{\psi}(y)\psi(x)\right|0\right\rangle \qquad \text{when } x^0 < y^0. \tag{3.3.2}$$

Problem 3.10. Show that this is equal to the following Feynman Green's function

$$\left\langle 0\left|\mathsf{T}\psi(x)\bar{\psi}(y)\right|0\right\rangle = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{i(\not{k} + m)}{k^2 - m^2 + i\epsilon}e^{-ik\cdot(x-y)} \tag{3.3.3}$$

$$\equiv S_F(x - y). \tag{3.3.4}$$

□

We may also see that

$$(i\partial\!\!\!/_x - m)S_F(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i(k\!\!\!/ - m)(k\!\!\!/ + m)}{k^2 - m^2 + i\epsilon}e^{-ik\cdot(x-y)} \tag{3.3.5}$$

$$= i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k\!\!\!/k\!\!\!/ - m^2}{k^2 - m^2 + i\epsilon}e^{-ik\cdot(x-y)} \tag{3.3.6}$$

$$= i\delta^{(4)}(x-y). \tag{3.3.7}$$

Note: $k\!\!\!/^2 = (1/2)\{\gamma^\mu, \gamma^\nu\}k_\mu k_\nu = k^2$.

To introduce a path integral for fermions, we first have to introduce Grassmann numbers. The basic rule is that Grassmann numbers anti-commute: $\theta\eta = -\eta\theta$, so for instance $\theta^2 = 0$. Also, for complex numbers A and B and Grassmann number α, Berezin defined

$$\int \mathrm{d}\alpha(A + B\alpha) \equiv B. \tag{3.3.8}$$

For spinor field $\psi(x)$ we expand as

$$\psi(x) = \sum_{\mathrm{I}} \theta_{\mathrm{I}}\Psi_{\mathrm{I}}(x), \tag{3.3.9}$$

where θ_{I} are Grassmann variables while $\Psi_{\mathrm{I}}(x)$ are the basis (non-Grassmannian) spinor fields.

The corresponding Fermionic path integral that generates all n point functions is

$$Z_0[\bar{J}, J] \equiv \frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[i\int_x \left(\bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{J}\psi + \bar{\psi}J\right)\right]}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[i\int_x \bar{\psi}(i\partial\!\!\!/ - m)\psi\right]} \tag{3.3.10}$$

Let us shift

$$\psi(x) \to \psi(x) + i\int_{x'} S_F(x-x')J(x'), \tag{3.3.11}$$

$$\overline{\psi}(x) \to \overline{\psi}(x) - i\int_{x'} J(x')^\dagger S_F(x-x')^\dagger\gamma^0. \tag{3.3.12}$$

Recalling $(\gamma^\mu)^\dagger\gamma^0 = \gamma^0\gamma^\mu$,

$$S_F(x-x')^\dagger\gamma^0 = -\int_k \frac{i(k\!\!\!/ + m)^\dagger\gamma^0}{k^2 - m^2 - i\epsilon}e^{ik\cdot(x-x')} \tag{3.3.13}$$

$$= -\gamma^0\int_k \frac{i(k\!\!\!/ + m)}{k^2 - m^2 - i\epsilon}e^{ik\cdot(x-x')} \equiv -\gamma^0 S_{\bar{F}}(x-x'). \tag{3.3.14}$$

Thus

$$\overline{\psi}(x) \to \overline{\psi}(x) + i\int_{x'} \overline{J}(x')S_{\bar{F}}(x-x'). \tag{3.3.15}$$

The quadratic in ψ portion of the action becomes

$$S_2 \to \int_x \left(\overline{\psi}(x) + i\int_{x'} \overline{J}(x')S_{\bar{F}}(x-x')\right)(i\partial\!\!\!/_x - m)\left(\psi(x) + i\int_{x''} S_F(x-x'')J(x'')\right) \quad (3.3.16)$$

$$= \int_x \left(\overline{\psi}(x) + i\int_{x'} \overline{J}(x')S_{\bar{F}}(x-x')\right)\big((i\partial\!\!\!/_x - m)\psi(x) - J(x)\big) \quad (3.3.17)$$

$$= \int_x \overline{\psi}(x)(i\partial\!\!\!/_x - m)\psi(x) + i\int_x\int_{x'} \overline{J}(x')(-i\partial_{x^\mu} S_{\bar{F}}(x-x')\gamma^\mu - mS_{\bar{F}}(x-x'))\psi(x) \quad (3.3.18)$$

$$- \int_x \left(\overline{\psi}(x)J(x) + i\int_{x'} \overline{J}(x')S_{\bar{F}}(x-x')J(x)\right). \quad (3.3.19)$$

We see that

$$-i\partial_{x^\mu} S_{\bar{F}}(x-x')\gamma^\mu - mS_{\bar{F}}(x-x') = \int_k \frac{i}{k^2-m^2-i\epsilon}(k\!\!\!/ + m)(k\!\!\!/ - m)e^{ik\cdot(x-x')} \quad (3.3.20)$$

$$= i\delta^{(4)}[x-x']; \quad (3.3.21)$$

and hence

$$S_2 \to \int_x \overline{\psi}(x)(i\partial\!\!\!/_x - m)\psi(x) - \int_x \big(\overline{J}(x)\psi(x) + \overline{\psi}(x)J(x)\big) \quad (3.3.22)$$

$$- i\int_x\int_{x'} \overline{J}(x')S_{\bar{F}}(x-x')J(x). \quad (3.3.23)$$

The linear in ψ portion becomes

$$S_1 \to \int_x \big(\overline{J}(x)\psi(x) + \overline{\psi}(x)J(x)\big) \quad (3.3.24)$$

$$+ \int_x\int_{x'} \overline{J}(x)iS_F(x-x')J(x') + \int_x\int_{x'} \overline{J}(x')iS_{\bar{F}}(x-x')J(x). \quad (3.3.25)$$

The full action becomes

$$S = \int_x \overline{\psi}(x)(i\partial\!\!\!/_x - m)\psi(x) + \int_x\int_{x'} \overline{J}(x)iS_F(x-x')J(x'). \quad (3.3.26)$$

We gather

$$Z_0[\bar{J}, J] = \exp\left(\int_x\int_{x'} i\overline{J}(x)S_F(x-x')iJ(x')\right). \quad (3.3.27)$$

We follow the convention in Peskin and Schroeder; if θ and η are Grassmann variables,

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\theta\eta = -\frac{\mathrm{d}}{\mathrm{d}\eta}\eta\theta = -\theta. \quad (3.3.28)$$

Consider

$$\left.\frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \frac{\delta}{\delta(iJ^B(x_2))} \exp\left[i\int_x \left(\bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{J}\psi + \bar{\psi}J\right)\right]}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[i\int_x \bar{\psi}(i\partial\!\!\!/ - m)\psi\right]}\right|_{\bar{J}=J=0} \quad (3.3.29)$$

$$= \frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi(-)\overline{\psi}_B(x_2) \exp\left[i\int_x \left(\bar{\psi}(i\partial\!\!\!/ - m)\psi\right)\right]}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[i\int_x \bar{\psi}(i\partial\!\!\!/ - m)\psi\right]} \quad (3.3.30)$$

and

$$\left.\frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \frac{\delta}{\delta(i\bar{J}^a(x_1))}\frac{\delta}{\delta(iJ^B(x_2))}\exp\left[i\int_x\left(\bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{J}\psi + \bar{\psi}J\right)\right]}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[i\int_x \bar{\psi}(i\partial\!\!\!/ - m)\psi\right]}\right|_{\bar{J}=J=0} \tag{3.3.31}$$

$$= -\frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\psi_A(x_1)\overline{\psi}_B(x_2)\exp\left[i\int_x\left(\bar{\psi}(i\partial\!\!\!/ - m)\psi\right)\right]}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[i\int_x \bar{\psi}(i\partial\!\!\!/ - m)\psi\right]} \tag{3.3.32}$$

The two point function is

$$\langle 0|\,\mathsf{T}\psi(x_1)\bar{\psi}(x_2)\,|0\rangle = -\left.\frac{\delta}{\delta(i\overline{J}(x_1))}\frac{\delta}{\delta(iJ(x_2))}\exp\left(\int_x\int_{x'} i\overline{J}(x)S_F(x-x')iJ(x')\right)\right|_{\bar{J}=J=0}$$

$$= \frac{\delta}{\delta(i\overline{J}(x_1))}\int_x i\overline{J}(x)S_F(x-x_2) \tag{3.3.33}$$

$$= S_F(x_1 - x_2). \tag{3.3.34}$$

Wick contractions always occur between ψ and $\overline{\psi}$; contractions between ψ and ψ or between $\overline{\psi}$ and $\overline{\psi}$ are zero.

Problem 3.11. Write down the 4− and 6−point fermion correlation function. □

Problem 3.12. Consider the theory defined by the Lagrangian density

$$\mathcal{L} \equiv \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{g}{M^2}\left(\bar{\psi}\psi\right)^2, \tag{3.3.35}$$

where M is a mass scale and g is a dimensionless coupling constant. Write down

$$\left\langle\Omega\left|\mathsf{T}\psi(x)\bar{\psi}(x')\right|\Omega\right\rangle \tag{3.3.36}$$

up to one loop order, in terms of the fermion Feynman Green's function S_F; you do not have to do the integrals. □

3.4 Scattering, Decay of Scalars and/or Fermions: Feynman Rules

Chapter 4.7 of P&S discusses Feynman rules for fermions, including Yukawa theory.

Incoming States The contraction of ψ (which contains a) with an incoming momentum p/fermion is

$$\psi\,|p,s\rangle \qquad \to \qquad u_p^s. \tag{3.4.1}$$

The contraction of $\bar{\psi}$ (which contains $a^\dagger$) with an incoming momentum/anti-fermion is

$$\bar{\psi}\,|p,s\rangle \qquad \to \qquad \bar{v}_p^s. \tag{3.4.2}$$

The convention is: the fermion flow is going *outwards* for an *incoming* anti-fermion.

Outgoing States The contraction of ψ (which contains $b^\dagger$) with an outgoing momentum/anti-fermion is

$$\langle p,s|\,\psi \qquad \to \qquad v_p^s. \tag{3.4.3}$$

The contraction of $\bar{\psi}$ (which contains $a^\dagger$) with an outgoing momentum/fermion is

$$\langle p, s| \bar{\psi} \qquad \to \qquad \bar{u}_p^s. \tag{3.4.4}$$

The convention is: the fermion flow is going *inwards* for an *outgoing* anti-fermion.

Internal Lines Momentum always flows along particle number flow.

Raising and Lowering Operators (The convention is: the fermion flow is going *outwards* for an *outgoing* fermion.) Also, since the order of the creation operators matter, we shall employ the convention

$$|p_1, s_1; p_2, s_2; \dots; p_n, s_n\rangle \equiv \sqrt{2E_{p_1}} \dots \sqrt{2E_{p_n}} (a_{p_1}^{s_1})^\dagger \dots (a_{p_n}^{s_n})^\dagger |0\rangle ; \tag{3.4.5}$$

which in turn implies the convention

$$\langle p_1, s_1; p_2, s_2; \dots; p_n, s_n| = \sqrt{2E_{p_1}} \dots \sqrt{2E_{p_n}} \langle 0| (a_{p_n}^{s_n})^\dagger \dots (a_{p_1}^{s_1})^\dagger. \tag{3.4.6}$$

Example: 4-Fermion Interaction Since Lagrangian densities in 4D must of mass dimensions 4, we have $[\bar{\psi}\psi] = 3$ and $[\psi] = 3/2$. Consider now the theory

$$\mathcal{L} = \bar{\psi}_1(i\not{\partial} - m_1)\psi_1 + \bar{\psi}_2(i\not{\partial} - m_2)\psi_2 + \mathcal{L}_\text{I}, \tag{3.4.7}$$

$$\mathcal{L}_\text{I} \equiv \frac{g}{M^2}\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2. \tag{3.4.8}$$

The $\bar{\psi}_i\psi_i$ (for $i = 1, 2$) can therefore be viewed as a vertex for outgoing fermion and anti-fermion; incoming fermion and anti-fermion; incoming fermion and outgoing fermion; or, finally, incoming anti-fermion and outgoing anti-fermion. In all of these processes, fermion number minus anti-fermion number is conserved.

At leading order,

$$\psi_1(k, s) + \bar{\psi}_1(k', s') \to \psi_2(p, r) + \bar{\psi}_2(p', r') \tag{3.4.9}$$

has amplitude – modulo the overall momentum-conserving $\delta-$functions –

$$i\mathcal{M} = \langle p, r; p', r'|i\frac{g}{M^2} \int \bar{\psi}_1\psi_1\bar{\psi}_2\psi_2|k, s; k', s'\rangle \tag{3.4.10}$$

$$= i\frac{g}{M^2}(\bar{v}_{k'}^{s'}u_k^s)(\bar{u}_p^r v_{p'}^{r'}). \tag{3.4.11}$$

Example: Yukawa Theory Now consider a fermion coupled to one scalar.

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m_\psi)\psi + \frac{1}{2}(\partial\varphi)^2 - \frac{m_\varphi^2}{2}\varphi^2 + \mathcal{L}_\text{I}, \tag{3.4.12}$$

$$\mathcal{L}_\text{I} \equiv g\varphi\bar{\psi}\psi. \tag{3.4.13}$$

Fermion-Antifermion Annihilation to Scalars A pair of fermion-antifermion may annihilate and produce a pair of scalars. At leading order,

$$\left\langle p_1, p_2 \left| \frac{(ig)^2}{2} \int_{x,y} \varphi_x\bar{\psi}_x\psi_x\varphi_y\bar{\psi}_y\psi_y \right| p_\psi, s; p_{\bar{\psi}}, \bar{s} \right\rangle_c$$

$$= -ig^2 \left(\bar{v}_{p_{\bar{\psi}}}^{\bar{s}} \frac{\not{k} + m_\psi}{k^2 - m_\psi^2 + i\epsilon} u_{p_\psi}^s + (p_1 \leftrightarrow p_2) \right) (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_\psi - p_{\bar{\psi}}). \tag{3.4.14}$$

where $k = p_\psi - p_1$ for the first term and $k = p_\psi - p_2$ for the second.

Fermion-Fermion Scattering Let us now consider scattering 2 fermions into another 2 fermions.

$$\left\langle p_1', s_1'; p_2', s_2' \left| \frac{(ig)^2}{2} \int_{x,y} \varphi_x \bar{\psi}_x \psi_x \varphi_y \bar{\psi}_y \psi_y \right| p_1, s_1; p_2, s_2 \right\rangle_c$$
$$= ig^2 \left(\frac{(\bar{u}_{p_2'}^{s_2'} u_{p_2}^{s_2})(\bar{u}_{p_1'}^{s_1'} u_{p_1}^{s_1})}{k^2 - m_\varphi^2 + i\epsilon} - \frac{(\bar{u}_{p_1'}^{s_1'} u_{p_2}^{s_2})(\bar{u}_{p_2'}^{s_2'} u_{p_1}^{s_1})}{k^2 - m_\varphi^2 + i\epsilon} \right) (2\pi)^4 \delta^{(4)} (p_1' + p_2' - p_1 - p_2) . \tag{3.4.15}$$

Decay of Scalar If the scalar is more than twice as massive as the fermion, $m_\varphi > 2m_\psi$, it can decay into a fermion-antifermion pair. To leading order, we need to study the transition amplitude

$$\left\langle p_\psi, s; p_{\bar{\psi}}, \bar{s} \left| ig \int_x \varphi_x \bar{\psi}_x \psi_x \right| p_\varphi \right\rangle_c = ig(2\pi)^4 \delta^{(4)}(p_\varphi - p_\psi - \psi_{\bar{\psi}}) \bar{u}_{p_\psi}^s v_{p_{\bar{\psi}}}^{\bar{s}}; \tag{3.4.16}$$

where s is the spin of the fermion and $\bar{s}$ is that of antifermion. The leading order amplitude is (up to overall $-$ signs)

$$\mathcal{M}(\varphi \to \psi + \bar{\psi}) = g \bar{u}_{p_\psi}^s v_{p_{\bar{\psi}}}^{\bar{s}}. \tag{3.4.17}$$

3.5 4-Fermi Theory

The 4-Fermi theory is defined by the Lagrangian density

$$\mathcal{L} = \sum_{\mathrm{I}=1}^{2} \left\{ \bar{\psi}_{\mathrm{I}} \left(i\partial\!\!\!/ - m_{\mathrm{I}} \right) \psi_{\mathrm{I}} + \bar{\nu}_{\mathrm{I}} i\partial\!\!\!/ \nu_{\mathrm{I}} \right\} + \frac{g^2}{M^2} \left\{ (\bar{\psi}_1 \nu_1)(\bar{\nu}_2 \psi_2) + \text{h.c.} \right\} . \tag{3.5.1}$$

This is a toy model for beta decay (or muon decay), where M is closely related to the $W^\pm$ and Z boson masses. Muon decay goes as $\mu^- \to \nu_\mu + W^-(\text{virtual})$ followed by $W^-(\text{virtual}) \to \bar{\nu}_e + e^-$; but the W is heavy enough that, to a good degree of accuracy, its propagator can be replaced with $i\eta_{\mu\nu}/M_W^2$. This leads to an effective 4-fermion interaction.

Beta Decay Assuming $m_1 > m_2$, compute – to leading order in g^2– the decay rate of the process

$$\psi_1 \to \nu_1 + \psi_2 + \overline{\nu}_2 \tag{3.5.2}$$

due to the interaction in eq. (3.5.26). Muon and quark decays are special cases of this process. Note that, from the properties of $\{\gamma^\mu\}$ and γ^5,

$$\left(\overline{\psi} \gamma^\mu (c - c'\gamma^5) \nu \right)^\dagger = \overline{\nu} \gamma^\mu (c - c'\gamma^5) \psi. \tag{3.5.3}$$

Therefore,

$$\mathcal{L}_{\mathrm{I}} = \frac{g^2}{M^2} \left\{ \overline{\psi}_1 \nu_1 \overline{\nu}_2 \psi_2 + \overline{\nu}_1 \psi_1 \overline{\psi}_2 \nu_2 \right\}$$

Let us denote the momentum of ψ_1 as $k^\mu = (m_1, \vec{0})$, i.e., in its rest frame; and let its spin be r. Let the spin-momentum of ψ_2, $\overline{\nu}_2$ and ν_1 be, respectively, (s_2', p_2), (s_2, q_2) and (s_1, q_1). To contract against the 'incoming' $\psi_1(r, k)$, we need the ψ_1 operator. The relevant matrix element is

$$\frac{ig^2}{M^2}\left\langle q_1, s_1; q_2, s_2; p_2, s_2' \left| \int \bar{\psi}_2 \nu_2 \bar{\nu}_1 \psi_1 \right| k, r \right\rangle . \tag{3.5.4}$$

Hence

$$|\mathcal{M}| = \frac{g^2}{M^2}\left| \overline{u}^{s_1}_{q_1} u^r_k \overline{u}^{s_2'}_{p_2} v^{s_2}_{q_2} \right| . \tag{3.5.5}$$

Let us average over the initial spins and sum over the final ones.

$$\frac{1}{2}\sum_{r,s_1,s_2,s_2'} |\mathcal{M}|^2 = \frac{g^4}{2M^4}\sum_{r,s_1,s_2,s_2'} \overline{u}^{s_1}_{q_1} u^r_k \overline{u}^r_k u^{s_1}_{q_1} \overline{u}^{s_2'}_{p_2} v^{s_2}_{q_2} \overline{v}^{s_2}_{q_2} u^{s_2'}_{p_2} \tag{3.5.6}$$

$$= \frac{g^4}{2M^4}\mathrm{Tr}\left[\not{q}_1 (\not{k} + m_1) \right] \mathrm{Tr}\left[(\not{p}_2 + m_2) \not{q}_2 \right] \tag{3.5.7}$$

$$= \frac{g^4}{2M^4} 16 (q_1 \cdot k)(p_2 \cdot q_2) = \frac{8g^4}{M^4}(q_1 \cdot k)(p_2 \cdot q_2). \tag{3.5.8}$$

Note that $q_1 \cdot k = m_1 |\vec{q}_1| \equiv m_1 q_1$ and conservation of momentum tells us

$$(q_2 + p_2)^2 = m_2^2 + 2 q_2 \cdot p_2 = (k - q_1)^2 = m_1^2 - 2 m_1 q_1, \tag{3.5.9}$$

$$q_2 \cdot p_2 = \frac{m_1^2 - m_2^2}{2} - m_1 q_1. \tag{3.5.10}$$

The decay rate is given by the integral

$$\begin{aligned}
\Gamma &= \frac{1}{2m_1}\int \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2E_{p_2}} \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3 2q_1} \int \frac{\mathrm{d}^3 q_2}{(2\pi)^3 2q_2} \frac{1}{2}\sum_{\text{spins}} |\mathcal{M}|^2 \\
&\quad \times (2\pi)^4 \delta(m_1 - q_1 - q_2 - E_{p_2}) \delta^{(3)}(\vec{q}_1 + \vec{q}_2 + \vec{p}_2) \\
&= \int \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2E_{p_2}} \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3 2q_1} \int \frac{\mathrm{d}^3 q_2}{(2\pi)^3 2q_2} (2\pi)^4 \delta(m_1 - q_1 - q_2 - E_{p_2}) \delta^{(3)}(\vec{q}_1 + \vec{q}_2 + \vec{p}_2) \\
&\qquad \times \frac{4g^4}{M^4} q_1 \left(\frac{m_1^2 - m_2^2}{2} - m_1 q_1 \right).
\end{aligned} \tag{3.5.11}$$

We have $\Gamma = 4(g/M)^4 I$, where

$$\begin{aligned}
I &\equiv \frac{1}{2}\int \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2E_{p_2}} \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3} \int \frac{\mathrm{d}^3 q_2}{(2\pi)^3 2q_2} (2\pi)^4 \delta(m_1 - q_1 - q_2 - E_{p_2}) \delta^{(3)}(\vec{q}_1 + \vec{q}_2 + \vec{p}_2) \\
&\qquad \times \left(\frac{m_1^2 - m_2^2}{2} - m_1 q_1 \right) \\
&= \frac{1}{2}\int \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2E_{p_2}} \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3} \frac{1}{2|\vec{q}_1 + \vec{p}_2|} (2\pi) \delta(m_1 - q_1 - q_2 - E_{p_2}) \left(\frac{m_1^2 - m_2^2}{2} - m_1 q_1 \right)
\end{aligned} \tag{3.5.12}$$

$$= \frac{1}{2} \int_0^\infty \frac{\mathrm{d}p_2 p_2^2}{(2\pi)^2 2E_{p_2}} \int_{-1}^{+1} \mathrm{d}\chi \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3} \frac{1}{2\sqrt{q_1^2 + p_2^2 - 2q_1 p_2 \chi}}$$
$$\times (2\pi)\delta\left(m_1 - q_1 - E_{p_2} - \sqrt{q_1^2 + p_2^2 - 2q_1 p_2 \chi}\right)\left(\frac{m_1^2 - m_2^2}{2} - m_1 q_1\right) \quad (3.5.13)$$

Put

$$q_2[\chi] \equiv \sqrt{q_1^2 + p_2^2 - 2q_1 p_2 \chi}, \quad (3.5.14)$$
$$\mathrm{d}q_2 = -\frac{q_1 p_2}{q_2} \mathrm{d}\chi. \quad (3.5.15)$$

$$I = \frac{1}{4} \int_0^\infty \frac{\mathrm{d}p_2 p_2^2}{(2\pi)^2 2E_{p_2}} \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3} \int_{|q_1 - p_2|}^{q_1 + p_2} \frac{\mathrm{d}q_2}{q_1 p_2}$$
$$\times (2\pi)\delta\left(m_1 - q_1 - q_2 - E_{p_2}\right)\left(\frac{m_1^2 - m_2^2}{2} - m_1 q_1\right) \quad (3.5.16)$$
$$= \frac{\pi}{2} \int_0^\infty \frac{\mathrm{d}p_2 p_2^2}{(2\pi) E_{p_2}} \int_0^\infty \frac{\mathrm{d}q_1 q_1^2}{(2\pi)^3} \frac{1}{q_1 p_2} \left(\frac{m_1^2 - m_2^2}{2} - m_1 q_1\right)$$
$$\times \Theta\left(q_1 + p_2 - (m_1 - q_1 - E_{p_2})\right) \Theta\left(m_1 - q_1 - E_{p_2} - |q_1 - p_2|\right) \quad (3.5.17)$$
$$= \frac{1}{4} \int_0^\infty \frac{\mathrm{d}p_2 p_2}{E_{p_2}} \int_0^\infty \frac{\mathrm{d}q_1 q_1}{(2\pi)^3} \left(\frac{m_1^2 - m_2^2}{2} - m_1 q_1\right)$$
$$\times \Theta\left(2q_1 + p_2 - m_1 + E_{p_2}\right) \Theta\left(m_1 - q_1 - E_{p_2} - |q_1 - p_2|\right) \quad (3.5.18)$$

In the limit where $m_1 \gg m_2$, let us put $m_2 = 0$ as a leading order approximation.

$$I = \frac{1}{4} \int_0^\infty \mathrm{d}p_2 \int_0^\infty \frac{\mathrm{d}q_1 q_1}{(2\pi)^3} \left(\frac{m_1^2}{2} - m_1 q_1\right)$$
$$\times \Theta\left(2q_1 + 2p_2 - m_1\right) \Theta\left(m_1 - q_1 - p_2 - |q_1 - p_2|\right) \quad (3.5.19)$$

The first step function says $q_1 + p_2 \geq m_1/2$. For the second step function, we have

$$m_1 - q_1 - p_2 - |q_1 - p_2| = m_1 - 2q_1 \qquad \text{if } q_1 > p_2 \quad (3.5.20)$$
$$= m_1 - 2p_2 \qquad \text{if } q_1 < p_2. \quad (3.5.21)$$

The step function $\Theta(m_1 - q_1 - p_2 - |q_1 - p_2|)$ together with $q_1, p_2 \geq 0$ therefore defines the square on the first quadrant of side $m_1/2$ with its lower left vertex at the origin. Since $q_1 + p_2 \geq m_1/2$, we must therefore integrate over the triangle defined by the vertices $(m_1/2, m_1/2)$, $(0, m_1/2)$ and $(m_1/2, 0)$.

Put $k_\pm \equiv q_1 \pm p_2$; so that $k_+ \geq 0$ and $-k_+ \leq k_- \leq k_+$. Then $\mathrm{d}k_+ \wedge \mathrm{d}k_- = 2\mathrm{d}q_1 \wedge \mathrm{d}p_2$.

$$I = \frac{1}{8} \int_{q_1, p_2 \geq 0} \frac{\mathrm{d}k_+ \mathrm{d}k_-}{(2\pi)^3} \frac{k_+ + k_-}{2} \left(\frac{m_1^2}{2} - m_1 \frac{k_+ + k_-}{2}\right)$$
$$\times \Theta\left(k_+ - \frac{m_1}{2}\right) \Theta\left(m_1 - k_+ - |k_-|\right) \quad (3.5.22)$$

$$= \frac{1}{8}\int_0^{m_1/2}\int_{-k_+}^{+k_+}\frac{\mathrm{d}k_+\mathrm{d}k_-}{(2\pi)^3}\frac{k_+ + k_-}{2}\left(\frac{m_1^2}{2} - m_1\frac{k_+ + k_-}{2}\right)$$
$$= \frac{m_1^5}{6144\pi^3}. \tag{3.5.23}$$

Drawing a plot will show you the relevant regions of integration are

$$I = \frac{1}{4}\left(\int_0^{m_1/2}\frac{\mathrm{d}k_-}{(2\pi)^3}\int_{m_1/2}^{m_1-k_-}\frac{\mathrm{d}k_+}{2} + \int_{-m_1/2}^{0}\frac{\mathrm{d}k_-}{(2\pi)^3}\int_{m_1/2}^{m_1+k_-}\frac{\mathrm{d}k_+}{2}\right)$$
$$\times \frac{k_+ - k_-}{2}\left(\frac{m_1^2}{2} - m_1\frac{k_+ - k_-}{2}\right)$$
$$\approx \frac{m_1^5}{6144\pi^3}\left(1 + \mathcal{O}\left((m_2/m_1)^2\right)\right) \tag{3.5.24}$$

We have arrived at, for $m_1 \gg m_2 \approx 0$, the decay rate

$$\Gamma = \frac{g^4}{M^4}\frac{m_1^5}{1536\pi^3}. \tag{3.5.25}$$

Problem 3.13. Lepton/Quark Decays Consider the following Lagrangian density describing 2 massive and 2 massless fermions ($\psi_{1,2}$ and $\nu_{1,2}$ respectively); and their 4-fermion interaction:

$$\mathcal{L} = \sum_{\mathrm{I}=1}^{2}\left\{\overline{\psi}_{\mathrm{I}}\left(i\partial\!\!\!/ - m_{\mathrm{I}}\right)\psi_{\mathrm{I}} + \overline{\nu}_{\mathrm{I}} i\partial\!\!\!/ \nu_{\mathrm{I}}\right\}$$
$$+ \frac{g^2}{M^2}\overline{\psi}_1\gamma^\mu\left(c_1 - c_2\gamma^5\right)\nu_1\overline{\nu}_2\gamma_\mu\left(c_3 - c_4\gamma^5\right)\psi_2 + \text{h.c.}, \qquad c_{1,2,3,4} \in \mathbb{R}. \tag{3.5.26}$$

Assuming $m_1 > m_2$, compute – to leading order in g^2– the decay rate of the process

$$\psi_1 \to \nu_1 + \psi_2 + \overline{\nu}_2 \tag{3.5.27}$$

due to the interaction in eq. (3.5.26). Muon and quark decays are special cases of this process. □

Problem 3.14. Lepton/Quark Polarized Scattering Using the model in Problem (3.13), analyze the polarization dependence of the annihilation

$$\psi_1 + \bar{\psi}_2 \to \nu_1 + \bar{\nu}_2. \tag{3.5.28}$$

Employ whatever approximation you deem necessary (for e.g., non-relativistic or ultra-relativistic $\psi_{1,2}$'s) but be sure to apply it consistently. □

3.6 Additional Problems

Problem 3.15. Massive Dirac field Use the Noether method to derive – up to an additive identically-divergence-free current – the energy-momentum-shear-stress tensor of the Dirac field. Assume its Lagrangian is

$$\mathcal{L} = -\frac{i}{2}(\partial_\mu\bar{\psi})\gamma^\mu\psi + \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi. \tag{3.6.1}$$

Bonus: Also work out the angular momentum Noether tensor $M^{\mu\alpha\beta}$ by considering an infinitesimal Lorentz transformation. □

Problem 3.16. Gaussian Integrals Compute the integrals involving real variables x^i and symmetric matrix A_{ij}:

$$\langle x^{i_1} x^{i_2} \rangle \equiv \int_{\mathbb{R}} \exp\left(i x^i A_{ij} x^j\right) x^{i_1} x^{i_2} \mathrm{d}^D \vec{x}, \tag{3.6.2}$$

$$\langle x^{i_1} x^{i_2} x^{i_3} \rangle \equiv \int_{\mathbb{R}} \exp\left(i x^i A_{ij} x^j\right) x^{i_1} x^{i_2} x^{i_3} \mathrm{d}^D \vec{x}, \tag{3.6.3}$$

$$\langle x^{i_1} x^{i_2} x^{i_3} x^{i_4} \rangle \equiv \int_{\mathbb{R}} \exp\left(i x^i A_{ij} x^j\right) x^{i_1} x^{i_2} x^{i_3} x^{i_4} \mathrm{d}^D \vec{x}. \tag{3.6.4}$$

Explain how to obtain the general even-point function $\langle x^{i_1} \dots x^{i_{2n}} \rangle$ and odd-point function $\langle x^{i_1} \dots x^{i_{2n+1}} \rangle$. □

Problem 3.17. Grassmannian Integrals Compute the integrals involving complex Grassmann variables θ^i and Hermitian matrix H_{ij}:

$$\langle \theta^{i_1} \overline{\theta^{i_2}} \rangle \equiv \int_{\mathbb{R}} \exp\left(i \overline{\theta^i} H_{ij} \theta^j\right) \theta^{i_1} \overline{\theta^{i_2}} \mathrm{d}^D \vec{\theta} \mathrm{d}^D \vec{\theta}^*, \tag{3.6.5}$$

$$\langle \theta^{i_1} \overline{\theta^{i_2}} \theta^{i_3} \overline{\theta^{i_4}} \rangle \equiv \int_{\mathbb{R}} \exp\left(i \overline{\theta^i} H_{ij} \theta^j\right) \theta^{i_1} \overline{\theta^{i_2}} \theta^{i_3} \overline{\theta^{i_4}} \mathrm{d}^D \vec{\theta} \mathrm{d}^D \vec{\theta}^*. \tag{3.6.6}$$

Also evaluate the integrals involving only θ or only its complex conjugate $\overline{\theta} \equiv \theta^*$, namely $\langle \theta^{i_1} \dots \theta^{i_N} \rangle$ and $\langle \overline{\theta^{i_1}} \dots \overline{\theta^{i_N}} \rangle$. Can you explain how to get the general function $\langle \theta^{i_1} \overline{\theta^{j_1}} \dots \theta^{i_n} \overline{\theta^{j_n}} \rangle$? This and the above real variable case is the discrete analog of QFT and its Wick's theorem. □

Problem 3.18. Fractional Fermion Number on Scalar Kinks in (1+1)D In this problem we will explore, in (1+1)D with Cartesian coordinates $x^\mu \equiv (t, x)$, how the number operator of a fermion ψ can acquire a fractional eigenvalue in the presence of a background static 'kink' solution to the scalar field ϕ. We start with the total Lagrangian

$$\mathcal{L} \equiv \mathcal{L}_\phi + \mathcal{L}_\psi, \tag{3.6.7}$$

$$\mathcal{L}_\phi \equiv \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\lambda}{4} \left(\phi^2 - \eta^2\right)^2, \tag{3.6.8}$$

$$\mathcal{L}_\psi \equiv \overline{\psi} \left(i \partial\!\!\!/ - g\phi\right) \psi. \tag{3.6.9}$$

Note that the Dirac spinor is a 2−component object in $(1+1)D$. Explain why the following choice is a valid one for the γ^μ matrices. Denoting the Pauli matrices by $\{\sigma^i | i = 1, 2, 3\}$, define

$$\gamma^0 \equiv \sigma^3 \qquad \text{and} \qquad \gamma^1 \equiv i\sigma^1. \tag{3.6.10}$$

Scalar Solutions First focus on the scalar sector. Verify that the following are static solutions. You may be able to actually *derive* them; try plotting the potential.

$$\phi_{\mathrm{V}} = \pm\eta \tag{3.6.11}$$

$$\phi_{\mathrm{K}} = \pm\eta \tanh\left(\eta\sqrt{\lambda/2} \cdot x\right). \tag{3.6.12}$$

Fermion Solutions Next, solve the Dirac equation

$$\left(i\partial\!\!\!/ - g\phi_{\mathrm{V}}\right)\psi = 0. \tag{3.6.13}$$

And, derive the *zero energy* solutions to

$$\left(i\partial\!\!\!/ - g\phi_{\mathrm{K}}\right)\psi = 0. \tag{3.6.14}$$

Fractional Fermion Number Quantize the Dirac fermion in eq. (3.6.14) and explain why the existence of zero energy solutions on a kink background ϕ_{K} leads to half-integer eigenvalues for the fermion number operator. □

4 Classical Electromagnetism

4.1 Minkowski: $d-$dimensions

In this section we will discuss in some detail Minkowski spacetime electromagnetism to illustrate both its Lorentz and gauge symmetries. It will also provide us the opportunity to introduce the action principle, which is key formulating both classical and quantum field theories.

Maxwell & Lorentz We begin with Maxwell's equations in the following Lorentz covariant form, written in Cartesian coordinates $\{x^\mu\}$ so that $g_{\mu\nu} = \eta_{\mu\nu}$:

$$\partial_\mu F^{\mu\nu} = J^\nu, \qquad \partial_{[\mu} F_{\alpha\beta]} = 0, \qquad F_{\mu\nu} = -F_{\nu\mu}. \tag{4.1.1}$$

(When we quantize the theory, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$; but for now we may view $F_{\mu\nu}$ as independent fields.) The $J^\mu \equiv \rho v^\mu$ is the electromagnetic current. Assuming J^μ is timelike, v^μ is its $d-$proper velocity with $v^2 \equiv v^\mu v_\mu = 1$; and $\rho \equiv J^\mu v_\mu$ is the electric charge in the (local) rest frame where $v^\mu = \delta^\mu_0$. Defined this way, ρ is a Lorentz scalar and J^μ is a Lorentz vector since v^μ is a Lorentz vector. It is then reasonable to suppose $F_{\mu\nu}$ is a rank-2 Lorentz tensor. Specifically, let two inertial frames $\{x^\mu\}$ and $\{x'^\mu\}$ be related via the Lorentz transformation

$$x^\mu = \Lambda^\mu{}_\alpha x'^\alpha, \qquad \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta_{\mu\nu} = \eta_{\alpha\beta}. \tag{4.1.2}$$

The same rules would apply to any vector v^μ; specifically,

$$v^\mu(x^\alpha = \Lambda^\alpha{}_\beta x'^\beta) = \Lambda^\mu{}_\nu v^{\nu'}(x'), \tag{4.1.3}$$

$$v^{\mu'}(x') = (\Lambda^{-1})^\mu{}_\nu v^\nu(x^\alpha = \Lambda^\alpha{}_\beta x'^\beta). \tag{4.1.4}$$

(Note: $v^\mu(x)$ is the μ-th component of the vector in the $\{x^\alpha\}$ frame; while $v^{\mu'}(x')$ is the μ-th component of the vector in the $\{x'^\alpha\}$ frame.) Then the Faraday tensor transforms as

$$F_{\alpha'\beta'}(x') = F_{\mu\nu}\left(x(x') = \Lambda \cdot x'\right) \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \tag{4.1.5}$$

Its derivatives are also Lorentz covariant, for keeping in mind eq. (4.1.2),

$$\partial_{\lambda'} F_{\alpha'\beta'}(x') = \frac{\partial x^\sigma}{\partial x'^\lambda} \partial_\sigma F_{\mu\nu}\left(x(x') = \Lambda \cdot x'\right) \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \tag{4.1.6}$$

$$= \Lambda^\sigma{}_\lambda \partial_\sigma F_{\mu\nu}\left(x(x')\right) \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta. \tag{4.1.7}$$

This immediately tells us $\partial_\mu F^{\mu\nu} = \eta^{\mu\alpha}\partial_\mu F_{\alpha\beta}\eta^{\beta\nu}$ in eq. (4.1.1) is a Lorentz vector.

(3+1)D Lorentz Force Law & $F^{\mu\nu}$ vs $(\vec{E}, \vec{B})$ The electromagnetic force on a point charge q of mass m must take a Lorentz covariant form $m \mathrm{d}z^\mu/\mathrm{d}\tau^2 = f^\mu$, where $z^\mu(\tau)$ is its worldline trajectory and τ is the associated proper time. The force f^μ itself must be built out of $F^{\mu\nu}$ and the proper velocity, if we are to recover the $\vec{E} + \vec{v} \times \vec{B}$ in the non-relativistic limit. A good guess would be to assert

$$m\frac{\mathrm{d}^2 z^\mu}{\mathrm{d}\tau^2} = qF^\mu{}_\nu v^\nu. \tag{4.1.8}$$

Taking the non-relativistic limit, and focusing only on the spatial components,

$$m\frac{\mathrm{d}^2 z^\mu}{\mathrm{d}t} = q\left(F^i{}_0 + F^i{}_j\frac{\mathrm{d}z^i}{\mathrm{d}t}\right); \tag{4.1.9}$$

allowing us to identify the electric field as

$$E^i = F^{i0} \tag{4.1.10}$$

and the magnetic field as

$$\epsilon^{ijk}B^k = -F^{ij} \qquad \Leftrightarrow \qquad B^k = \frac{1}{2}\epsilon^{ijk}F^{ij}, \qquad \epsilon^{123} \equiv 1. \tag{4.1.11}$$

Notice: we are able to convert the rank 2 object F^{ij} into a spatial vector B^k and vice versa, because the Levi-Civita symbol has 3 indices in 3D space. In spatial dimension D, the magnetic field would be associated with the $(D^2-D)/2 = D(D-1)/2$ number of independent components of F^{ij}. For e.g., in 2D space, the magnetic field is a pseudo-scalar $B = (1/2)\epsilon^{ij}F^{ij}$; in 4D space, we have $4 \cdot 3/2 = 6$ components for the magnetic field.

Problem 4.1. 4D Maxwell's Equations in term of $(\vec{E}, \vec{B})$ Let us check that eq. (4.1.1) does in fact reproduce Maxwell's equations in terms of electric E^i and magnetic B^i fields in 4D. Given a Lorentzian inertial frame, define

$$F^{i0} \equiv E^i \qquad \text{and} \qquad F^{ij} \equiv \epsilon^{ijk}B^k; \tag{4.1.12}$$

with $\epsilon^{123} \equiv -1$. Show that the $\partial_\mu F^{\mu\nu} = J^\nu$ from eq. (4.1.1) translates to

$$\vec{\nabla} \cdot \vec{E} = J^0 \qquad \text{and} \qquad \vec{\nabla} \times \vec{B} - \partial_t \vec{E} = \vec{J}. \tag{4.1.13}$$

(The over-arrow refers to the spatial components; for instance $\vec{B} = (B^1, B^2, B^3)$.) The $\partial_{[\alpha}F_{\mu\nu]} = 0$ from eq. (4.1.1) translates to

$$\partial_t \vec{B} + \vec{\nabla} \times \vec{E} = 0 \qquad \text{and} \qquad \vec{\nabla} \cdot \vec{B} = 0. \tag{4.1.14}$$

Hint: Note that $(\vec{\nabla} \times \vec{A})^i = -\epsilon^{ijk}\partial_j A^k$, for any Cartesian vector $\vec{A}$. Also, when you compute $\partial_{[i}F_{jk]}$, you simply need to set $\{i,j,k\}$ to be any distinct permutation of $\{1,2,3\}$. (Why?)

Next, verify the Lorentz invariant relations, with $\epsilon^{0123} \equiv -1$:

$$F_{\mu\nu}F^{\mu\nu} = -2\left(\vec{E}^2 - \vec{B}^2\right), \qquad \vec{E}^2 \equiv E^i E^i,\ \vec{B}^2 \equiv B^i B^i, \tag{4.1.15}$$

$$\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} = 4\partial_\mu\left(\epsilon^{\mu\nu\alpha\beta}A_\nu\partial_\alpha A_\beta\right) = 8\vec{E}\cdot\vec{B}. \tag{4.1.16}$$

How does $F_{\mu\nu}F^{\mu\nu}$ transform under time reversal, $t \equiv x^0 \to -t$? How does it transform under parity flips, $x^i \to -x^i$ (for a fixed i)? Answer the same questions for $\widetilde{F}^{\mu\nu}F_{\mu\nu}$, where the dual of $F_{\mu\nu}$ is

$$\widetilde{F}^{\mu\nu} \equiv \frac{1}{2}\widetilde{\epsilon}^{\mu\nu\alpha\beta}F_{\alpha\beta}. \tag{4.1.17}$$

$d \neq 4$ Can you comment what the analog of the magnetic field ought to be in spacetime dimensions different from 4 – is it still a 'vector'? – and what is the lowest dimension that the magnetic field still exists? How many components does the electric field have in 1+1 dimensions?

□

Problem 4.2. 4D Lorentz boosts of electric and magnetic fields Consider a Lorentz boost along the 1−direction, where

$$x^\mu = \Lambda^\mu{}_\nu x'^\nu, \tag{4.1.18}$$

and

$$\Lambda^\mu{}_\nu = \begin{bmatrix} \frac{1}{\sqrt{1-v^2}} & -\frac{v}{\sqrt{1-v^2}} & 0 & 0 \\ -\frac{v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad 0 \leq |v| < 1. \tag{4.1.19}$$

Write down the relationship between the electric $\vec{E}(x) = (E^1, E^2, E^3)$ and magnetic $\vec{B}(x) = (B^1, B^2, B^3)$ fields in the $\{x^\mu\}$ frame and the $\vec{E}'(x') = (E^{1'}, E^{2'}, E^{3'})$ and magnetic $\vec{B}'(x') = (B^{1'}, B^{2'}, B^{3'})$ in the $\{x'^\mu\}$ frame. Hint: Use the Lorentz transformation rules for $F^{\mu\nu}$; and assume $E^i \equiv F^{i0}$ and $F^{ij} \equiv \epsilon^{ijk} B^k$ (with $\epsilon^{123} \equiv -1$) holds in all inertial frames. □

Current conservation Taking the divergence of $\partial_\mu F^{\mu\nu} = J^\nu$ yields the conservation of the electric current as a consistency condition. For, by the antisymmetry $F_{\mu\nu} = -F_{\nu\mu}$, $\partial_\nu \partial_\mu F^{\mu\nu} = (1/2)\partial_\nu \partial_\mu F^{\mu\nu} - (1/2)\partial_\mu \partial_\nu F^{\nu\mu} = 0$.

$$\partial_\mu J^\mu = 0. \tag{4.1.20}$$

Problem 4.3. Total charge is constant in all inertial frames Even though we defined ρ in the $J^\mu = \rho v^\mu$ as the charge density in the local rest frame of the electric current itself, we may also define the charge density $J^{\widehat{0}} \equiv u_\mu J^\mu$ in the rest frame of an arbitrary family of inertial time-like observers whose worldlines' tangent vector is $u^\mu \partial_\mu = \partial_\tau$. (In other words, in their frame, the spacetime metric is $\mathrm{d}s^2 = (\mathrm{d}\tau)^2 - \mathrm{d}\vec{x} \cdot \mathrm{d}\vec{x}$.) Show that total charge is independent of the Lorentz frame by demonstrating that

$$Q \equiv \int_{\mathbb{R}^D} \mathrm{d}^D \Sigma_\mu J^\mu, \qquad \mathrm{d}^D \Sigma_\mu \equiv \mathrm{d}^D \vec{x} u_\mu, \quad D \equiv d - 1, \tag{4.1.21}$$

is a constant. □

Vector Potential & Gauge Symmetry The other Maxwell equation (cf eq. (4.1.1)) leads us to introduce a vector potential A_μ. For $\partial_{[\mu} F_{\alpha\beta]} = 0 \Leftrightarrow \mathrm{d}F = 0$ tells us, by the Poincaré lemma, that

$$F = \mathrm{d}A \qquad \Leftrightarrow \qquad F_{\mu\nu} = \partial_{[\mu} A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4.1.22}$$

Notice the dynamics in eq. (4.1.1) is not altered if we add to A_μ any object L_μ that obeys $\mathrm{d}L = 0$, because that does not alter the Faraday tensor: $F = \mathrm{d}(A + L) = F + \mathrm{d}L = F$. Now, $\mathrm{d}L = 0$ means, again by the Poincaré lemma, that $L_\mu = \partial_\mu L$, where L on the right hand side is a scalar. *Gauge symmetry*, in the context of electromagnetism, is the statement that the following replacement involving the gauge potential

$$A_\mu(x) \to A_\mu(x) + \partial_\mu L(x) \tag{4.1.23}$$

leaves the dynamics encoded in Maxwell's equations (4.1.1) unchanged.

The use of the gauge potential A_μ makes the $\mathrm{d}F = 0$ portion of the dynamics in eq. (4.1.1) redundant; and what remains is the vector equation

$$\partial_\mu F^{\mu\nu} = \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = J^\nu. \tag{4.1.24}$$

The symmetry under the gauge transformation of eq. (4.1.23) means that solutions to eq. (4.1.24) cannot be unique – in particular, since A_μ and $A_\mu + \partial_\mu L$ are simultaneously solutions, there really is an infinity of solutions parametrized by the arbitrary function L. In this same vein, by going to Fourier space, namely

$$A_\mu(x) \equiv \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \widetilde{A}_\mu(k) e^{-ik_\mu x^\mu} \qquad \text{and} \qquad J_\mu(x) \equiv \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \widetilde{J}_\mu(k) e^{-ik_\mu x^\mu}, \tag{4.1.25}$$

we may see that the differential operator in eq. (4.1.24) cannot be inverted because it has a zero eigenvalue. Firstly, the Fourier version of eq. (4.1.24) reads

$$-K^{\mu\nu}\widetilde{A}_\mu = \widetilde{J}^\nu, \tag{4.1.26}$$

$$K^{\mu\nu} \equiv k_\sigma k^\sigma \eta^{\mu\nu} - k^\nu k^\mu. \tag{4.1.27}$$

If K^{-1} exists, the solution in Fourier space would be (schematically) $\widetilde{A} = -K^{-1}\widetilde{J}$. However, since $K^{\mu\nu} = K^{\nu\mu}$ is a real symmetric matrix, it must be diagonalizable via an orthogonal transformation, with $\det K^{\mu\nu}$ equal to the product of its eigenvalues. That $\det K^{\mu\nu} = 0$ and therefore K^{-1} does not exist can now be seen by observing that k_μ is in fact its null eigenvector:

$$K^{\mu\nu}k_\mu = (k_\sigma k^\sigma)k^\nu - k^\nu k^\mu k_\mu = 0. \tag{4.1.28}$$

Problem 4.4. Can you explain why eq. (4.1.28) amounts to the statement that $F_{\mu\nu}$ is invariant under the gauge transformation of eq. (4.1.23)? Hint: Consider eq. (4.1.23) in Fourier space.

Lorenz gauge To make $K^{\mu\nu}$ invertible, one *fixes a gauge.* A common choice is the Lorenz gauge; in Fourier spacetime:

$$k^\mu \widetilde{A}_\mu = 0. \tag{4.1.29}$$

In 'position'/real spacetime, this reads instead

$$\partial^\mu A_\mu = 0 \qquad \text{(Lorenz gauge)}. \tag{4.1.30}$$

With the constraint in eq. (4.1.29), Maxwell's equations in eq. (4.1.26) becomes

$$-\left(k_\sigma k^\sigma \widetilde{A}^\nu - k^\nu(k^\mu \widetilde{A}_\mu)\right) = -k_\sigma k^\sigma \widetilde{A}^\nu = \widetilde{J}^\nu. \tag{4.1.31}$$

Now, Maxwell's equations have become invertible:

$$\widetilde{A}_\mu(k) = \frac{\widetilde{J}_\mu(k)}{-k^2}, \qquad k^2 \equiv k_\sigma k^\sigma, \qquad \text{(Lorenz gauge)}. \tag{4.1.32}$$

In position/real spacetime, eq. (4.1.31) is equivalent to

$$\partial^2 A^\nu(x) = J^\nu(x) \qquad \partial^2 \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu. \tag{4.1.33}$$

Problem 4.5. Lorentz covariance Suppose $\Lambda^\alpha{}_\mu$ is a Lorentz transformation; let two inertial frames $\{x^\mu\}$ and $\{x'^\mu\}$ be related via

$$x^\mu = \Lambda^\mu{}_\alpha x'^\alpha + a^\mu, \tag{4.1.34}$$

where a^μ is a constant vector. Suppose we solved the Lorenz gauge Maxwell's equations in the $\{x^\mu\}$ frame, namely

$$\frac{\partial A^\mu(x)}{\partial x^\mu} = 0, \qquad \eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}A_\alpha(x) = J_\alpha(x). \tag{4.1.35}$$

Explain how to solve $A_{\alpha'}(x')$, the solution in the $\{x'^\mu\}$ frame. □

Problem 4.6. Poincaré transformations in Fourier spacetime Given a 1−form $A_\mu(x)$ in Minkowski spacetime – not necessarily the vector potential – let us define its Fourier transform as

$$A_\mu(x) = \int_{\mathbb{R}^{d-1,1}} \frac{\mathrm{d}^d k}{(2\pi)^d} \widetilde{A}_\mu(k) e^{-ik\cdot x}, \qquad k\cdot x \equiv k_\sigma x^\sigma. \tag{4.1.36}$$

Show that, under a Poincaré transformation,

$$x^\mu = \Lambda^\mu{}_\nu x'^\nu + a^\mu, \tag{4.1.37}$$

the Fourier coefficient in the $\{x'\}$ frame

$$A_{\mu'}(x') = \int_{\mathbb{R}^{d-1,1}} \frac{\mathrm{d}^d k'}{(2\pi)^d} \widetilde{A}_{\mu'}(k') e^{-ik'\cdot x'}, \qquad k'\cdot x' \equiv k'_\sigma x'^\sigma, \tag{4.1.38}$$

is related to the one in the $\{x\}$ frame (cf. (4.1.36)) through

$$\widetilde{A}_{\mu'}(k') = \widetilde{A}_\alpha(k) e^{-ik\cdot a} \Lambda^\alpha{}_\mu \Big|_{k_\alpha = k'_\mu (\Lambda^{-1})^\mu{}_\alpha}. \tag{4.1.39}$$

Because the equation obeyed by the Lorenz gauge vector potential, namely eq. (4.1.33), is Lorentz covariant – its Fourier coefficients must transform according to eq. (4.1.39). □

[5] In the Lorenz gauge, we have d Minkowski scalar wave equations, one for each Cartesian component. We may express its position spacetime solution by inverting the Fourier transform in eq. (4.1.32):

$$A_\mu(x) = \int_{\mathbb{R}^{d-1,1}} \mathrm{d}^d x' G^+_d(x-x') J_{\mu'}(x'), \tag{4.1.40}$$

$$G^+_d(x-x') \equiv \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{e^{-ik\cdot(x-x')}}{-k^2}. \tag{4.1.41}$$

[5] Eq. (4.1.33) is valid in any dimension $d \geq 3$. In 2D, the $\mathrm{d}F = 0$ portion of Maxwell's equations is trivial – i.e., *any* F would satisfy it – because there cannot be three distinct indices in $\partial_{[\mu} F_{\alpha\beta]} = 0$.

Because A_μ is not gauge-invariant, its physical interpretation can be ambiguous. Classically it is the electromagnetic fields $F_{\mu\nu}$ that exert forces on charges/currents, so we need its solution. In fact, we may take the curl of eq. (4.1.33) to see that

$$\partial^2 F_{\mu\nu} = \partial_{[\mu} J_{\nu]}; \tag{4.1.42}$$

this means, using the same Green's function in eq. (4.1.41):

$$F_{\mu\nu}(x) = \int_{\mathbb{R}^{d-1,1}} \mathrm{d}^d x' G_d^+(x-x') \partial_{[\mu'} J_{\nu']}(x'). \tag{4.1.43}$$

We may verify that equations (4.1.40) and (4.1.41) solve eq. (4.1.33) readily:

$$\begin{aligned} \partial_x^2 G_d^+(x-x') &= \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\partial_\sigma \partial^\sigma e^{-ik\cdot(x-x')}}{-k^2} \\ &= \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\partial_\sigma (-ik^\rho \delta^\sigma_\rho e^{-ik\cdot(x-x')})}{-k^2} \\ &= \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\partial_\sigma (-ik^\sigma e^{-ik\cdot(x-x')})}{-k^2} \\ &= \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{(-ik_\sigma)(-ik^\sigma) e^{-ik\cdot(x-x')}}{-k^2} \\ &= \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik\cdot(x-x')} = \delta^{(d)}(x-x'); \end{aligned} \tag{4.1.44}$$

with a similar calculation showing $\partial_{x'}^2 G_d^+(x-x') = \delta^{(d)}(x-x')$. To sum,

$$\partial_x^2 G_d^+(x-x') = \partial_{x'}^2 G_d^+(x-x') = \delta^{(d)}(x-x'); \tag{4.1.45}$$

Moreover, comparing each Cartesian component of the wave equation in eq. (4.1.33) with the one obeyed by the Green's function in eq. (4.1.45), we may identify the source J of the Green's function itself to be a unit strength spacetime point source at some fixed location x'. It is often useful to think of x as the spacetime location of some observer; so $x^0 - x'^0 \equiv t - t'$ is the time elapsed while $|\vec{x} - \vec{x}'|$ is the observer-source spatial distance. Altogether, we may now view the solution in eq. (4.1.40) as the sum of the field generated by all spacetime point sources, weighted by the physical electric current $J_\mu(x')$.

We now may verify directly that eq. (4.1.40) is indeed a solution to eq. (4.1.33).

$$\begin{aligned} \partial_x^2 A_\mu(x) = \partial_x^2 \left(\int_{\mathbb{R}^{d-1,1}} \mathrm{d}^d x' G_d^+(x-x') J_{\mu'}(x') \right) &= \int_{\mathbb{R}^{d-1,1}} \mathrm{d}^d x' \delta^{(d)}(x-x') J_{\mu'}(x') \\ &= J_\mu(x). \end{aligned} \tag{4.1.46}$$

Lorenz gauge: Existence That we have managed to solve Maxwell's equations using the Lorenz gauge, likely convinces the practical physicist that the Lorenz gauge itself surely exists. However, it is certainly possible to provide a general argument. For suppose $\partial^\mu A_\mu$ were not zero, then all one has to show is that we may perform a gauge transformation (cf. (4.1.23)) that would render the new gauge potential $A'_\mu \equiv A_\mu - \partial_\mu L$ satisfy

$$\partial^\mu A'_\mu = \partial^\mu A_\mu - \partial^2 L = 0. \tag{4.1.47}$$

But all that means is, we have to solve $\partial^2 L = \partial^\mu A_\mu$; and since the Green's function $1/\partial^2$ exists, we have proved the assertion.

Lorenz gauge and current conservation You may have noticed, by taking the divergence of both sides of eq. (4.1.33),

$$\partial^2 \left(\partial^\sigma A_\sigma\right) = \partial^\sigma J_\sigma. \tag{4.1.48}$$

This teaches us the consistency of the Lorenz gauge is intimately tied to the conservation of the electric current $\partial^\sigma J_\sigma = 0$. Another way to see this, is to take the time derivative of the divergence of the vector potential, followed by subtracting and adding the spatial Laplacian of A_0 so that $\partial^2 A_0 = J_0$ may be employed:

$$\begin{aligned} \partial^\sigma \dot{A}_\sigma = \ddot{A}_0 + \partial^i \dot{A}_i &= \partial^0\partial_0 A_0 + \partial^i\partial_i A_0 + \partial^i\partial_0 A_i - \partial^i\partial_i A_0 \\ &= \partial^2 A_0 - \partial^i\left(\partial_i A_0 - \partial_0 A_i\right) \\ \partial_0\left(\partial^\sigma A_\sigma\right) &= J_0 - \partial^i F_{i0}. \end{aligned} \tag{4.1.49}$$

Notice the right hand side of the last line is zero if the $\nu = 0$ component of $\partial_\mu F^{\mu\nu} = J^\nu$ is obeyed – and if the latter is obeyed the 'left-hand-side' of Lorenz gauge condition $\partial_\mu A^\mu$ is a time independent quantity.

4.2 Gauge Invariant Variables for Electromagnetic Vector Potential

Although the vector potential A_μ itself is not a gauge invariant object, we will now exploit the spatial translation symmetry of Minkowski spacetime to seek a gauge-invariant set of partial differential equations involving a "scalar-vector" decomposition of A_μ. There are at least two reasons for doing so.

- We will witness how, for a given inertial frame, the only portion of the vector potential A_μ that obeys a wave equation is its gauge-invariant "transverse" spatial portion. (Even though every component of A_μ in the Lorenz gauge (cf. eq. (4.1.33)) obeys the wave equation, remember such a statement is not gauge-invariant.) We shall also identify a gauge-invariant scalar potential sourced by charge density.

- This will be a warm-up to an analogous analysis for gravitation linearized about a Minkowski "background" spacetime.

Scalar-Vector Decomposition The scalar-vector decomposition is the statement that the spatial components of the vector potential may be expressed as a gradient of a scalar α plus a transverse vector α_i:

$$A_i = \partial_i \alpha + \alpha_i, \tag{4.2.1}$$

where by "transverse" we mean

$$\partial_i \alpha_i = 0. \tag{4.2.2}$$

To demonstrate the generality of eq. (4.2.1) we shall first write A_i in Fourier space

$$A_i(t, \vec{x}) = \int_{\mathbb{R}^D} \frac{\mathrm{d}^D \vec{k}}{(2\pi)^D} \widetilde{A}_i(t, \vec{k}) e^{i\vec{k}\cdot\vec{x}}; \tag{4.2.3}$$

where $\vec{k} \cdot \vec{x} \equiv \delta_{ij} k^i x^j = -k_j x^j$. Every spatial derivative ∂_j acting on $A_i(t, \vec{x})$ becomes in Fourier space a $-ik_j$, since

$$\begin{aligned}
\partial_j A_i &= \int_{\mathbb{R}^D} \frac{\mathrm{d}^D \vec{k}}{(2\pi)^D} \partial_j \left(i\delta_{ab} k^a x^b\right) \widetilde{A}_i(t, \vec{k}) e^{i\vec{k}\cdot\vec{x}} \\
&= \int_{\mathbb{R}^D} \frac{\mathrm{d}^D \vec{k}}{(2\pi)^D} \left(i\delta_{ab} k^a \delta^b_j\right) \widetilde{A}_i(t, \vec{k}) e^{i\vec{k}\cdot\vec{x}} \\
&= \int_{\mathbb{R}^D} \frac{\mathrm{d}^D \vec{k}}{(2\pi)^D} i k^j \widetilde{A}_i(t, \vec{k}) e^{i\vec{k}\cdot\vec{x}} \\
&= \int_{\mathbb{R}^D} \frac{\mathrm{d}^D \vec{k}}{(2\pi)^D} (-ik_j) \widetilde{A}_i(t, \vec{k}) e^{i\vec{k}\cdot\vec{x}}.
\end{aligned} \tag{4.2.4}$$

As such, the transverse property of $\alpha_i(t, \vec{x})$ would in Fourier space become

$$-ik_i \widetilde{\alpha}_i(t, \vec{k}) = 0. \tag{4.2.5}$$

At this point we simply write down

$$\widetilde{A}_i(t, \vec{k}) = \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}\right) \widetilde{A}_j(t, \vec{k}) + \frac{k_i k_j}{\vec{k}^2} \widetilde{A}_j(t, \vec{k}). \tag{4.2.6}$$

This is mere tautology, of course. However, we may now check that the first term on the left hand side of eq. (4.2.6) is transverse:

$$-ik_i \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}\right) \widetilde{A}_j(t, \vec{k}) = -i\left(k_j - \frac{\vec{k}^2 k_j}{\vec{k}^2}\right) \widetilde{A}_j(t, \vec{k}) = 0. \tag{4.2.7}$$

The second term on the right hand side of eq. (4.2.6) is a gradient because it is

$$-ik_i \left(\frac{ik_j}{\vec{k}^2} \widetilde{A}_j\right). \tag{4.2.8}$$

To sum, we have identified the α and α_i terms of eq. (4.2.1) as

$$\alpha(t, \vec{x}) = \int_{\mathbb{R}^D} \frac{\mathrm{d}^D \vec{k}}{(2\pi)^D} \frac{ik_j}{\vec{k}^2} \widetilde{A}_j(t, \vec{k}) e^{i\vec{k}\cdot\vec{x}}; \tag{4.2.9}$$

and the transverse portion as

$$\alpha_i(t,\vec{x}) = \int_{\mathbb{R}^D} \frac{\mathrm{d}^D\vec{k}}{(2\pi)^D} P_{ij}(\vec{k})\widetilde{A}_j(t,\vec{k})e^{i\vec{k}\cdot\vec{x}},$$
$$P_{ij}(\vec{k}) \equiv \delta_{ij} - \frac{k_i k_j}{\vec{k}^2}. \tag{4.2.10}$$

Notice it is really the projector P_{ij} that is "transverse"; i.e.

$$k_i P_{ij}(\vec{k}) = 0. \tag{4.2.11}$$

Let us also note that this scalar-vector decomposition is unique, in that – if we have the Fourier-space equation

$$-ik_i\widetilde{\alpha} + \widetilde{\alpha}_i = -ik_i\widetilde{\beta} + \widetilde{\beta}_i, \tag{4.2.12}$$

where $k_i\widetilde{\alpha}_i = k_i\widetilde{\beta}_i = 0$, then

$$\widetilde{\alpha} = \widetilde{\beta} \qquad \text{and} \qquad \widetilde{\alpha}_i = \widetilde{\beta}_i. \tag{4.2.13}$$

For, we may first "dot" both sides of eq. (4.2.12) with $\vec{k}$ and see that – for $\vec{k} \neq \vec{0}$,

$$\vec{k}^2\widetilde{\alpha} = \vec{k}^2\widetilde{\beta} \qquad \Leftrightarrow \qquad \widetilde{\alpha} = \widetilde{\beta}. \tag{4.2.14}$$

Plugging this result back into eq. (4.2.12), we also conclude $\widetilde{\alpha}_i = \widetilde{\beta}_i$.

Now, this scalar-vector decomposition is really just a mathematical fact, and may even be performed in a curved space – as long as the latter is infinite – since it depends on the existence of the Fourier transform and not on the metric structure. (A finite space would call for a discrete Fourier-like series of sorts.) However, to determine its usefulness, we would need to insert it into the partial differential equations obeyed by A_i, where the metric structure does matter. As we now turn to examine, because of the spatial translation symmetry of Minkowski spacetime, Maxwell's equations themselves admit a scalar-vector decomposition. This, in turn, would lead to PDEs for the gauge-invariant portions of A_μ.

Problem 4.7. Gauge transformations We first define

$$\Phi \equiv A_0 - \dot{\alpha}, \tag{4.2.15}$$
$$A_i^{\mathrm{T}} \equiv \alpha_i. \tag{4.2.16}$$

By first doing a scalar-vector decomposition of the gauge transformation rules $A_\mu \to A_\mu + \partial_\mu\Lambda$, show that Φ and A_i^{T} are gauge-invariant. (Recall the uniqueness discussion above.) Proceed to show that

$$F_{0i} = \dot{\alpha}_i - \partial_i\Phi \tag{4.2.17}$$
$$F_{ij} = \partial_{[i}\alpha_{j]}. \tag{4.2.18}$$

Electric current We also need to perform a scalar-vector decomposition of the electric current

$$J_\mu \equiv (\rho_{\rm E}, \partial_i \mathcal{J} + \mathcal{J}_i) . \tag{4.2.19}$$

Show that its conservation, namely $\partial^\mu J_\mu$, leads to

$$\dot{\rho}_{\rm E} = \vec{\nabla}^2 \mathcal{J} . \tag{4.2.20}$$

Notice the transverse portion of the current does not appear in the conservation law. □

Maxwell's Equations At this point, we are ready to write down Maxwell's equations $\partial^\mu F_{\mu\nu} = J_\nu$. From eq. (4.2.17), the $\nu = 0$ component is

$$-\partial_i F_{i0} = \partial_i (\dot{\alpha}_i - \partial_i \Phi) = -\vec{\nabla}^2 \Phi = \rho_{\rm E} . \tag{4.2.21}$$

The $\nu = i$ component of $\partial^\mu F_{\mu\nu} = J_\nu$, according to eq. (4.2.17) and (4.2.18),

$$\partial_0 F_{0i} - \partial_j F_{ji} = \partial_i \mathcal{J} + \mathcal{J}_i \tag{4.2.22}$$

$$\ddot{\alpha}_i - \partial_i \dot{\Phi} - \partial_j \left(\partial_j \alpha_i - \partial_i \alpha_j \right) = \partial_i \mathcal{J} + \mathcal{J}_i \tag{4.2.23}$$

$$\partial^2 \alpha_i - \partial_i \dot{\Phi} = \mathcal{J}_i + \partial_i \mathcal{J} . \tag{4.2.24}$$

As already advertised, we see that the spatial components of Maxwell's equations does admit a scalar-vector decomposition. By the uniqueness argument above, we may read off the "transverse-vector" portion to be

$$\partial^2 \alpha_i = \mathcal{J}_i . \tag{4.2.25}$$

and the "scalar" portion to be

$$-\dot{\Phi} = \mathcal{J} . \tag{4.2.26}$$

We have gotten 3 (groups of) equations – (4.2.21), (4.2.25), (4.2.26) – for 2 sets of variables (Φ, α_i). Let us argue that eq. (4.2.26) is actually redundant. Taking into account eq. (4.2.20), we may take a time derivative of both sides of eq. (4.2.21),

$$-\vec{\nabla}^2 \dot{\Phi} = \dot{\rho}_{\rm E} = \vec{\nabla}^2 \mathcal{J} . \tag{4.2.27}$$

For the physically realistic case of isolated electric currents, where we may assume implies both $\dot{\Phi} \to 0$ and $\mathcal{J} \to 0$ as the observer-J_i distance goes to infinity, the solution to this above Poisson equation is then unique. This hands us eq. (4.2.26).

> **Gauge-Invariant Formalism** To sum: for physically realistic situations in Minkowski spacetime, if we perform a scalar-vector decomposition of the photon vector potential A_μ through eq. (4.2.1) and that of the current J_μ through eq. (4.2.19), we find a gauge-invariant Poisson equation
>
> $$-\vec{\nabla}^2 \Phi = \rho_{\rm E} \tag{4.2.28}$$

for the scalar potential Φ sourced by the electric charge density ρ_{E}; as well as a gauge-invariant wave equation

$$\partial^2 \alpha_i = \mathcal{J}_i; \tag{4.2.29}$$

for the transverse photon α_i sourced by the transverse portion of the electric current $\mathcal{J}_i$. The gauge-invariant scalar Φ and photon $A_i^{\mathrm{T}} \equiv \alpha_i$ are defined in equations (4.2.15) and (4.2.16).

These illuminate the theoretical structure of electromagnetism.[6]

Vacuum solution & Spin/Helicity-1 As we now turn to study, the vacuum solutions to the transverse-vector portion of A_μ may be identified with massless spin-1 photons in (3+1)-dimensions. By vacuum we mean the absence of any electric current, namely $J_\mu = 0$. For a given inertial frame, eq. (4.2.28) tells us $\Phi = 0$ (assuming $\Phi \to 0$ at spatial infinity); and eq. (4.2.29) translates to the vacuum wave equation

$$\partial^2 \alpha_i = 0. \tag{4.2.30}$$

In Fourier space we may immediately write down

$$\alpha_i(t, \vec{x}) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 \vec{k}}{(2\pi)^{d-1}} \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2} \right) \left\{ \epsilon_j(\vec{k}) e^{-ik\cdot x} + \text{c.c.} \right\}, \tag{4.2.31}$$

$$e^{-ik\cdot x} = e^{-i|\vec{k}|t + i\vec{k}\cdot\vec{x}}. \tag{4.2.32}$$

Here, the ϵ_j is an arbitrary $\vec{k}-$dependent $(d-1)$-component object and 'c.c.' is the complex conjugate of the preceding term.

Let us examine a single $\vec{k}-$mode and suppose k_i points in the positive $3-$axis, so that

$$k_\mu = k(1, 0, 0, -1) \qquad \text{and} \qquad k^\mu = k(1, 0, 0, 1). \tag{4.2.33}$$

This means the plane wave itself becomes

$$\exp(-ik_\mu x^\mu) = \exp(-ik(t - x^3)); \tag{4.2.34}$$

i.e., it indeed describes propagation in the positive $3-$direction. The polarization vector may then be decomposed as follows:

$$\epsilon_j = \kappa \cdot \epsilon^{\parallel}{}_j + a_+ \epsilon^+{}_j + a_- \epsilon^-{}_j; \tag{4.2.35}$$

where the κ and a's are (scalar) complex amplitudes; while the basis vectors $\epsilon^{\pm}$ are

$$\epsilon^{\parallel}{}_j \equiv (0, 0, 1)^T, \tag{4.2.36}$$

$$\epsilon^{\pm}{}_\mu \equiv \frac{1}{\sqrt{2}} (0, \mp 1, i, 0)^T. \tag{4.2.37}$$

[6] *(1+1)D remark:* The one constraint $\partial_i \alpha_i = 0$ obeyed by the spin-1 photon α_i means it has really $D-1 = d-2$ independent components, since in Fourier space $k_i \widetilde{\alpha}_i = 0$ implies (for $\vec{k} \neq 0$) the $\{\widetilde{\alpha}_i\}$ are linearly dependent. In particular, in $(1+1)$D $k_1 \widetilde{\alpha}_1 = 0$ and as long as $k_1 \neq 0$, the spin-1 photon itself is trivial: $\widetilde{\alpha}_1 = 0$.

Problem 4.8. Show that for $k_\mu = k(1,0,0,-1)$ and referring to eq. (4.2.35),

$$\left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}\right)\epsilon_j = a_+ {\epsilon^+}_i + a_- {\epsilon^-}_i. \tag{4.2.38}$$

That is, in a given inertial frame, the projector in eq. (4.2.31) selects only the 2D space of polarization vectors perpendicular to $\vec{k}$. □

Now, under the following rotation on the $(1,2)$-plane orthogonal to $\vec{k}$, namely

$$\widehat{R}(\theta)_{ij} \doteq \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.2.39}$$

the spatial polarizations in eq. (4.2.37) transform as

$$\widehat{R}(\theta)_{ij} {\epsilon^\pm}_j = \left(e^{-i\theta J}\right)_{ij} {\epsilon^\pm}_j = e^{\pm i\theta} {\epsilon^\pm}_j. \tag{4.2.40}$$

These $\epsilon^\pm$ are the spin-1 modes because they are eigenvectors of the (Hermitian) generator of rotations on the 2D (x^1, x^2) plane.

Problem 4.9. Verify eq. (4.2.40). □

(3+1)D Spin-1 Waves To sum, given an inertial frame, the electromagnetic vector potential A_μ in vacuum is given by the following superposition of spin-1 waves:

$$\alpha_j(t, \vec{x}) = \text{Re} \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \left(a_+ {\epsilon^+}_j(\vec{k}) + a_- {\epsilon^-}_j(\vec{k})\right) e^{-ik\cdot x}, \tag{4.2.41}$$

where ${\epsilon^\pm}_\mu$ are spatial polarization tensors orthogonal to the k_i; and, under a rotation by an angle θ around the plane perpendicular to k_i transforms as $\epsilon^\pm \to \exp(\pm i\theta)\epsilon^\pm$.

Problem 4.10. Circularly polarized light from 4D spin-1 Consider a single spin-1 vacuum plane wave (cf. (4.2.37)) propagating along the 3−axis, with $k_\mu = k(1,0,0,-1)$:

$$\alpha_j^\pm(t,x,y,z) \equiv \text{Re}\left\{a_\pm {\epsilon^\pm}_j e^{-ik(t-z)}\right\}, \qquad a_\pm \in \mathbb{R}. \tag{4.2.42}$$

Compute the electric field ${}_\pm E^i = F^{i0}$ and show that these plane waves give rise to circularly polarized light, i.e., for either a fixed time t or spatial location z – the electric field direction rotates in a circular fashion:

$${}_\pm E^i = -\frac{k a_\pm}{\sqrt{2}} \left(\pm \sin(k(t-z))\widehat{x}^i + \cos(k(t-z))\widehat{y}^i\right), \tag{4.2.43}$$

where $\widehat{x}$ and $\widehat{y}$ are the unit vectors in the 1− and 2−directions:

$$\widehat{x}^i \doteq (1,0,0) \qquad \text{and} \qquad \widehat{y}^i \doteq (0,1,0). \tag{4.2.44}$$

□

4.3 4 dimensions

4D Maxwell We now focus on the physically most relevant case of $(3+1)D$. In 4D, the wave operator ∂^2 has the following inverse – i.e., retarded Green's function – that obeys causality:

$$G_4^+(x-x') \equiv \frac{\delta(t-t'-|\vec{x}-\vec{x}'|)}{4\pi|\vec{x}-\vec{x}'|}, \qquad x^\mu = (t,\vec{x}),\ x'^\mu = (t',\vec{x}'), \tag{4.3.1}$$

$$\partial_x^2 G_4^+(x-x') = \partial_{x'}^2 G_4^+(x-x') = \delta^{(4)}(x-x'), \tag{4.3.2}$$

$$\partial_x^2 \equiv \eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} \qquad \partial_{x'}^2 \equiv \eta^{\mu\nu}\frac{\partial}{\partial x'^\mu}\frac{\partial}{\partial x'^\nu}. \tag{4.3.3}$$

To see that G_4^+ obeys causality, that it respects the principle that cause precedes effect, one merely needs to focus on the δ-function in eq. (4.3.1). It is non-zero only when the time elapsed $t-t'$ is precisely equal to the observer-source distance $|\vec{x}-\vec{x}'|$. That is, if the source is located at a spatial distance $R = |\vec{x}-\vec{x}'|$ away from the observer, and if the source emitted an instantaneous flash at time t', then the observer would see a signal at time R later (i.e., at $t = t'+R$). In other words, the retarded Green's function propagates signals on the *forward* light cone of the source.[7]

Problem 4.11. Analogy: Driven Simple Harmonic Oscillator Suppose we only Fourier-transformed the spatial coordinates in the Lorenz gauge Maxwell eq. (4.1.33). Show that this leads to

$$\ddot{\widetilde{A}}_\mu(t,\vec{k}) + k^2\widetilde{A}_\mu(t,\vec{k}) = \widetilde{J}_\mu(t,\vec{k}), \qquad k \equiv |\vec{k}|. \tag{4.3.4}$$

[8]Compare this to the simple harmonic oscillator (in flat space), with Cartesian coordinate vector $\vec{q}(t)$, mass m, spring constant σ, and driven by an external force $\vec{f}$:

$$m\ddot{\vec{q}} + \sigma\vec{q} = \vec{f}, \tag{4.3.5}$$

where each over-dot corresponds to a time derivative. Identify k^2 and $\widetilde{J}$ in eq. (4.3.4) with the appropriate quantities in eq. (4.3.5). Even though the Lorenz gauge Maxwell equations are fully relativistic, notice the analogy with the non-relativistic driven harmonic oscillator! In particular, when the electric current is not present (i.e., $J_\mu = 0$), the 'mixed-space' equations of (4.3.4) are in fact a collection of free simple harmonic oscillators.

Now, how does one solve eq. (4.3.5)? Explain why the inverse of $(\mathrm{d}/\mathrm{d}t)^2 + k^2$ is

$$G_{\text{SHO}}(t-t',k) = -\int_{\mathbb{R}}\frac{\mathrm{d}\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega^2-k^2}. \tag{4.3.6}$$

That is, verify that this equation satisfies

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}+k^2\right)G_{\text{SHO}}(t-t',k) = \left(\frac{\mathrm{d}^2}{\mathrm{d}t'^2}+k^2\right)G_{\text{SHO}}(t-t',k) = \delta(t-t'). \tag{4.3.7}$$

---

[7]The advanced Green's function $G_4^-(x-x') = \delta(t-t'+|\vec{x}-\vec{x}'|)/(4\pi|\vec{x}-\vec{x}'|)$ also solves eq. (4.1.45), but propagates signals on the past light cone: $t = t'-R$.

[8]This equation actually holds in all dimensions $d \geq 3$.

If one tries to integrate ω over the real line in eq. (4.3.6), one runs into trouble – explain the issue. In other words, eq. (4.3.6) is actually ambiguous as it stands.

Now evaluate the Green's function G^+_{SHO} in eq. (4.3.6) using the contour running just slightly above the real line, i.e., $\omega \in (-\infty + i0^+, +\infty + i0^+)$. You should find

$$G^+_{\text{SHO}}(t-t',k) = \Theta(t-t')\frac{\sin\left(k(t-t')\right)}{k}. \tag{4.3.8}$$

Here, Θ is the step function

$$\Theta(x) = 1, \qquad \text{if } x > 0, \tag{4.3.9}$$
$$= 0, \qquad \text{if } x < 0. \tag{4.3.10}$$

Hence, the mixed-space Maxwell's equations have the solution

$$\widetilde{A}_\mu(t,\vec{k}) = \int_{-\infty}^{t} \mathrm{d}t' G^+_{\text{SHO}}(t-t',k)\widetilde{J}_\mu(t',\vec{k}). \tag{4.3.11}$$

By performing an inverse-Fourier transform, namely

$$A_\mu(x) = \int_{\mathbb{R}^{3,1}} \mathrm{d}^4x' G^+_4(x-x')J_{\mu'}(x'), \tag{4.3.12}$$

arrive at the expression in eq. (4.3.1) $\square$

Doppler Shift For each Lorenz-gauge plane wave in an inertial frame $\{x^\mu = (t,\vec{x})\}$,

$$\epsilon^{\pm}{}_\mu(k)\exp(-ik\cdot x) = \epsilon^{\pm}{}_\mu(k)\exp(-ik_jx^j)\exp(-i\omega t), \qquad \omega \equiv |\vec{k}|, \tag{4.3.13}$$

we may read off its frequency ω as the coefficient of the time coordinate t. Quantum mechanics tells us ω is also the energy of the associated photon. Suppose a different Lorentz inertial frame $\{x'\}$ is related to the previous through the Lorentz transformation $\Lambda^\alpha{}_\mu$: $x^\alpha = \Lambda^\alpha{}_\mu x'^\mu$. Because the phase in the plane wave solution of eq. (4.3.13) is a scalar, in the $\{x'\}$ Lorentz frame

$$-ik_\alpha x^\alpha = -ik_\alpha\Lambda^\alpha{}_\mu x'^\mu = -i(k_\alpha\Lambda^\alpha{}_0)t' - i(k_\alpha\Lambda^\alpha{}_i)x'^i. \tag{4.3.14}$$

The frequency ω' and hence the photon's energy in this $\{x'\}$ frame is therefore

$$\omega' = k_\alpha\Lambda^\alpha{}_0 = \omega\left(\Lambda^0{}_0 + \widehat{k}_i\Lambda^i{}_0\right) \tag{4.3.15}$$
$$\widehat{k}_i \equiv k_i/|\vec{k}| = k_i/\omega. \tag{4.3.16}$$

There is a slightly different way to express this redshift result that would help us generalize the analysis to curved spacetime, at least in the high frequency 'JWKB' limit. To extract the frequency directly from the phase $S \equiv k\cdot x$, we may take its time derivative using the unit norm vector $u \equiv \partial_t = \partial_0$ that we may associate with the worldlines of observers at rest in the $\{x\}$ frame:

$$u^\mu\partial_\mu S = \partial_0(k_\alpha x^\alpha) = \omega. \tag{4.3.17}$$

The observers at rest in the $\{x'\}$ frame have $u' \equiv \partial_{t'} = \partial_{0'}$ as their timelike unit norm tangent vector. (Note: $x^\alpha = \Lambda^\alpha{}_\mu x'^\mu \Leftrightarrow \partial_{\mu'} = \Lambda^\alpha{}_\mu\partial_\alpha$.) The energy of the photon is then

$$u'^\alpha\partial_{\alpha'}S = \partial_{t'}S = \Lambda^\alpha{}_0\partial_\alpha(k\cdot x)$$
$$= \Lambda^\alpha{}_0 k_\alpha = \omega\left(\Lambda^0{}_0 + \widehat{k}_i\Lambda^i{}_0\right). \tag{4.3.18}$$

Problem 4.12. Consider a single photon with wave vector $k_\mu = \omega(1, \widehat{n}_i)$ (where $\widehat{n}_i \widehat{n}_j \delta^{ij} = 1$) in some inertial frame $\{x^\mu\}$. Let a family of inertial observers be moving with constant velocity $v^\mu \equiv (1, v^i)$ with respect to the frame $\{x^\mu\}$. What is the photon's frequency ω' in their frame? Compute the redshift formula for ω'/ω. Comment on the redshift result when v^i is (anti)parallel to $\widehat{n}_i$ and when v^i is perpendicular to $\widehat{n}_i$. □

4.4 Symmetry and Conservation Laws for Free Photons

Free Photons We turn to the free photon, satisfying the free Maxwell equation

$$\partial_\mu F^{\mu\nu} = 0 \qquad \text{with} \qquad F_{\mu\nu} \equiv \partial_{[\mu} A_{\nu]}, \tag{4.4.1}$$

$$0 = \partial_{[\alpha} F_{\beta\gamma]}; \tag{4.4.2}$$

which in turn arises from the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{4.4.3}$$

Here, there is a subtlety. While $F_{\mu\nu}$ is a Lorentz rank−2 tensor; the transformation properties of A_μ depends on the choice of gauge. If a non-covariant gauge is chosen, there is no reason to claim A_μ is a rank−1 Lorentz tensor. To this end, for technical convenience, we shall choose the Lorentz covariant Lorenz gauge

$$\partial^\mu A_\mu = 0. \tag{4.4.4}$$

Translations Under spacetime translations, the vector potential then transforms as follows.

$$A_\mu(x) \to A_\mu(x + a) = A_\mu(x) + a^\nu \partial_\nu A_\mu(x) \tag{4.4.5}$$

$$\partial_\mu A_\nu(x) \to \partial_\mu A_\nu(x + a) = \partial_\mu A_\nu(x) + a^\sigma \partial_\sigma \partial_\mu A_\nu(x). \tag{4.4.6}$$

The Lagrangian density transforms as

$$\mathcal{L}(x) \to \mathcal{L}(x) + a^\sigma \partial_\sigma \mathcal{L}(x) \tag{4.4.7}$$

$$\to \mathcal{L} + \partial_\mu (a^\sigma \delta^\mu_\sigma \mathcal{L}). \tag{4.4.8}$$

Whereas, if we had expanded it via its fields,

$$\mathcal{L} \to \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \partial_\lambda A_\tau} a^\sigma \partial_\sigma \partial_\lambda A_\tau(x) \tag{4.4.9}$$

$$= \mathcal{L} + \partial_\lambda \left(\frac{\partial \mathcal{L}}{\partial \partial_\lambda A_\tau} a^\sigma \partial_\sigma A_\tau(x) \right) - \partial_\lambda \left(\frac{\partial \mathcal{L}}{\partial \partial_\lambda A_\tau} \right) a^\sigma \partial_\sigma A_\tau(x). \tag{4.4.10}$$

The Noether current evaluated on the EoM $\partial_\lambda(\partial \mathcal{L}/\partial(\partial_\lambda A_\tau)) = 0$ is

$$J^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\tau} \partial_\nu A_\tau(x) - \delta^\mu{}_\nu \mathcal{L} \tag{4.4.11}$$

$$= -F^{\mu\tau} \partial_\nu A_\tau + \frac{1}{4} \delta^\mu{}_\nu F_{\lambda\tau} F^{\lambda\tau}. \tag{4.4.12}$$

Notice this is not symmetric nor gauge-invariant![9] Let's backtrack a little.

$$a^\sigma \partial_\sigma \mathcal{L} = \delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\tau)} a^\sigma \partial_\sigma \partial_\lambda A_\tau \tag{4.4.13}$$

$$= -a^\sigma F^{\lambda\tau} \partial_\sigma \partial_\lambda A_\tau = -\frac{1}{2} a^\sigma F^{\lambda\tau} \partial_\sigma F_{\lambda\tau}. \tag{4.4.14}$$

Let us use the Bianchi identity $\partial_{[\sigma} F_{\lambda\tau]} = 0$ to deduce

$$\partial_\sigma F_{\lambda\tau} = \partial_\lambda F_{\sigma\tau} + \partial_\tau F_{\lambda\sigma} = -\partial_{[\lambda} F_{\tau]\sigma}. \tag{4.4.15}$$

Inserting this back into the first order variation of the Lagrangian,

$$a^\sigma \partial_\mu (\delta^\mu_\sigma \mathcal{L}) = \frac{1}{2} a^\sigma F^{\lambda\tau} \partial_{[\lambda} F_{\tau]\sigma} \tag{4.4.16}$$

$$= a^\sigma F^{\lambda\tau} \partial_\lambda F_{\tau\sigma} = -a^\sigma F^{\lambda\tau} \partial_\lambda F_{\sigma\tau} \tag{4.4.17}$$

$$= -a^\nu \partial_\mu \left(F^{\mu\sigma} F_{\nu\sigma} \right). \tag{4.4.18}$$

Our gauge invariant and symmetric Noether current is therefore

$$J^{\mu\nu} = -F^{\mu\sigma} F^\nu{}_\sigma + \frac{1}{4} \eta^{\mu\nu} F^{\sigma\rho} F_{\sigma\rho} \equiv T^{\mu\nu} \tag{4.4.19}$$

$$0 = \partial_\mu J^{\mu\nu}. \tag{4.4.20}$$

Lorentz transformations We now turn to the Noether current of Lorentz transformations. For *fixed* (α, β), we have

$$\partial_{\mu'} A_{\nu'}(x') = \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial x^\rho}{\partial x'^\nu} \partial_\sigma A_\rho(x = \Lambda x') \tag{4.4.21}$$

$$= \partial_\mu A_\nu(x \to x') + i\omega \left(\widehat{J}^{\alpha\beta} \right)^\sigma{}_\mu \partial_\sigma A_\nu(x \to x') + i\omega \left(\widehat{J}^{\alpha\beta} \right)^\sigma{}_\nu \partial_\mu A_\sigma(x \to x')$$
$$- i\omega \left(\widehat{J}^{\alpha\beta} \right)^\sigma{}_\rho x'^\rho \partial_{\sigma'} \partial_\mu A_\nu(x \to x'). \tag{4.4.22}$$

The Lagrangian, being a Lorentz scalar, transforms as

$$\mathcal{L}(x = \Lambda x') = \mathcal{L}(x \to x') - i\omega \left(\widehat{J}^{\alpha\beta} \right)^\sigma{}_\rho x'^\rho \partial_{\sigma'} \mathcal{L}(x \to x'). \tag{4.4.23}$$

This means, under $x \to \Lambda x$, we have

$$x^{[\beta} \partial^{\alpha]} \mathcal{L} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \left(\delta^{[\alpha}_\mu \partial^{\beta]} A_\nu + \delta^{[\alpha}_\nu \partial_\mu A^{\beta]} + x^{[\beta} \partial^{\alpha]} \partial_\mu A_\nu \right) \tag{4.4.24}$$

$$= -F^{\mu\nu} \left(\delta^{[\alpha}_\mu \partial^{\beta]} A_\nu + \delta^{[\alpha}_\nu \partial_\mu A^{\beta]} + x^{[\beta} \partial^{\alpha]} \partial_\mu A_\nu \right). \tag{4.4.25}$$

[9] Several sources, including the *An Introduction to Quantum Field Theory* textbook by Peskin and Schroeder and the *Classical Field Theory* book by Davison Soper, stops here. The latter goes on to assert the electromagnetic Noether current generated by spacetime translations is not gauge-invariant nor symmetric because of 'internal spin'. It is true that the Noether current is not unique; a $\partial_\sigma K^{\mu\sigma\nu}$ may always be added to $J^{\mu\nu}$, while still maintaining $\partial_\mu J^{\mu\nu} = 0$, if K is anti-symmetric in the first two indices: $K^{\mu\sigma\nu} = -K^{\sigma\mu\nu}$. However, we will soon witness, if the index anti-symmetries and Bianchi identities are proprerly employed, a symmetric and gauge invariant free Maxwell stress tensor may be obtained.

Using the explicit expression $F^{\mu\nu} = \partial^{[\mu} A^{\nu]}$, and via a direct calculation, we may verify the first two terms on the last line vanish.

$$\partial_\mu \left(x^{[\beta} \eta^{\alpha]\mu} \mathcal{L} \right) = -F^{\mu\nu} x^{[\beta} \partial^{\alpha]} \partial_\mu A_\nu = -\frac{1}{2} F^{\mu\nu} x^{[\beta} \partial^{\alpha]} F_{\mu\nu} \tag{4.4.26}$$

$$= \frac{1}{2} F^{\mu\nu} x^{[\beta} \partial_{[\mu} F_{\nu]}{}^{\alpha]} = -F^{\mu\nu} x^{[\beta} \partial_\mu F^{\alpha]}{}_\nu. \tag{4.4.27}$$

Bianchi was employed in the second line. Recalling the free photon EoM,

$$\partial_\mu \left(x^{[\beta} \eta^{\alpha]\mu} \mathcal{L} \right) = -\partial_\mu \left(F^{\mu\nu} x^{[\beta} F^{\alpha]}{}_\nu \right) + F^{\mu\nu} \delta^{[\beta}_\mu F^{\alpha]}{}_\nu \tag{4.4.28}$$

$$= \partial_\mu \left(-x^{[\beta} F^{\alpha]}{}_\nu F^{\mu\nu} \right) + F^{[\beta}{}_\mu F^{\alpha]}{}_\nu \eta^{\mu\nu}. \tag{4.4.29}$$

The second term on the final line is zero, because we are anti-symmetrizing the indices of a symmetric tensor. Comparison with the stress tensor of the free electromagnetic field, we obtain the angular momentum Noether tensor

$$M^{\mu\alpha\beta} = x^{[\alpha} T^{\beta]\mu} = x^{[\alpha} \left(-F^{\beta]\sigma} F^\mu{}_\sigma + \frac{1}{4} \eta^{\beta]\mu} F_{\sigma\rho} F^{\sigma\rho} \right) \tag{4.4.30}$$

$$0 = \partial_\mu M^{\mu\alpha\beta}. \tag{4.4.31}$$

5 Quantum Field Theory of Photons

5.1 Path Integral for Photons

The action for pure photons is

$$S_\gamma \equiv -\frac{1}{4}\int \mathrm{d}^d x F_{\mu\nu}F^{\mu\nu}, \tag{5.1.1}$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{5.1.2}$$

In Fourier spacetime,

$$S_\gamma = -\frac{1}{4}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\left(k_\mu \widetilde{A}_\nu(k) - k_\nu \widetilde{A}_\mu(k)\right)\left(k^\mu \widetilde{A}^\nu(-k) - k^\nu \widetilde{A}^\mu(-k)\right) \tag{5.1.3}$$

$$= -\frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\widetilde{A}_\mu(k)\left(k^2\eta^{\mu\nu} - k^\mu k^\nu\right)\widetilde{A}_\nu(-k). \tag{5.1.4}$$

Viewed as a linear operator,

$$M^\mu{}_\nu \equiv k^2\delta^\mu{}_\nu - k^\mu k_\nu, \tag{5.1.5}$$

notice k^μ is its null eigenvector

$$M^\mu{}_\nu k^\nu = 0. \tag{5.1.6}$$

Fadeev-Popov Gauge-Fixing Procedure This in turn means $M^\mu{}_\nu$ is not invertible. It also means the portion of $\widetilde{A}(k)$ parallel to k^μ does not show up in the exponential $\exp[iS_\gamma]$. These two issues are related for performing the path integral of photons. Consider the following

$$\int \mathcal{D}\Lambda \det\left[\frac{\delta(\partial^2\Lambda)}{\delta\Lambda}\right]\delta\left[\partial_\mu\left(A^\mu - \partial^\mu\Lambda\right) - \alpha\right] = \int \mathcal{D}\left(\partial^2\Lambda\right)\delta\left[\partial_\mu\left(A^\mu - \partial^\mu\Lambda\right) - \alpha\right] \tag{5.1.7}$$

$$= \int \mathcal{D}\Gamma\delta\left[\partial_\mu A^\mu - \alpha - \Gamma\right] = \int \mathcal{D}\Gamma'\delta\left[\Gamma'\right] = \mathbb{I}. \tag{5.1.8}$$

That means we may insert this into the path integral for photons without altering the result.

$$\left(\prod_\sigma \int \mathcal{D}A_\sigma\right) e^{iS_\gamma} = \left(\prod_\sigma \int \mathcal{D}A_\sigma\right)\int \mathcal{D}\Lambda \det\left[\partial^2\right]\delta\left[\partial_\mu\left(A^\mu - \partial^\mu\Lambda\right) - \alpha\right] e^{iS_\gamma} \tag{5.1.9}$$

Under gauge transformations $A_\sigma \to A_\sigma - \partial_\sigma F$, for arbitrary function F, both the measure $\mathcal{D}A_\sigma$ and the action S_γ remains invariant. Moreover, the insertion of unity is also gauge invariant:

$$\int \mathcal{D}\Lambda \det\left[\partial^2\right]\delta\left[\partial_\mu\left(A^\mu - \partial^\mu\Lambda\right) - \alpha\right] \to \det\left[\partial^2\right]\int \mathcal{D}\Lambda\delta\left[\partial_\mu\left(A^\mu - \partial^\mu(\Lambda + F)\right) - \alpha\right] \tag{5.1.10}$$

$$= \det\left[\partial^2\right]\int \mathcal{D}\Lambda'\delta\left[\partial_\mu\left(A^\mu - \partial^\mu\Lambda'\right) - \alpha\right]. \tag{5.1.11}$$

At this point, we have a fully gauge-invariant path integral. To 'fix a gauge' we massage the integral further by introducing an integral over α involving a Gaussian:

$$\int \mathcal{D}\alpha e^{-\frac{i}{2\xi}\int \mathrm{d}^d x \alpha^2} \left(\prod_\sigma \int \mathcal{D}A_\sigma\right) \int \mathcal{D}\Lambda \det\left[\partial^2\right] \delta\left[\partial_\mu \left(A^\mu - \partial^\mu \Lambda\right) - \alpha\right] e^{iS_\gamma}. \tag{5.1.12}$$

If we performed the $\alpha-$integral last, it would merely introduce an overall constant and therefore does not alter the content of the theory itself. But if we now shift $A \to A + \partial\Lambda$ followed by performing the α integral:

$$\int \mathcal{D}\alpha e^{-\frac{i}{2\xi}\int \mathrm{d}^d x \alpha^2} \left(\prod_\sigma \int \mathcal{D}A_\sigma\right) \int \mathcal{D}\Lambda \det\left[\partial^2\right] \delta\left[\partial_\mu \left(A^\mu - \partial^\mu \Lambda\right) - \alpha\right] e^{iS_\gamma} \tag{5.1.13}$$

$$= \left(\prod_\sigma \int \mathcal{D}A'_\sigma\right) \int \mathcal{D}\alpha e^{-\frac{i}{2\xi}\int \mathrm{d}^d x \alpha^2} \int \mathcal{D}\Lambda \det\left[\partial^2\right] \delta\left[\partial_\mu A'^\mu - \alpha\right] e^{iS_\gamma[A']} \tag{5.1.14}$$

$$= \left(\int \mathcal{D}\Lambda\right) \det\left[\partial^2\right] \left(\prod_\sigma \int \mathcal{D}A'_\sigma\right) \exp\left[iS_\gamma[A'] - \frac{i}{2\xi}\int \mathrm{d}^d x (\partial_\mu A'^\mu)^2\right]. \tag{5.1.15}$$

The Λ integral is now wildly divergent. We may attribute it to the fact that the original $A-$integral had divergences arising from integrating over the components parallel to k_μ; i.e., heuristically, $\int \mathcal{D}\Lambda \sim \int \prod \mathcal{D}A^{\parallel}$. To corroborate this interpretation, let us examine the remaining $A-$dependent integrand, and witness that the resulting analog to $M^\mu{}_\nu$ is now invertible.

$$\begin{aligned} S_\gamma - (2\xi)^{-1}\int \mathrm{d}^d x \, (\partial A)^2 &= -\frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} \widetilde{A}_\mu(k) \left(k^2 \eta^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) k^\mu k^\nu\right) \widetilde{A}_\mu(-k) \\ &\equiv -\frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} \widetilde{A}_\mu(k) M'^{\mu\nu} \widetilde{A}_\mu(-k). \end{aligned} \tag{5.1.16}$$

Let us now proceed to introduce Schwinger's trick; but with the additional constraint that $\partial_\mu J^\mu = 0$ so that $A_\mu J^\mu$ is gauge invariant. To generate the $n-$point photon function, we define

$$Z_J \equiv \frac{\left(\prod_\sigma \int \mathcal{D}A_\sigma\right) \exp\left[iS_\gamma - i(2\xi)^{-1}\int (\partial A)^2 + i\int_x J_\mu(x) A^\mu(x)\right]}{\left(\prod_\sigma \int \mathcal{D}A_\sigma\right) \exp\left[iS_\gamma - i(2\xi)^{-1}\int (\partial A)^2\right]}. \tag{5.1.17}$$

so that nth term in the Taylor series is

$$\frac{i^n}{n!}\int \mathrm{d}^d x_1 \dots \mathrm{d}^d x_n J^{\alpha_1}(x_1) \dots J^{\alpha_n}(x_n) \left\langle 0 \left| \mathsf{T}\{A_{\alpha_1}(x_1) \dots A_{\alpha_n}(x_n)\} \right| 0 \right\rangle. \tag{5.1.18}$$

Shifting the field by

$$\widetilde{A}_\nu \to \widetilde{A}_\nu + i\widetilde{G}_{\nu\lambda}(k)\widetilde{J}^\lambda(k), \tag{5.1.19}$$

the measure remains un-changed while the action now reads

$$S_\gamma - (2\xi)^{-1}\int \mathrm{d}^d x \, (\partial A)^2 + \int J^\mu A_\mu \tag{5.1.20}$$

$$= -\frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} \left(\widetilde{A}_\mu(k) + i\widetilde{G}_{\mu\lambda}(k)\widetilde{J}^\lambda(k)\right) M^{\mu\nu} \left(\widetilde{A}_\nu(-k) + i\widetilde{G}_{\nu\rho}(-k)\widetilde{J}^\rho(-k)\right) \tag{5.1.21}$$

$$+ \frac{1}{2}\int_k \left(\widetilde{J}^\mu(-k)\widetilde{A}_\mu(k) + \widetilde{J}^\mu(-k)i\widetilde{G}_{\mu\lambda}(k)\widetilde{J}^\lambda(k) + \widetilde{J}^\mu(k)\widetilde{A}_\mu(-k) + \widetilde{J}^\mu(k)i\widetilde{G}_{\mu\lambda}(-k)\widetilde{J}^\lambda(-k)\right)$$

$$= -\frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} \Bigg(\widetilde{A}_\mu(k)M^{\mu\nu}\widetilde{A}_\nu(-k) + i\widetilde{G}_{\mu\lambda}(k)\widetilde{J}^\lambda(k)M^{\mu\nu}i\widetilde{G}_{\nu\rho}(-k)\widetilde{J}^\rho(-k)$$

$$- \left(\widetilde{J}^\mu(-k)i\widetilde{G}_{\mu\lambda}(k)\widetilde{J}^\lambda(k) + \widetilde{J}^\mu(k)i\widetilde{G}_{\mu\lambda}(-k)\widetilde{J}^\lambda(-k)\right)$$

$$- \widetilde{A}_\mu(k)\left(\delta^\mu_\rho - M^{\mu\nu}i\widetilde{G}_{\nu\rho}(-k)\right)\widetilde{J}^\rho(-k) - \left(\widetilde{J}^\mu(k)\delta^\nu_\mu - i\widetilde{G}_{\mu\lambda}(k)\widetilde{J}^\lambda(k)M^{\mu\nu}\right)\widetilde{A}_\nu(-k)\Bigg).$$

Problem 5.1. Define $\widetilde{G}_{\mu\nu}$ to be the solution to the equation

$$M'^{\mu\nu}\widetilde{G}_{\nu\lambda} = -i\delta^\mu{}_\lambda, \tag{5.1.22}$$

where $M'^{\mu\nu}$ can be found in eq. (5.1.16). Verify that

$$\widetilde{G}_{\nu\lambda}(k) = -\frac{i}{k^2}\left(\eta_{\nu\lambda} + (\xi - 1)\frac{k_\nu k_\lambda}{k^2}\right). \tag{5.1.23}$$

Note that it is symmetric $\widetilde{G}_{\nu\lambda}(k) = \widetilde{G}_{\lambda\nu}(k)$ and invariant under $k \leftrightarrow -k$. □

At this point, we have

$$iS_\gamma - i(2\xi)^{-1}\int \mathrm{d}^d x\,(\partial A)^2 + i\int J^\mu A_\mu \tag{5.1.24}$$

$$= -\frac{i}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\widetilde{A}_\mu(k)M^{\mu\nu}\widetilde{A}_\nu(-k) + \frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} i\widetilde{J}^\lambda(k)\widetilde{G}_{\nu\rho}(k)i\widetilde{J}^\rho(-k). \tag{5.1.25}$$

The generating function is now

$$Z_J = \exp\left[\frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} i\widetilde{J}^\lambda(k)\widetilde{G}_{\nu\rho}(k)i\widetilde{J}^\rho(-k)\right] \tag{5.1.26}$$

$$= \exp\left[\frac{1}{2}\int \mathrm{d}^d x \int \mathrm{d}^d x'\; iJ^\lambda(x)G_{\nu\rho}(x - x')iJ^\rho(x')\right]. \tag{5.1.27}$$

In particular,

$$\frac{i^2}{2!}\int \mathrm{d}^d x \mathrm{d}^d x' J_\alpha(x)J_\beta(x')\left\langle 0\left|\mathsf{T}A^\alpha(x)A^\beta(x')\right|0\right\rangle = \frac{i^2}{2!}\int \mathrm{d}^d x \mathrm{d}^d x' J_\alpha(x)J_\beta(x')G_F^{\alpha\beta}(x - x'),$$

$$G_F^{\alpha\beta}(x - x') = \int_k \frac{-i\eta^{\alpha\beta}}{k^2 + i\epsilon}e^{-ik\cdot(x-x')}. \tag{5.1.28}$$

Problem 5.2. Gauge-Fixing for General Relativity (GR) The Einstein-Hilbert action for General Relativity is given by

$$S_{\text{EH}} \equiv -2M_{\text{pl}}^2\int \mathrm{d}^d x\sqrt{|g|}\mathcal{R} + \text{surface term}, \tag{5.1.29}$$

where $M_{\rm pl} \equiv (32\pi G_{\rm N})^{-1/2}$, $\mathcal{R}$ is the Ricci scalar built out of the metric $g_{\mu\nu}$ and $\sqrt{|g|}$ is the square root of the absolute value of its determinant. Despite what you may read in popular-science media, it is entirely possible to quantize GR – at least at low energies / long distances. The problem with quantum GR only occurs at very short distances, of order $1/M_{\rm pl}$ (in 4D), where humanity still does not have the resources to probe properly.

When perturbed about flat spacetime,

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_{\rm pl}^{(d/2)-1}}, \tag{5.1.30}$$

you would discover – bonus points if you are able to derive it! – that the quadratic-in-h terms of the action read

$$iS_{\rm EH} = iS_2 + \mathcal{O}\left(h^3\right), \tag{5.1.31}$$

$$S_2 \equiv \int \frac{{\rm d}^d k}{(2\pi)^d}\left(\frac{k^2}{2}\left(\widetilde{h}_{\rho\sigma}(\widetilde{h}^{\rho\sigma})^* - \widetilde{h}^\sigma{}_\sigma(\widetilde{h}^\rho{}_\rho)^*\right)\right.$$
$$\left. + \frac{1}{2}k^\alpha k^\beta \widetilde{h}_{\alpha\beta}(\widetilde{h}^\rho{}_\rho)^* + \frac{1}{2}k^\alpha k^\beta (\widetilde{h}_{\alpha\beta})^*\widetilde{h}^\rho{}_\rho - k^\alpha k^\beta \widetilde{h}_{\alpha\sigma}(\widetilde{h}_\beta{}^\sigma)^*\right) \tag{5.1.32}$$

$$\equiv \int \frac{{\rm d}^d k}{(2\pi)^d}\widetilde{h}_{\alpha\beta}(k) M^{\alpha\beta\mu\nu}(k)\widetilde{h}_{\mu\nu}(k)^*. \tag{5.1.33}$$

- Argue that $M^{\mu\nu\alpha\beta}$ is not invertible; hence, the need to 'gauge-fix'. Hint: For *fixed* $\mu = 0, 1, 2, 3$,

$$V^{\alpha\beta}_{(\mu)} \equiv k^\alpha \delta^\beta_\mu + k^\beta \delta^\alpha_\mu; \tag{5.1.34}$$

  verify

$$M^{\alpha\beta}{}_{\sigma\rho} V^{\sigma\rho}_{(\mu)} = 0. \tag{5.1.35}$$

- Argue why the path integral generating functional for GR may be massaged to take the following form:

$$\text{Numerator} \equiv \left(\prod_{\kappa=0}^{d-1}\int \mathcal{D}C^\kappa e^{i\int {\rm d}^d x C^\tau C_\tau}\right)\prod_{\sigma=0}^{d-1}\int \mathcal{D}\Lambda^\sigma \det\left[\frac{\delta(\partial_\mu \Pi^{\mu\alpha})}{\delta\Lambda^\beta}\right]\delta\left[\partial_\mu\left(\bar{h}^{\mu\sigma} - \Pi^{\nu\sigma}\right) - C^\sigma\right]$$
$$\times \prod_{\rho,\gamma}\int \mathcal{D}h_{\rho\gamma}\exp\left[iS_2 + \mathcal{O}(h^3)\right]. \tag{5.1.36}$$

  where all indices are now moved with the flat metric $\eta_{\mu\nu}$ and we have defined

$$\Pi^{\mu\sigma} \equiv \partial^\mu\Lambda^\sigma + \partial^\sigma\Lambda^\mu - \eta^{\sigma\mu}\partial^\nu\Lambda_\nu \tag{5.1.37}$$

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - (1/2)\eta_{\mu\nu}h^\rho{}_\rho. \tag{5.1.38}$$

- Introduce Schwinger's 'source' into the 'Numerator' to generate the $n-$point graviton function, *after* carrying out the integrals over the auxiliary variables. You should now obtain

$$Y_J \equiv \left(\prod_{\sigma=0}^{d-1} \int \mathcal{D}\Lambda^\sigma \det \left[\frac{\delta(\partial_\mu \Pi^{\mu\alpha})}{\delta \Lambda^\beta} \right] \right) \prod_{\rho,\gamma} \int \mathcal{D}h_{\rho\gamma} \tag{5.1.39}$$
$$\times \exp\left[iS_2 + i \int \mathrm{d}^d x \eta_{\alpha\beta} \partial_\mu \bar{h}^{\mu\alpha} \partial_\nu \bar{h}^{\nu\beta} + \mathcal{O}(h^3) + i \int \mathrm{d}^d x h_{\alpha\beta}(x) J^{\alpha\beta}(x) \right].$$

- Finally, complete-the-square to arrive at

$$Z_J \equiv \frac{Y_J}{Y_{J=0}} = \exp\left[\frac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} i \widetilde{J}_{\alpha\beta}(k) (\widetilde{G}_{\mathrm{F}})^{\alpha\beta\mu\nu}(k) i \widetilde{J}_{\mu\nu}(k)^* + \dots \right] \tag{5.1.40}$$
$$(\widetilde{G}_{\mathrm{F}})_{\alpha\beta\mu\nu}(k) \equiv \frac{i}{k^2} \left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{2}{d-2}\eta_{\mu\nu}\eta_{\alpha\beta} \right). \tag{5.1.41}$$

Here, the ... indicates the higher order terms arising from the nonlinearities in GR.

□

5.2 Canonical Quantization: Gauge Invariant Approach

We have seen that the gauge potential enjoys a gauge freedom; i.e., under the replacement

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x), \tag{5.2.1}$$

the field strength

$$F_{\mu\nu} \equiv \partial_{[\mu} A_{\nu]} \to F_{\mu\nu} \tag{5.2.2}$$

remains invariant – and, hence, so does the Lagrangian density

$$\mathcal{L} \equiv -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \to \mathcal{L}. \tag{5.2.3}$$

In this section, we shall seek a *gauge-invariant* approach – an approach independent of the choice of the Λ above – to quantize the free photon. The key observation is that it is possible to decompose the gauge potential in such a manner that we can identify its gauge-invariant constituents.

Scalar-Vector Decomposition The primary technical insight is that any spatial vector (or $1-$form) may be decomposed in Fourier space into components parallel and perpendicular to its wave vector $\vec{k}$. For instance,

$$A_i(t, \vec{x}) = \int \frac{\mathrm{d}^3 \vec{k}}{(2\pi)^3} \left(\widetilde{P}_{ij}(\vec{k}) + \frac{k_i k_j}{\vec{k}^2} \right) \widetilde{A}_j(t, \vec{k}) e^{i\vec{k}\cdot\vec{x}}, \tag{5.2.4}$$
$$\widetilde{P}_{ij} \equiv \delta_{ij} - \frac{k_i k_j}{\vec{k}^2}; \tag{5.2.5}$$

where $\widetilde{P}_{ij}$ may be regarded as the flat space metric perpendicular to $\widehat{k}_i \equiv k_i/|\vec{k}|$, because $\widetilde{P}_{ij}k_j = 0$. Moreover, every $-ik_j = +ik^j$ may be identified as a spatial derivative ∂_i. Therefore, we have

$$A_i = A_i^{\mathrm{T}} + \partial_i \alpha; \tag{5.2.6}$$

where the former is the transverse component

$$A_i^{\mathrm{T}}(t,\vec{x}) \equiv \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \widetilde{P}_{ij}(\vec{k}) \widetilde{A}_j(t,\vec{k}) e^{i\vec{k}\cdot\vec{x}}, \tag{5.2.7}$$

and the latter the longitudinal component

$$A_i^{\parallel} \equiv \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{k_i k_j}{\vec{k}^2} \widetilde{A}_j(t,\vec{k}) e^{i\vec{k}\cdot\vec{x}} \equiv \partial_i \alpha. \tag{5.2.8}$$

Problem 5.3. Explain why $\partial_i A_i^{\mathrm{T}} = 0$; i.e., 'transverse' in momentum space corresponds to divergence-free in position space. □

Problem 5.4. Show that

$$\widetilde{P}_{ij}\widetilde{P}_{jk} = \widetilde{P}_{ik}. \tag{5.2.9}$$

This property justifies the nomenclature that $\widetilde{P}$ is a projector. □

Let us now show that this decomposition into transverse and longitudinal components is *unique* for non-zero wavelength modes (i.e., for all $\vec{k} \neq \vec{0}$). That is, if we were given

$$-ik_i\widetilde{\alpha} + \widetilde{P}_{ij}\widetilde{A}_j = -ik_i\widetilde{\beta} + \widetilde{P}_{ij}\widetilde{B}_j; \tag{5.2.10}$$

then

$$\widetilde{\alpha} = \widetilde{\beta} \tag{5.2.11}$$

and therefore $\widetilde{P}_{ij}\widetilde{A}_j = \widetilde{P}_{ij}\widetilde{B}_j$. To see this, simply dot both sides with ik_i, and divide throughout by k^2 since we have assumed $k \neq 0$. In fact, we may also act both sides with $\widetilde{P}$ and arrive at the same uniqueness conclusion.

2 d.o.f.s for transverse vector That the 3−component A_i^{T} is subject to one constraint $\partial_i A_i^{\mathrm{T}} = 0$ strongly suggests it has only two independent components.

Gauge-Transformation in Fourier Space In Fourier space, the gauge transformation goes as

$$\widetilde{A}_0(t,\vec{k}) \to \widetilde{A}_0(t,\vec{k}) + \partial_t \widetilde{\Lambda}(t,\vec{k}), \tag{5.2.12}$$

$$\widetilde{A}_i(t,\vec{k}) \to \widetilde{A}_i(t,\vec{k}) - ik_i \widetilde{\Lambda}(t,\vec{k}); \tag{5.2.13}$$

But as discussed, the spatial components may be decomposed as

$$-ik_i\widetilde{\alpha} + \widetilde{A}_i^{\mathrm{T}} \to -ik_i(\widetilde{\alpha} + \widetilde{\Lambda}) + \widetilde{A}_i^{\mathrm{T}}. \tag{5.2.14}$$

Since this decomposition is unique, we may associate gauge transformations on the spatial components solely to the longitudinal ones – at least for $\vec{k} \neq \vec{0}$:

$$A_i^{\mathrm{T}} \to A_i^{\mathrm{T}} \qquad \text{and} \qquad \alpha \to \alpha + \Lambda. \tag{5.2.15}$$

Additionally, let us observe that

$$\Phi \equiv A_0 - \dot{\alpha} \to A_0 + \dot{\Lambda} - (\dot{\alpha} + \dot{\Lambda}) \tag{5.2.16}$$

is also a gauge-invariant combination formed from A_μ.

Problem 5.5. Electromagnetic Fields Since $F_{\mu\nu}$ is gauge-invariant, it should be possible to express it in terms of (Φ, A_i^{T}). Show that

$$F_{0i} = \partial_t A_i^{\mathrm{T}} - \partial_i \Phi, \tag{5.2.17}$$

$$F_{ij} = \partial_{[i} A_{j]}^{\mathrm{T}}; \tag{5.2.18}$$

and therefore the Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\left(\partial_t A_j^{\mathrm{T}} \partial_t A_j^{\mathrm{T}} - \partial_i A_j^{\mathrm{T}} \partial_i A_j^{\mathrm{T}} + \partial_i \Phi \partial_i \Phi\right). \tag{5.2.19}$$

Notice *only* A_i^{T} has time derivatives acting on it. □

The (Heisenberg) equations of motion are

$$\partial^2 A_i^{\mathrm{T}} = 0 = \vec{\nabla}^2 \Phi. \tag{5.2.20}$$

Since we are dealing with free photons – i.e., where the electric current is zero – we have $\Phi = 0$ and

$$A_i^{\mathrm{T}}(t, \vec{x}) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 \vec{k}}{(2\pi)^3 \sqrt{2k}} \sum_{\mathrm{A}=\{1,2\}} \left(a_{\vec{k}}^{\mathrm{A}} \epsilon_i^{\mathrm{A}} e^{-ik\cdot x} + \text{h.c.}\right). \tag{5.2.21}$$

Because $\widetilde{P}_{ij}$ projects out the components of a vector parallel to $\vec{k}$ that means the unit vectors $\{\epsilon_i^{\mathrm{A}}\}$ must spanned by a 2D space. For instance, if $\vec{k}$ were pointing in the 3−direction, two basis vectors can be chosen to be

$$\epsilon^{\pm}{}_i \equiv \frac{1}{\sqrt{2}}\left(1, \pm i, 0\right)^T. \tag{5.2.22}$$

Problem 5.6. Show that, for a counter-clockwise rotation on the $(1,2)$−plane

$$\widehat{R}_{ij} = \left[\begin{array}{ccc} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{array}\right]; \tag{5.2.23}$$

that

$$\widehat{R}_{ij} \epsilon_j^{\pm} = e^{-i(\pm 1)\theta} \epsilon_j^{\pm}. \tag{5.2.24}$$

These are the $n = \pm 1$ helicity eigenstates arising from the SO_2 little group of the (3+1)D Poincaré algebra. □

If we treat each component of A_i^{T} as a SHO, we would recognize the conjugate field momenta to be

$$\Pi_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{A}_i^{\mathrm{T}}} = \partial_t A_i^{\mathrm{T}}; \tag{5.2.25}$$

and hence the equal time commutation relations to be

$$\left[A_i^{\mathrm{T}}(t, \vec{x}), \Pi_j(t, \vec{x}')\right] = i \left(\delta_{ij} - \frac{\partial_i \partial_j}{\vec{\nabla}^2}\right) \delta^{(3)}[\vec{x} - \vec{x}']. \tag{5.2.26}$$

The derivative structure on the right hand side is to ensure that the divergence-free constrains are maintained.

Problem 5.7. Show that, if we impose

$$\left[a_{\vec{k}}^{\mathrm{A}}, (a_{\vec{k}'}^{\mathrm{B}})^\dagger\right] = \delta^{\mathrm{AB}} \delta^{(3)}[\vec{k} - \vec{k}'], \tag{5.2.27}$$

eq. (5.2.26) would be recovered. The physical interpretation is:

> $\sqrt{2k} \cdot (a_{\vec{k}}^{\mathrm{A}})^\dagger$ acting on the vacuum produces a single photon state $\left|\vec{k}, \mathrm{A}\right\rangle$, where $k_\mu k^\mu = 0$ and A labels one of the two possible helicities.

Hint: You may need to employ the relation

$$\sum_{\mathrm{A} \in \{1,2\}} \epsilon_i^{\mathrm{A}} (\epsilon_j^{\mathrm{A}})^* = \widetilde{P}_{ij}(k). \tag{5.2.28}$$

Why is this true? □

5.3 Canonical Quantization: Lorenz ($\xi = 1$) Gauge

We will now proceed to quantize the photon in a Lorentz covariant manner by employing the Lorenz gauge. The starting point is the wave equation

$$\partial^\mu F_{\mu\nu} = \partial^2 A_\nu = 0, \tag{5.3.1}$$

where the Lorenz gauge

$$\partial^\mu A_\mu = 0 \tag{5.3.2}$$

has been imposed. The covariant commutation relations we shall assume are

$$i\left[A_\alpha(x), A_\beta(x')\right] = (G_{\mathrm{ret}})_{\alpha\beta}(x - x') - (G_{\mathrm{adv}})_{\alpha\beta}(x - x') \tag{5.3.3}$$

$$= \mathrm{sgn}(t - t') \eta_{\alpha\beta} \int \frac{\mathrm{d}^D \vec{k}}{(2\pi)^3} \frac{\sin(k(t - t'))}{k} e^{i\vec{k}\cdot(\vec{x} - \vec{x}')}. \tag{5.3.4}$$

The vector potential can therefore be expanded as

$$A_\mu(x) = \int \frac{\mathrm{d}^D \vec{k}}{(2\pi)^D \sqrt{2k}} \sum_{s=1}^{3} \left(a_k^s \epsilon^{(s)}{}_\mu e^{-ik\cdot x} + \mathrm{h.c.}\right). \tag{5.3.5}$$

Feynman Rules The contraction of a photon A_μ with an external line with incoming momentum/photon is given by

$$\epsilon_\mu(p) \tag{5.3.6}$$

whereas the contraction of a photon A_μ with an external line with outgoing momentum/photon is given by

$$\epsilon_\mu(p)^*. \tag{5.3.7}$$

6 Quantum Electrodynamics (QED)

6.1 Motivation: U_1 Symmetry

If a Lagrangian is invariant, up to an additive total divergence, under a transformation of the fields and/or spacetime coordinates – then there exists a corresponding conserved current when it is evaluated on the equations-of-motion. This is known as Noether's theorem.

Dirac Lagrangians and U_1 Symmetry Let us notice, under a global U_1 phase rotation

$$\psi \to e^{i\theta}\psi \qquad \text{and} \qquad \psi^\dagger \to \psi^\dagger e^{-i\theta}; \tag{6.1.1}$$

the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi \tag{6.1.2}$$

is invariant. We may identify

$$\delta\psi = i\theta\psi \qquad \text{and} \qquad \delta\psi^\dagger = -i\theta\psi^\dagger. \tag{6.1.3}$$

Varying the Lagrangian directly,

$$0 = -i\theta\bar{\psi}(i\not{D} - m)\psi + i^2\theta\partial_\mu\left(\psi^\dagger\gamma^0\gamma^\mu\psi\right) + i\left\{(-i\partial_\mu + eA_\mu)\psi^\dagger\right\}\gamma^0\gamma^\mu\psi + i\theta\psi^\dagger\gamma^0(-m)\psi. \tag{6.1.4}$$

Assuming the EoM holds, $(i\not{D} - m)\psi = 0$,

$$0 = i^2\theta\partial_\mu\left(\psi^\dagger\gamma^0\gamma^\mu\psi\right) + i\left\{(\gamma^\mu(i\partial_\mu + eA_\mu) - m)\psi\right\}^\dagger\gamma^0\psi \tag{6.1.5}$$

$$= i^2\theta\partial_\mu\left(\bar{\psi}\gamma^\mu\psi\right). \tag{6.1.6}$$

We have the Noether current of U_1 symmetry:

$$\partial_\mu J^\mu = 0 \qquad \text{and} \qquad J^\mu = \bar{\psi}\gamma^\mu\psi. \tag{6.1.7}$$

Talk about divergence of current using Dirac equation. Also chiral current. $\partial_\mu J^{\mu 5} = 2im\bar{\psi}\gamma^5\psi$. $\psi \to e^{i\alpha\gamma^5}\psi$.

YZ: I of II

The Lagrangian for the Dirac equation coupled to an electromagnetic vector potential is

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi. \tag{6.1.8}$$

Here, the covariant derivative is

$$D_\mu \equiv \partial_\mu + ieA_\mu \tag{6.1.9}$$

and $\not{D} \equiv \gamma^\mu D_\mu$. The mass is m and $\bar{\psi} \equiv \psi^\dagger\gamma^0$ and $\{\gamma^\mu\}$ are the Dirac matrices obeying the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. Furthermore, in the chiral basis, which may be expressed in terms of the Pauli matrices $\sigma^\mu = (\mathbb{I}, \sigma^i)$ and $\bar{\sigma}^\mu = (\mathbb{I}, -\sigma^i)$ as

$$\gamma^\mu \equiv \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}; \tag{6.1.10}$$

we may split the 4-component spinor into left and right handed Weyl spinors $\psi = (\psi_\mathrm{L}\ \psi_\mathrm{R})^\mathrm{T}$ and recognize

$$\mathcal{L} = \psi_\mathrm{L}^\dagger \bar{\sigma}^\mu D_\mu \psi_\mathrm{L} + \psi_\mathrm{R}^\dagger \sigma^\mu D_\mu \psi_\mathrm{R} - m\left(\psi_\mathrm{L}^\dagger \psi_\mathrm{R} + \psi_\mathrm{R}^\dagger \psi_\mathrm{L}\right). \tag{6.1.11}$$

Those familiar with the $\mathrm{SL}_{2,\mathbb{C}}$ transformations would recognize this to be manifestly invariant under the Lorentz group. In particular, the mass term couples the left- and right-handed Weyl spinors.

Because ψ is complex, we may vary ψ and $\psi^\dagger$ independently – for e.g., the complex number $z = x + iy$ may be varied by altering x and y independently or z and z^* independently. Doing so would yield

$$\int \mathrm{d}^4x' \delta_\psi \mathcal{L} = \int \mathrm{d}^4x' \psi^\dagger \gamma^0 (i\not{D} - m)\delta\psi \tag{6.1.12}$$

$$= \int \mathrm{d}^4x' \left\{(-i\partial_\mu + eA_\mu)\psi^\dagger\right\} \gamma^0\gamma^\mu \delta\psi - m\psi^\dagger\gamma^0\delta\psi + \text{surface term}. \tag{6.1.13}$$

Because

$$\gamma^0\gamma^\mu = \begin{bmatrix} \bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{bmatrix}, \tag{6.1.14}$$

this means $\gamma^0\gamma^\mu$ is hermitian:

$$\gamma^0\gamma^\mu = (\gamma^\mu)^\dagger\gamma^0. \tag{6.1.15}$$

And

$$\int \mathrm{d}^4x' \delta_\psi \mathcal{L} = \int \mathrm{d}^4x' \left\{\gamma^\mu(i\partial_\mu + eA_\mu)\psi - m\psi\right\}^\dagger \gamma^0\delta\psi + \text{surface term}. \tag{6.1.16}$$

This leads us to the Dirac equation

$$(i\not{D} - m)\psi = 0. \tag{6.1.17}$$

We may also vary $\psi^\dagger$ to deduce this same equation more readily.

$$\int \mathrm{d}^4x' \delta_{\psi^\dagger} \mathcal{L} = \int \mathrm{d}^4x' \delta\psi^\dagger \gamma^0 (i\not{D} - m)\psi. \tag{6.1.18}$$

Gauge Invariance Note that this Dirac Lagrangian is invariant under the simultaneous replacements

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \theta(x), \tag{6.1.19}$$

$$\psi \to e^{ie\theta}\psi. \tag{6.1.20}$$

This guarantees the gauge-invariance of the Dirac equation $(i\not{D} - m)\psi = 0$.

YZ: II of II

U_1 Symmetry & Covariance Notice all fermion Lorentz invariants are invariant under the U_1 phase rotations

$$\psi \to e^{i\theta}\psi \qquad \text{and} \qquad \psi^\dagger \to \psi^\dagger e^{-i\theta} \tag{6.1.21}$$

because the $\psi^\dagger$ and ψ occurs in pairs. This is certainly true when θ is a mere constant – i.e., when the U_1 transformation is *global* – but what happens when $\theta = \theta(x)$ is *local*, meaning it depends on the spacetime location?

To study this question, let us remember, under a gauge transformation, the vector potential itself transforms as

$$A_\mu \to A_\mu + \partial_\mu \theta(x). \tag{6.1.22}$$

Let us now consider acting the covariant derivative on the spinor field, followed by rotating ψ by a local U_1 phase rotation.

$$D_\mu \psi \to (\partial_\mu + ieA_\mu)\left(e^{ie\theta(x)}\psi\right) \tag{6.1.23}$$

$$= e^{ie\theta}(\partial_\mu - ie\partial_\mu\theta + ieA_\mu)\psi \tag{6.1.24}$$

$$= e^{ie\theta}\left(\partial_\mu - ie(\partial_\mu\theta - A_\mu)\right)\psi. \tag{6.1.25}$$

We see that, if A_μ shifted by the gauge transformation in eq. (6.1.22), the covariant derivative of ψ would transform *covariantly* under local U_1 transformations. Namely, under the simultaneous transformations

$$\psi \to e^{ie\theta}\psi \qquad \text{and} \qquad A_\mu \to A_\mu + \partial_\mu\theta; \tag{6.1.26}$$

we have

$$D_\mu\psi \to e^{ie\theta}D_\mu\psi. \tag{6.1.27}$$

This tells us the full QED action

$$S_{\text{QED}} = \int \mathrm{d}^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi\right) \tag{6.1.28}$$

is not only globally, but also locally U_1 'gauge-invariant'.

Problem 6.1. Noether Current of Poincaré Symmetry Work out the Noether currents ${T^\mu}_\alpha$ of spacetime translations for the action in eq. (3.1.32). Do the same for Lorentz transformations. Can you derive a relationship like that in eq. (1.3.28)? □

Problem 6.2. Commutator of Covariant Derivatives Show that

$$[D_\mu, D_\nu]\psi = ieF_{\mu\nu}\psi, \tag{6.1.29}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor. This serves as an analog to the commutator of covariant derivatives in differential geometry, which yields the Riemann curvature tensor. □

Problem 6.3. Abelian Higgs Model We may provide a similar argument as above, to construct a local U_1 invariant complex scalar field theory coupled to electromagnetic fields. The resulting Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\varphi\overline{D^\mu\varphi} - V(|\varphi|^2) \tag{6.1.30}$$

$$V(|\varphi|^2) \equiv \frac{\lambda}{4}\left(|\varphi|^2 - \nu^2\right)^2, \qquad \nu \in \mathbb{R}. \tag{6.1.31}$$

- Identify the electromagnetic current. Write down the equations of motion for both the electromagnetic fields and for the scalar field φ.
- Work out the Noether current of *global* U_1 symmetry for φ. A complex scalar field with global U_1 symmetry, just like its fermionic counterpart, has a conserved number current.
- Verify that $A_\mu = 0$ and $\varphi = \nu e^{i\chi}$ (for real constant χ) is a solution. Describe the trajectory of $\varphi = \nu e^{i\chi}$ within the potential $V(|\varphi|^2)$ versus φ field space 3D plot, as χ is varied from 0 to 2π.
- Consider inserting $\varphi = (\nu + \rho(x))e^{ie\phi(x)}$, where ρ and ϕ both depend on spacetime, into the Lagrangian. Explain how A_μ may be transformed to remove ϕ. Then work out the portion of the Lagrangian quadratic in A_μ and ρ. What are the (classical) masses of A_μ and ρ?
- *Remarks*: We know a massless photon has 2 degrees-of-freedom; while a complex scalar should have 2. But due to local gauge-invariance, what you are showing is that the dynamics may be converted into a massive gauge boson (with 3 degrees of freedom); and one real scalar ρ. The photon acquires a mass – this is a model for the Meissner effect in superconductors.

Problem 6.4. (1+1)D Dirac Equation Coupled to a Scalar Kink In this problem we will study the Dirac equation for ψ coupled to a scalar field ϕ in $(1+1)$D flat spacetime with Cartesian coordinates $x^\mu = (t, x)$.

- To build the Dirac equation in 2D, we need to find the appropriate $\gamma-$matrices. Explain why the following choice is a valid one. Denoting the Pauli matrices by $\{\sigma^i | i = 1, 2, 3\}$, define

  $$\gamma^0 \equiv \sigma^3 \qquad \text{and} \qquad \gamma^1 \equiv i\sigma^1. \tag{6.1.32}$$

  Since Pauli matrices are 2×2 matrices, this tells us the Dirac spinor ψ is a 2-component object.

- We will now solve for the stationary state wave functions of the 2D Dirac equation coupled to a scalar field ϕ,

$$(i\gamma^\mu \partial_\mu - g\phi)\psi = 0, \tag{6.1.33}$$

with the scalar field taking the specific form

$$\phi(x) = \nu \tanh(mx/2), \qquad \nu, m \in \mathbb{R}. \tag{6.1.34}$$

Such a ϕ, known as a $\mathbb{Z}_2$ kink, arises as a solution to a relativistic field theory with a double well potential.

Consider the stationary state wave function ansatz

$$\psi(t,x) = \frac{e^{-iEt}}{\sqrt{2}} \begin{bmatrix} \beta_+(x) - \beta_-(x) \\ \beta_+(x) + \beta_-(x) \end{bmatrix} \tag{6.1.35}$$

Show that, upon performing the re-scaling

$$x \equiv \frac{2z}{m}, \qquad g \equiv \frac{em}{2\nu}, \qquad E \equiv \frac{m}{2}\bar{E}, \tag{6.1.36}$$

the following pair of Schrödinger equations are obtained:

$$-\partial_z^2 \beta_\pm(z) + \left(e^2 - e(e \pm 1)\text{sech}^2(z)\right)\beta_\pm(z) = \bar{E}^2 \beta_\pm(z). \tag{6.1.37}$$

- The fermion ψ may bind to the $\mathbb{Z}_2$ kink. The energy levels of these bound states are

$$\bar{E}_n^\pm = \pm\sqrt{n(2e-n)}, \qquad 0 \leq n < e, \qquad n = 0,1,2,3,\dots. \tag{6.1.38}$$

Hence, we see that every positive energy solution has a negative energy counterpart – except for the zero energy mode $\bar{E}_0 = 0$. Solve the $\beta_\pm(z)$ for this zero mode, and write down the corresponding Dirac spinor $\psi_0 \equiv \psi(t,x;n=0)$.

Bonus Can you solve the $\beta_\pm$ (and, hence, ψ_n) for all the bound state wave functions? (Warning: It's hard work, involving the hypergeometric function ${}_2F_1$.)

Problem 6.5. (2+1)D Dirac Equation

- Show that $\gamma^0 \equiv \sigma^3$, $\gamma^1 \equiv i\sigma^1$ and $\gamma^2 \equiv i\sigma^2$ satisfies the Clifford Algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}$.

6.2 Basic Scattering and Decay Processes

Chapter 4.8 covers . Chapter 5 of P&S covers basic QED processes. In this section we shall consider Quantum Electrodynamics (QED) defined by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{\text{I}} \left\{\bar{\psi}_{\text{I}}\left(i\partial\!\!\!/ - m_{\text{I}}\right)\psi_{\text{I}} - e_{\text{I}}\bar{\psi}_{\text{I}}A\!\!\!/\psi_{\text{I}}\right\}. \tag{6.2.1}$$

In what follows, it is common to define

$$\alpha \equiv \frac{e^2}{4\pi} \approx \frac{1}{137}. \tag{6.2.2}$$

For the I-th species of fermion, the theory has a single interaction vertex: photon-fermion-fermion. In the Lorenz gauge the photon propagator is

$$\langle \widetilde{A}_\mu(p) \widetilde{A}_\nu(p') \rangle = \frac{-i\eta_{\mu\nu}}{p^2 + i\epsilon} (2\pi)^4 \delta^{(4)}(p + p'). \tag{6.2.3}$$

The fermion propagator is

$$\langle \widetilde{\psi}_{\rm I}(p) \widetilde{\overline{\psi}}_{\rm I}(p') \rangle = \frac{i(\not{p} + m)}{p^2 - m_{\rm I}^2 + i\epsilon} (2\pi)^4 \delta^{(4)}(p + p'). \tag{6.2.4}$$

Feynman Rules See Peskin+Schröder or Griffiths' *Introduction to Elementary Particles.*

- Assign a momentum to each external line. For internal lines use momentum conservation. Any un-determined momentum is to be integrated over; though this does not occur for tree level amplitudes.
- For fermion lines, there is an extra flow of fermions – i.e., in addition to momentum – that needs to be accounted for. Fermion number is conserved due to U_1 symmetry. For internal lines, fermion number flows parallel to momentum.
- Fermion External states: incoming u and outgoing $\bar{u}$.
- AntiFermion External states: incoming $\bar{v}$ and outgoing v.
- Photon: incoming $\epsilon^s_\mu(p)$ and outgoing $\overline{\epsilon^s_\mu}(p')$. Strictly speaking, these polarizations need to be purely spatial and transverse to their corresponding momentum; i.e., helicity $s = \pm 1$.
- For QED the vertex is $-ie\gamma^\mu$.
- Internal fermion propagator with momentum k: $i(\not{k} + m)/(k^2 - m^2 + i\epsilon)$.
- Internal Lorenz gauge photon propagator with momentum k: $-i\eta_{\mu\nu}/(k^2 + i\epsilon)$.
- Fermi statistics: Make sure there is a relative $-$ sign between any two diagrams that differ *only* by (I) interchange between two incoming (anti)fermions; (II) interchange between two outgoing (anti)fermions; (III) incoming fermion with outgoing antifermion; or incoming antifermion with outgoing fermion.

Compton Scattering Incoming electron (s_1, p_1) and photon (ϵ_2, p_2), outgoing electron (s_4, p_4) and photon (ϵ_3, p_3).

$$i\mathcal{M} = ie^2 \frac{\bar{u}[4] \not{\epsilon}[2] (\not{p}_1 - \not{p}_3 + m_e) \not{\epsilon}^*[3] u[1]}{(p_1 - p_3)^2 - m_e^2 + i\epsilon} + ie^2 \frac{\bar{u}[4] \not{\epsilon}[3]^* (\not{p}_1 + \not{p}_2 + m_e) \not{\epsilon}[2] u[1]}{(p_1 + p_2)^2 - m_e^2 + i\epsilon}. \tag{6.2.5}$$

Electron-Muon Scattering Incoming muon (s_2, p_2) and outgoing muon (s_4, p_4); incoming electron (s_1, p_1) and outgoing (s_4, p_4).

$$i\mathcal{M} = \bar{u}[3]ie\gamma^\mu u[1]\frac{-i\eta_{\mu\nu}}{(p_1-p_3)^2}\bar{u}[4]ie\gamma^\nu u[2]. \tag{6.2.6}$$

Electron-Electron Scattering Incoming electrons (s_1, p_1) and (s_2, p_2); outgoing electrons (s_3, p_3) and (s_4, p_4). Fermionic statistics leads to two different contributions, leading to anti-symmetric amplitude.

$$i\mathcal{M} = \bar{u}[3]ie\gamma^\mu u[1]\frac{-i\eta_{\mu\nu}}{(p_1-p_3)^2}\bar{u}[4]ie\gamma^\nu u[2] - \bar{u}[4]ie\gamma^\mu u[1]\frac{-i\eta_{\mu\nu}}{(p_1-p_4)^2}\bar{u}[3]ie\gamma^\nu u[2]. \tag{6.2.7}$$

Electron-Positron Scattering Incoming electron (s_1, p_1) and positron (s_2, p_2), outgoing electron (s_3, p_3) and positron (s_4, p_4).

$$i\mathcal{M} = ie^2\frac{\bar{u}[3]\gamma^\alpha u[1]\bar{v}[2]\gamma_\alpha v[4]}{(p_1-p_3)^2+i\epsilon} - ie^2\frac{\bar{u}[3]\gamma^\alpha v[4]\bar{v}[2]\gamma_\alpha u[1]}{(p_1+p_2)^2+i\epsilon}. \tag{6.2.8}$$

Comment on repulsive vs attractive interactions.

Electron-Positron to Muon-Antimuon The annihilation of electron-positron can produce a muon-antimuon pair at large center-of-mass energies. This is a simple process that illustrates *relativistic quantum field theory* in that we see not only its relativistic and probabilistic character, but also of its variable-body ($e^+e^- \to \mu^+\mu^-$) nature. Note that $m_e \approx 0.5$MeV whereas $m_\mu \approx 106$MeV; and therefore $m_\mu/m_e \approx 200$. The amplitude at leading order is given by

$$(ie)^2 \left\langle p_\mu, s_\mu; p_{\bar{\mu}}, s_{\bar{\mu}} \middle| \mathsf{T}\int_{x,y} \bar{\mu}_x \not{A}_x \mu_x \bar{e}_y \not{A}_y e_y \middle| p_e, p_{\bar{e}} \right\rangle_c \tag{6.2.9}$$

$$= -e^2 \bar{u}^{s_\mu}_{p_\mu}\gamma^\alpha v^{s_{\bar{\mu}}}_{p_{\bar{\mu}}} \frac{-i\eta_{\alpha\beta}}{s+i\epsilon}\bar{v}^{s_{\bar{e}}}_{p_{\bar{e}}}\gamma^\beta u^{s_e}_{p_e}(2\pi)^4\delta^{(4)}(p_e+p_{\bar{e}}-p_\mu-p_{\bar{\mu}}) \tag{6.2.10}$$

$$i\mathcal{M} = \frac{ie^2}{s+i\epsilon}\bar{u}^{s_\mu}_{p_\mu}\gamma^\alpha v^{s_{\bar{\mu}}}_{p_{\bar{\mu}}}\bar{v}^{s_{\bar{e}}}_{p_{\bar{e}}}\gamma_\alpha u^{s_e}_{p_e}. \tag{6.2.11}$$

We have introduced the Mandelstam variable

$$s \equiv (p_e+p_{\bar{e}})^2 = (p_\mu+p_{\bar{\mu}})^2. \tag{6.2.12}$$

Let's square the amplitude and average of the initial spins and sum over the final spins. We'll see it simplifies the end result.

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4}{4s^2}\sum_{\text{spins}}\bar{v}^{s_{\bar{\mu}}}_{p_{\bar{\mu}}}\gamma^\beta u^{s_\mu}_{p_\mu}\bar{u}^{s_\mu}_{p_\mu}\gamma^\alpha v^{s_{\bar{\mu}}}_{p_{\bar{\mu}}}\bar{v}^{s_{\bar{e}}}_{p_{\bar{e}}}\gamma_\alpha u^{s_e}_{p_e}\bar{u}^{s_e}_{p_e}\gamma_\beta v^{s_{\bar{e}}}_{p_{\bar{e}}} \tag{6.2.13}$$

$$= \frac{e^4}{4s^2}\mathrm{Tr}\left[(\not{p}_{\bar{\mu}}-m_\mu)\gamma^\beta(\not{p}_\mu+m_\mu)\gamma^\alpha\right]\mathrm{Tr}\left[(\not{p}_{\bar{e}}-m_e)\gamma_\alpha(\not{p}_e+m_e)\gamma_\beta\right] \tag{6.2.14}$$

To see how the Tr is obtained, first recall the identities

$$\sum_s (u^s_p)_A(\bar{u}^s_p)_B = (\not{p}+m)_{AB}, \tag{6.2.15}$$

$$\sum_s (v^s_p)_A(\bar{v}^s_p)_B = (\not{p}-m)_{AB}. \tag{6.2.16}$$

So, for instance,

$$\sum_{s_{\bar{\mu}},s_\mu} (\bar{v}^{s_{\bar{\mu}}}_{p_{\bar{\mu}}})_A(\gamma^\beta)_{AB}(u^{s_\mu}_{p_\mu})_B(\bar{u}^{s_\mu}_{p_\mu})_E(\gamma^\alpha)_{EF}(v^{s_{\bar{\mu}}}_{p_{\bar{\mu}}})_F = (\gamma^\beta)_{AB}(\not{p}_\mu + m_\mu)_{BE}(\gamma^\alpha)_{EF}(\not{p}_{\bar{\mu}} - m_\mu)_{FA} \quad (6.2.17)$$

$$= \mathrm{Tr}\left[\gamma^\beta(\not{p}_\mu + m_\mu)\gamma^\alpha(\not{p}_{\bar{\mu}} - m_\mu)\right]. \quad (6.2.18)$$

Let's examine

$$\mathrm{Tr}\left[(\not{p}_{\bar{\mu}} - m_\mu)\gamma^\beta(\not{p}_\mu + m_\mu)\gamma^\alpha\right] \quad (6.2.19)$$

$$= \mathrm{Tr}\left[\not{p}_{\bar{\mu}}\gamma^\beta(\not{p}_\mu + m_\mu)\gamma^\alpha\right] - m_\mu\left(\mathrm{Tr}\left[\gamma^\beta\not{p}_\mu\gamma^\alpha\right] + m_\mu\mathrm{Tr}\left[\gamma^\beta\gamma^\alpha\right]\right) \quad (6.2.20)$$

$$= \mathrm{Tr}\left[\not{p}_{\bar{\mu}}\gamma^\beta\not{p}_\mu\gamma^\alpha\right] + m_\mu\mathrm{Tr}\left[\not{p}_{\bar{\mu}}\gamma^\beta\gamma^\alpha\right] - m_\mu^2\mathrm{Tr}\left[\gamma^\beta\gamma^\alpha\right] \quad (6.2.21)$$

$$= \mathrm{Tr}\left[\not{p}_{\bar{\mu}}\gamma^\beta\not{p}_\mu\gamma^\alpha\right] - m_\mu^2\mathrm{Tr}\left[\gamma^\beta\gamma^\alpha\right] \quad (6.2.22)$$

$$= 4\left(p_{\bar{\mu}}^{\{\beta}p_\mu^{\alpha\}} - (p_{\bar{\mu}}\cdot p_\mu)\eta^{\alpha\beta} - m_\mu^2\eta^{\alpha\beta}\right) \quad (6.2.23)$$

Next we have

$$\mathrm{Tr}\left[(\not{p}_{\bar{e}} - m_e)\gamma^\alpha(\not{p}_e + m_e)\gamma^\beta\right] = 4\left(p_{\bar{e}}^{\{\beta}p_e^{\alpha\}} - (p_{\bar{e}}\cdot p_e)\eta^{\alpha\beta} - m_\mu^2\eta^{\alpha\beta}\right). \quad (6.2.24)$$

We'll follow Peskin and Schröeder and simplify the analysis even further by assuming the scattering is occurring at high enough center-of-mass energy $E \gg m_e$ that the electron mass can be neglected. This leads us to

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{8e^4}{s^2}\left(m_\mu^2(p_e\cdot p_{\bar{e}}) + (p_e\cdot p_\mu)(p_{\bar{e}}\cdot p_{\bar{\mu}}) + (p_{\bar{e}}\cdot p_\mu)(p_e\cdot p_{\bar{\mu}})\right) \quad (6.2.25)$$

Furthermore, let us work in the COM frame, and take (remember $m_e = 0$)

$$p_e = E(1, \widehat{z}), \quad (6.2.26)$$

$$p_{\bar{e}} = E(1, -\widehat{z}); \quad (6.2.27)$$

and

$$p_\mu = (E, \vec{k}), \quad (6.2.28)$$

$$p_{\bar{\mu}} = (E, -\vec{k}), \qquad E^2 = k^2 + m_\mu^2. \quad (6.2.29)$$

Now, $(p_\mu + p_{\bar{\mu}})^2 = (p_e + p_{\bar{e}})^2 = (2E)^2 = s = 2m_\mu^2 + 2(p_\mu\cdot p_{\bar{\mu}})$.

$$p_e\cdot p_{\bar{e}} = 2E^2, \quad (6.2.30)$$

$$p_e\cdot p_{\bar{\mu}} = E^2 + E\vec{k}\cdot\widehat{z} \quad (6.2.31)$$

$$p_e\cdot p_\mu = E^2 - E\vec{k}\cdot\widehat{z} \quad (6.2.32)$$

$$p_{\bar{e}}\cdot p_{\bar{\mu}} = E^2 - E\vec{k}\cdot\widehat{z} \quad (6.2.33)$$

$$p_{\bar{e}} \cdot p_\mu = E^2 + E\vec{k} \cdot \widehat{z} \tag{6.2.34}$$

Hence, we have

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{16E^4}\left(m_\mu^2(2E^2) + E^2(E - \vec{k}\cdot\widehat{z})^2 + E^2(E + \vec{k}\cdot\widehat{z})^2\right) \tag{6.2.35}$$

$$= \frac{e^4}{2E^4}E^2\left(2m_\mu^2 + 2E^2 + 2(E^2 - m_\mu^2)(\widehat{k}\cdot\widehat{z})^2\right) \tag{6.2.36}$$

$$= e^4\left(1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right)(\widehat{k}\cdot\widehat{z})^2\right) \tag{6.2.37}$$

The differential cross section reads

$$\mathrm{d}^6\sigma = |\mathcal{M}|^2 \frac{S}{4\sqrt{(p_1\cdot p_2)^2 - (m_1 m_2)^2}} \prod_{i=1}^{n}\left(\frac{\mathrm{d}^3\vec{p}_i'}{(2\pi)^3 2E_i'}\right)(2\pi)^4\delta^{(4)}\left(p_1 + p_2 - \sum_{i=1}^{n} p_n'\right) \tag{6.2.38}$$

$$= \frac{e^4}{8E^2}\left(1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right)(\widehat{k}\cdot\widehat{z})^2\right)\frac{\mathrm{d}^3\vec{p}_\mu}{(2\pi)^3 2E_\mu}\frac{\mathrm{d}^3\vec{p}_{\bar{\mu}}}{(2\pi)^3 2E_{\bar{\mu}}} \tag{6.2.39}$$

$$\times (2\pi)^4\delta(2E - E_\mu - E_{\bar{\mu}})\delta^{(3)}(\vec{p}_\mu + \vec{p}_{\bar{\mu}}) \tag{6.2.40}$$

$$\mathrm{d}^3\sigma = \frac{e^4}{16E^2}\left(1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right)(\widehat{k}\cdot\widehat{z})^2\right)\frac{\mathrm{d}^2\widehat{k}\,\mathrm{d}E_\mu|\vec{k}|}{(2\pi)^3 4E_\mu}(2\pi)\delta(E - E_\mu) \tag{6.2.41}$$

$$\mathrm{d}^2\sigma = \frac{e^4}{16E^2}\left(1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right)(\widehat{k}\cdot\widehat{z})^2\right)\frac{\mathrm{d}^2\widehat{k}\sqrt{E^2 - m_\mu^2}}{(2\pi)^2(4E)} \tag{6.2.42}$$

Note: $k^2 = E^2 - m_\mu^2 \Rightarrow \mathrm{d}k \cdot k = E\mathrm{d}E \Rightarrow \mathrm{d}k \cdot k^2/E = k\mathrm{d}E$.

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{16E^2}\left(1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right)(\widehat{k}\cdot\widehat{z})^2\right)\sqrt{1 - \frac{m_\mu^2}{E^2}} \tag{6.2.43}$$

where we have used $\alpha^2 = e^4/(16\pi^2)$. The total cross section is

$$\sigma = 4\pi\frac{\alpha^2}{16E^2}\left(1 + \frac{m_\mu^2}{E^2} + \frac{1}{3}\left(1 - \frac{m_\mu^2}{E^2}\right)\right)\sqrt{1 - \frac{m_\mu^2}{E^2}} \tag{6.2.44}$$

$$= \frac{4\pi}{3}\frac{\alpha^2}{4E^2}\left(1 - \frac{1}{2}\frac{m_\mu^2}{E^2}\right)\sqrt{1 - \frac{m_\mu^2}{E^2}}. \tag{6.2.45}$$

6.3 More Physics

Non-Relativistic Limit Fourier transform of amplitude yields V. Yukawa potential. Coulomb potential from (anti)fermion-(anti)fermion or fermion-antifermion scattering.

Bremsstrahlung Radiation Accelerating charges emit photons. Both quantum and classical. Photon is massless, so they are emitted no matter what the energy scales are. IR divergence problem.

Angular Distributions

Radiative Corrections

Vertex Corrections $g-2$ and magnetic moment. Magnetic moment of the electron is

$$\vec{\mu} = g_e \frac{e}{2m_e} \vec{S}, \tag{6.3.1}$$

where $g_e = 2$ from the Dirac equation. However, quantum corrections from QED would alter this value; the first order result is in fact $g_e - 2 = \alpha/\pi$. Running of electric charge.

Quantum corrections to mass Two point function and its pole.

6.4 Appendix

$$\mathrm{Tr}\left[\mathbb{I}\right] = 4 \tag{6.4.1}$$

$$\mathrm{Tr}\left[\text{odd number of } \gamma \text{ matrices}\right] = 0 \tag{6.4.2}$$

$$\mathrm{Tr}\left[\gamma^\alpha \gamma^\beta\right] = 4\eta^{\alpha\beta} \tag{6.4.3}$$

$$\mathrm{Tr}\left[\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta\right] = 4\left(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\alpha\nu}\right) \tag{6.4.4}$$

$$\mathrm{Tr}\left[\gamma^\alpha \gamma^\beta \gamma^5\right] = 0 \tag{6.4.5}$$

$$\mathrm{Tr}\left[\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta \gamma^5\right] = -4i\epsilon^{\mu\nu\alpha\beta} \tag{6.4.6}$$

7 Acknowledgments

A Cross Sections, Lifetimes, Amplitudes

Cross Section Consider a parallel beam of particles zipping down an accelerator tunnel. If L denotes flux of particles (number of particles per unit time per unit area) across some infinitesimal cross sectional area $\mathrm{d}\sigma$, so that the number of outgoing/scattered particles is $\mathrm{d}N = L\mathrm{d}\sigma$, then the differential cross section is defined by the number of particles scattered into a given infinitesimal solid angle $\mathrm{d}\Omega$ divided by the luminosity L:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \equiv \frac{1}{L}\frac{\mathrm{d}N}{\mathrm{d}\Omega}. \tag{A.0.1}$$

This definition is consistent with how cross section is actually measured: L is set by the parameters of the accelerator setup, whereas $\mathrm{d}N/\mathrm{d}\Omega$ involves counting particles.

Decay lifetime If we have N number of unstable particles of the same species, the reciprocal of their decay lifetime τ is defined to be the fractional change of its population per unit time:

$$\frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}t} = -\frac{1}{\tau} \equiv -\Gamma. \tag{A.0.2}$$

The solution to this equation is

$$N(t) = N_0 \exp\left(-t/\tau\right). \tag{A.0.3}$$

In words: the number of particles $N(\tau)$ after time τ has elapsed is $1/e$ of the initial number of particles N_0.

Amplitudes to Differential Cross Sections & Decay Rates The S matrix for a process $\{p_i\} \to \{p'_j\}$ is defined by the transition amplitude

$$_{\mathrm{H}}\left\langle \{p'_j\}, t \to \infty \right| \{p_i\}, t \to -\infty \rangle_{\mathrm{H}} = {}_{\mathrm{S}}\left\langle \{p'_j\} \left| S \right| \{p_i\} \right\rangle_{\mathrm{S}}. \tag{A.0.4}$$

This is further split into the identity plus the T-matrix:

$$S \equiv \mathbb{I} + iT, \tag{A.0.5}$$

$$i\mathcal{M} \cdot (2\pi)^4 \delta^{(4)} \left(\sum_i p_i - \sum_j p'_j \right) = {}_{\mathrm{S}}\left\langle \{p'_j\} \left| iT \right| \{p_i\} \right\rangle_{\mathrm{S}}. \tag{A.0.6}$$

The physical meaning is that the amplitude $\mathcal{M}$ captures the non-trivial part of the interactions; all the relevant dynamics is contained within T.

Given the amplitude $\mathcal{M}$ for a 2 particle to $n-$particle scattering process, $p_1 + p_2 \to \{p'_i\}$, the differential cross section is given by

$$\mathrm{d}\sigma = |\mathcal{M}|^2 \frac{S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2}} \prod_{i=1}^{n} \left(\frac{\mathrm{d}^3 \vec{p}'_i}{(2\pi)^3 2E'_i} \right) (2\pi)^4 \delta^{(4)} \left(p_1 + p_2 - \sum_{i=1}^{n} p'_n \right). \tag{A.0.7}$$

(S is defined below.) In Peskin & Schroeder, the $\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2}$ in the denominator is written as $E_1 E_2 |v_1 - v_2|$, where $|v_1 - v_2|$ is the relative velocity of the 2 incoming particles.

Given the amplitude $\mathcal{M}$ for a particular decay process $1 \to 2 + 3 + \ldots n$, the 'Golden Rule' for decay rate of particle 1, which will assume is at rest (i.e., $p_1 = (m_1, \vec{0})$), is given by

$$\mathrm{d}\Gamma = |\mathcal{M}|^2 \frac{S}{2m_1} \prod_{i=2}^{n} \left(\frac{\mathrm{d}^3 \vec{p}_i}{(2\pi)^3 2E_i} \right) (2\pi)^4 \delta^{(4)} \left(p_1 - \sum_{i=2}^{n} p_n \right) \tag{A.0.8}$$

$$S = \prod_{j=1}^{N} \frac{1}{\ell_j !}. \tag{A.0.9}$$

If there are ℓ identical particles in the final decay product, we need to divide the answer by $\ell!$. The S refers to this – its definition is given by the product all such statistical factors, where ℓ_j is the number of identical particles in the jth group of decay products.

B Standard Model of Elementary Particle Interactions

Weak Interactions

C Last update: January 13, 2025

References