

Field Theory: Notes

Yi-Zen Chu

Contents

1	What is Quantum Field Theory? Why do We Need It? QM versus QFT	2
2	Classical Scalar Fields in Minkowski Spacetime	6
2.1	Action Principle and Symmetries	6
2.2	Local Conservation Laws	9
2.3	Noether: Continuous Symmetries and Conserved Currents	10
2.4	Hamiltonian Formulation	14
2.5	Fourier Space(time)	17
3	QFT of Massive Scalar Field	21
3.1	Non-Interacting Case	21
3.2	In- and Out-Vacuum (Heisenberg Picture)	42
3.3	Interactions: Correlations, Wick's Theorem & Path Integrals	44
3.4	T Matrix/Scattering Amplitudes	50
4	(3+1)D Spinor Fields, U_1 Symmetry – Quantum Electrodynamics (QED)	51
4.1	Invariants and Covariants	51
5	Quantum Field Theory of Fermions	55
5.1	Canonical Quantization & Fermi-Dirac Statistics	55
5.2	Fermionic Path Integrals	60
5.3	Appendix	62
6	Classical Electromagnetism	64
6.1	Minkowski: d -dimensions	64
6.2	Gauge Invariant Variables for Electromagnetic Vector Potential	70
6.3	4 dimensions	76
7	(Semi-)Classical Action Principles & Symmetries	79
7.1	Dirac Spinors in (3+1)D	79
7.2	Noether's Theorem & Internal Symmetries	80
7.3	Noether's Theorem & Energy-Momentum-Shear-Stress	81
8	QFT of Photons	83

9 Scattering & Decay Processes	86
9.1 1 Scalar + 1 Fermion: Yukawa Theory	86
9.2 Quantum Electrodynamics (QED)	86
9.3 4-Fermi Theory	89
9.4 Appendix	91
10 Acknowledgments	92
A Cross Sections, Lifetimes, Amplitudes	92
B Last update: November 19, 2022	93

Contents

1 What is Quantum Field Theory? Why do We Need It? QM versus QFT

¹What?

- The application of the rules of quantum mechanics to *fields* – i.e., beginning with those of photons – is Quantum Field Theory (QFT).
- The electron was discovered as a particle. Quantum mechanics (QM) gave it a wave function description – i.e., a field theory. But with Quantum Field Theory (QFT) this wave function is further promoted into an operator (sometimes dubbed ‘second quantization’).
- Photon – particle or wave? Both! – in QM. QFT provides the framework,
- Important physical example: Quantized fields of photons and electrons is dubbed *Quantum Electrodynamics*.
- **What are particles?** Particles are the vibrations/excitations of quantized fields.
- **Core Issues:**
 - Symmetries – particularly, Lorentz invariance (i.e., Special Relativity). Local and global gauge symmetries.
 - Variable-Body Quantum Physics. Creation and destruction of particles. Scattering, decay processes do not conserve particle number/
 - Combining relativity and QM. Relativistic QFT is widely viewed as the natural outcome of merging relativity with quantum mechanics.
 - Route: Particles \rightarrow Fields; or, Fields \rightarrow Particles.

¹A more authoritative discussion of the What & Why’s of QFT, including its historical developments, can be found in Weinberg’s QFT text Volume 1 Chapter 1.

Physics

- QFT-related phenomenon & concepts much richer than single or few body QM.
- Creation & Destruction of particles. Particle collisions/interactions can create and/or destroy particles. Particles can even be produced from nothing: e.g., Schwinger effect; particle production in cosmology; Hawking radiation.
- Is the quantum vacuum real? Change it slightly and see what happens – Casimir effect.
- In Minkowski spacetime: Particles as the irreducible representations of Lorentz symmetry. In curved spacetimes or time dependent backgrounds, notion of particles become murkier.
- ‘Vacuum’ or ‘nothingness’ becomes a much more subtle issue. Fluctuation of the vacuum. Examples: Casimir effect.
- Cosmology of the early universe. Inflation
- Classical symmetries can be broken by quantum effects.
- Condensed matter physics is often very much applied QFT.
- Effective Field Theory perspectives to QFT.
- Spin and Statistics. Fermions versus Bosons.
- Lagrangian/Path integral approach makes symmetries manifest; e.g. Lorentz invariant Lagrangian. Whereas Hamiltonian approach makes unitary evolution manifest.
- Scattering theory & S-matrix
- Symmetries: spacetime (Lorentz, parity, time reversal, Weyl/conformal), fermionic chiral symmetry, supersymmetry, gauge symmetries.
- Short range (massive) versus long range (massless) forces.
- Generically, everything couples to everything else – through loops.
- Renormalization, regularization. Renormalization group. (Actually not necessarily quantum.)

History

- de Broglie waves for electrons. Schrödinger equation; wave function for particles.
- de Broglie and Schrödinger were inspired by special relativity. But it is very difficult (impossible?) to construct a relativistic wave mechanics of sorts. This is what lead to QFT.
- Klein-Gordon equation: relativistic wave equation for particle of mass m . (Supposedly independently discovered by Schrödinger.)

- Spin of electron: introduced to account for the multiplicity of energy states in atomic spectra (Uhlenbeck and Goudsmit). Magnitude of Zeeman splitting gives spin 1/2 of electron.
- It was Paul Dirac who found the equation that incorporates relativity, spin and electromagnetism. $(i\gamma^\mu(\partial_\mu + ieA_\mu) - m_e)\psi = 0$.
- From a modern perspective, Dirac's equation transforms covariantly under the spin-1/2 representation of Lorentz group. Klein-Gordon (Higgs boson) is spin-0. Maxwell vector potential/gluons are (massless) spin-1; W, Z are massive spin-1. Weak field gravitation is (massless) spin-2.
- Dirac's equation admits negative energy solutions, whose interpretation eventually lead to the notion of anti-particles: every particle has a corresponding counter-part - $+e$ charge (particle) becomes $-e$ charge (anti-particle), m mass remains the same, etc. Electron-positrons may annihilate to produce photons.
- Pauli exclusion principle, devised to explain atomic shell structure; Dirac sea.
- Anderson discovers positron from cosmic rays.
- Born, Heisenberg, Jordan: quantized free photons. SHO. Infinite zero-point energy/CC problem.
- Dirac was first to properly treat spontaneous photon emission. (Example of particle number non-conservation) Of course, photons can be absorbed too.
- Jordan and Wigner: anti-commutation relations for electron fields. Pauli exclusion. Fierz and Pauli showed spin-statistics.
- Heisenberg-Pauli developed more general QFT.
- Fermi beta decay. Neutrinos. Particle physics and QFT.
- Pauli and Weisskopf resolved Dirac sea problem. Two sets of creation/destruction operators.
- Fields are promoted to operators. They are not probability amplitudes.
- New particles discovered, not account for by QED.
- Self-energy of electron. Infinities in QFT. UV divergences; ignorance of short distance.
- Light by light scattering. IR divergences; massless particles can be produced with arbitrarily low energy. Heisenberg proposed Fundamental Length, to cut off UV divergences. Renormalizable QFTs and infinities.
- Radiative corrections. Electron magnetic moment is not just $e\hbar/(2m_e c)$ but slightly larger. Schwinger. Lamb shift $2s_{1/2} - -2p_{1/2}$ shift in H; can be computed using QFT; intermediate states. ("Just because something is infinite does not mean it is zero.")

- Schwinger (operator method and action principle); Lorentz-invariant operator methods of Tomonaga; Feynman diagrams. Dyson the consolidator. (Also: Dyson and infinities/renormalizability.)
- QFT: Conservatives won.

Key Ingredients of QFT

- Study of spacetime, ‘internal’ and gauge symmetries. Group theory.
- Correlation functions.
- Path integrals.
- Scattering amplitudes – key for probing particle interactions.
- Quantum vacuum

Our Path to (Perturbative) QFT

- Classical Field Theory
- Group representation theory
- Quantum SHO. Canonical Quantization.
- Path integrals
- QED

2 Classical Scalar Fields in Minkowski Spacetime

Field theory in Minkowski spacetime indicates we wish to construct partial differential equations obeyed by fields such that they take the same form in all inertial frames – i.e., the PDEs are Lorentz covariant. As a warm-up, we shall in this section study the case of scalar fields.

A scalar field $\varphi(x)$ is an object that transforms, under Poincaré transformations

$$x^\mu = \Lambda^\mu{}_\nu x'^\nu + a^\mu \quad (2.0.1)$$

as simply

$$\varphi(x(x')) = \varphi(x^\mu = \Lambda^\mu{}_\nu x'^\nu + a^\mu) \equiv \varphi(x'). \quad (2.0.2)$$

To ensure that this is the case, we would like the PDE it obeys to take the same form in the two inertial frames $\{x^\mu\}$ and $\{x'^\mu\}$ related by eq. (2.0.1). The simplest example is the wave equation with some external scalar source $J(x)$. Let's first write it in the x^μ coordinate system.

$$\eta^{\mu\nu} \partial_\mu \partial_\nu \varphi(x) = J(x), \quad \partial_\mu \equiv \partial / \partial x^\mu. \quad (2.0.3)$$

If putting a prime on the index denotes derivative with respect to x'^μ , namely $\partial_{\mu'} \equiv \partial / \partial x'^\mu$, then by the chain rule,

$$\partial_{\mu'} = \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial}{\partial x^\sigma} = \partial_{\mu'} (\Lambda^\sigma{}_\rho x'^\rho + a^\sigma) \partial_\sigma \quad (2.0.4)$$

$$= \Lambda^\sigma{}_\mu \partial_\sigma. \quad (2.0.5)$$

Therefore the wave operator indeed takes the same form in both coordinate systems:

$$\eta^{\mu\nu} \partial_{\mu'} \partial_{\nu'} = \eta^{\mu\nu} \Lambda^\sigma{}_\mu \Lambda^\rho{}_\nu \partial_\sigma \partial_\rho \quad (2.0.6)$$

$$= \eta^{\sigma\rho} \partial_\sigma \partial_\rho. \quad (2.0.7)$$

because of Lorentz invariance

$$\eta^{\mu\nu} \Lambda^\sigma{}_\mu \Lambda^\rho{}_\nu = \eta^{\sigma\rho}. \quad (2.0.8)$$

A generalization of the wave equation in eq. (2.0.3) is to add a potential $V(\varphi)$:

$$\partial^2 \varphi + V'(\varphi) = J, \quad (2.0.9)$$

where $\partial^2 \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu$ and the prime is a derivative with respect to the argument.

2.1 Action Principle and Symmetries

There is in fact an efficient means to define a theory such that it would enjoy the symmetries one desires. This is the action principle. You may encounter it in (non-relativistic) Classical Mechanics, where Newton's second law emerges from demanding the integral

$$S \equiv \int_{t_i}^{t_f} L dt, \quad (2.1.1)$$

$$L \equiv \frac{1}{2} m \dot{\vec{x}}(t)^2 - V(\vec{x}(t)). \quad (2.1.2)$$

Here, L is called the Lagrangian, and in this context is the difference between the particle's kinetic and potential energy. The action of a field theory also plays a central role in its quantum theory when phrased in the path integral formulation; roughly speaking, $\exp(iS)$ is related to the infinitesimal quantum transition amplitude. For these reasons, we shall study the classical field theories – leading up to General Relativity itself – through the principle of stationary action.

Lorentz covariance In field theory one defines an object similar to the one in eq. (2.1.1), except the integrand \mathcal{L} is now a Lagrangian *density* (per unit spacetime volume). To obtain Lorentz covariant equations, we now demand that the Lagrangian density is, possibly up to a total divergence, a scalar under spacetime Lorentz transformations and other symmetry transformations relevant to the problem at hand.

$$S \equiv \int_{t_i}^{t_f} \mathcal{L} d^d x \quad (2.1.3)$$

One then demands that the action is extremized under the boundary conditions that the field configurations at some initial t_i and final time t_f are fixed. If the spatial boundaries of the spacetime are a finite distance away, one would also have to impose appropriate boundary conditions there; otherwise, if space is infinite, the fields are usually assumed to fall off to zero sufficiently quickly at spatial infinity – below, we will assume the latter for technical simplicity. (In particle mechanics, the action principle also assumes the initial and final positions of the particle are specified.)

Let us begin with a scalar field φ . For concreteness, we shall form its Lagrangian density $\mathcal{L}(\varphi, \partial_\alpha \varphi)$ out of φ and its first derivatives $\partial_\alpha \varphi$. Demanding the resulting action be extremized means its first order variation need to vanish. That is, we shall replace $\varphi \rightarrow \varphi + \delta\varphi$ (which also means $\partial_\alpha \varphi \rightarrow \partial_\alpha \varphi + \partial_\alpha \delta\varphi$) and demand that the portion of the action linear in $\delta\varphi$ be zero.

$$\begin{aligned} \delta_\varphi S &= \int_{t_i}^{t_f} d^d x \left(\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \varphi)} \partial_\alpha \delta\varphi \right) \\ &= \left[\int d^{d-1} \Sigma_\alpha \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \varphi)} \delta\varphi \right]_{t_i}^{t_f} + \int_{t_i}^{t_f} d^d x \delta\varphi \left(\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \varphi)} \right) \end{aligned} \quad (2.1.4)$$

Because the initial and final field configurations $\varphi(t_i)$ and $\varphi(t_f)$ are assumed fixed, their respective variations are zero by definition: $\delta\varphi(t_i) = \delta\varphi(t_f) = 0$. This sets to zero the first term on the second equality. At this point, the requirement that the action be stationary means $\delta_\varphi S$ be zero for any small but arbitrary $\delta\varphi$, which in turn implies the coefficient of $\delta\varphi$ must be zero. That leaves us with the Euler-Lagrangian equations

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \partial_\alpha \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \varphi)}. \quad (2.1.5)$$

We may now consider a coordinate transformation $x(x')$. Assuming \mathcal{L} is a coordinate scalar, this means the only ingredient that is not a scalar is the derivative with respect to $\partial_\alpha \varphi$. Since

$$\frac{\partial x^\alpha}{\partial x'^\mu} \partial_\alpha \varphi(x) = \partial_{\mu'} \varphi(x') \equiv \partial_{\mu'} \varphi(x(x')), \quad (2.1.6)$$

we have

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \varphi(x))} = \frac{\partial(\partial_{\mu'} \varphi(x'))}{\partial(\partial_\alpha \varphi(x))} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu'} \varphi(x'))} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu'} \varphi(x'))}. \quad (2.1.7)$$

That is, $\partial \mathcal{L} / \partial(\partial_\alpha \varphi(x))$ transforms as a rank-1 vector; and $\partial_\alpha \{ \partial \mathcal{L} / \partial(\partial_\alpha \varphi(x)) \}$ is its divergence, i.e., a scalar. Altogether, we have thus demonstrated that the Euler-Lagrange equations in eq. (2.1.5), for a scalar field φ , is itself a scalar. This is a direct consequence of the fact that \mathcal{L} is a coordinate scalar by construction. A common example of such a scalar action is

$$S[\varphi] \equiv \int d^d x \left(\frac{1}{2} \eta^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right), \quad (2.1.8)$$

where V is its scalar potential.

Problem 2.1. Show from eq. (2.1.5) that the equations derived from the action in eq. (2.1.8) is $\partial^2 \varphi = -V'(\varphi)$. \square

Internal Global O_N Symmetry To provide an example of a symmetry other than the invariance under coordinate transformations, let us consider the following action involving $N > 1$ scalar fields $\{\varphi^I | I = 1, 2, 3, \dots, N\}$:

$$S \equiv \int d^d x \mathcal{L} (\eta^{\mu\nu} \partial_\mu \varphi^I \partial_\nu \varphi^I, \varphi^I \varphi^I). \quad (2.1.9)$$

With summation convention in force, we see that the sum over the scalar field label ‘I’ is simply a dot product in ‘field space’. This in turn leads us to observe that the action is invariant under a global rotation:

$$\varphi^I \equiv \widehat{R}^I_J \varphi'^J, \quad (2.1.10)$$

where $\widehat{R}^I_A \widehat{R}^J_B \delta_{IJ} = \delta_{AB}$. (By ‘global’ rotation, we mean the rotation matrices $\{\widehat{R}^I_J\}$ do not depend on spacetime.) Explicitly,

$$\int d^d x \mathcal{L} (\eta^{\mu\nu} \partial_\mu \varphi^I \partial_\nu \varphi^I, \varphi^I \varphi^I) = \int d^d x \mathcal{L} (\eta^{\mu\nu} \partial_\mu \varphi'^I \partial_\nu \varphi'^I, \varphi'^I \varphi'^I). \quad (2.1.11)$$

Let us now witness, because we have constructed a Lagrangian density that is invariant under such an internal O_N symmetry, the resulting equations of motion transform covariantly under rotations. Firstly, the I-th Euler-Lagrange equation, gotten by varying eq. (2.1.9) with respect to φ^I , reads

$$\frac{\partial \mathcal{L}}{\partial \varphi^I} = \partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \varphi^I)}. \quad (2.1.12)$$

Under rotation, eq. (2.1.10) is equivalent to

$$\left(\widehat{R}^{-1} \right)^J_I \varphi^I = \varphi'^J, \quad (2.1.13)$$

which in turn tells us

$$\left(\widehat{R}^{-1}\right)_{\text{I}}^{\text{J}} \partial_{\alpha} \varphi^{\text{I}} = \partial_{\alpha} \varphi^{\text{J}}. \quad (2.1.14)$$

Therefore eq. (2.1.12) becomes

$$\frac{\partial \varphi^{\text{J}}}{\partial \varphi^{\text{I}}} \frac{\partial \mathcal{L}}{\partial \varphi^{\text{J}}} = \frac{\partial \partial_{\alpha} \varphi^{\text{J}}}{\partial \partial_{\alpha} \varphi^{\text{I}}} \partial_{\alpha} \frac{\partial \mathcal{L}}{\partial (\partial_{\alpha} \varphi^{\text{J}})}, \quad (2.1.15)$$

$$\left(\widehat{R}^{-1}\right)_{\text{I}}^{\text{J}} \frac{\partial \mathcal{L}}{\partial \varphi^{\text{J}}} = \left(\widehat{R}^{-1}\right)_{\text{I}}^{\text{J}} \partial_{\alpha} \frac{\partial \mathcal{L}}{\partial (\partial_{\alpha} \varphi^{\text{J}})}. \quad (2.1.16)$$

The PDEs for our O_N -invariant scalar field theory transforms covariantly as a vector under global rotation of the fields $\{\varphi^{\text{I}}\}$.

2.2 Local Conservation Laws

Non-relativistic You would be rightly shocked if you had stored a sealed tank of water on your rooftop only to find its contents gradually disappearing over time – the total mass of water ought to be a constant. Assuming a flat space geometry, if you had instead connected the tank to two pipes, one that pumps water into the tank and the other pumping water out of it, the rate of change of the total mass of the water

$$M \equiv \int_{\text{tank}} \rho(t, \vec{x}) d^3 \vec{x} \quad (2.2.1)$$

in the tank – where t is time, \vec{x} are Cartesian coordinates, and $\rho(t, \vec{x})$ is the water’s mass density – is

$$\frac{d}{dt} \int_{\text{tank}} \rho d^3 \vec{x} = - \left(\int_{\text{cross section of 'in' pipe}} + \int_{\text{cross section of 'out' pipe}} \right) d^2 \vec{\Sigma} \cdot (\rho \vec{v}). \quad (2.2.2)$$

Note that $d^2 \vec{\Sigma}$ points *outwards* from the tank, so at the ‘in’ pipe-tank interface, if the water were indeed following into the pipe, $-d^2 \vec{\Sigma} \cdot (\rho \vec{v}) > 0$ and its contribution to the rate of increase is positive. At the ‘out’ pipe-tank interface, if the water were indeed following out of the pipe, $-d^2 \vec{\Sigma} \cdot (\rho \vec{v}) < 0$. If we apply Gauss’ theorem,

$$\int_{\text{tank}} \dot{\rho} d^3 \vec{x} = - \int_{\text{tank}} d^3 \vec{x} \vec{\nabla} \cdot (\rho \vec{v}). \quad (2.2.3)$$

If we applied the same sort of reasoning to any infinitesimal packet of fluid, with some local mass density ρ , we would find the following local conservation law

$$\dot{\rho} = -\partial_i (\rho \cdot v^i). \quad (2.2.4)$$

This is a “local” conservation law in the sense that mass cannot simply vanish from one location and re-appear a finite distance away, without first flowing to a neighboring location.

Relativistic We have implicitly assumed a non-relativistic system, where $|\vec{v}| \ll 1$. This is an excellent approximation for most hydrodynamics problems. Strictly speaking, however,

relativistic effects – length contraction, in particular – imply that mass density is not a Lorentz scalar. If we define $\rho(t, \vec{x})$ to be the mass density at (t, \vec{x}) in a frame instantaneously at rest (aka ‘co-moving’) with the fluid packet, then the mass density current that is a locally conserved Lorentz vector is given by

$$J^\mu(t, \vec{x}) \equiv \rho(t, \vec{x})v^\mu(t, \vec{x}). \quad (2.2.5)$$

Along its integral curve v^μ should be viewed as the proper velocity $d(t, \vec{x})^\mu/d\tau$ of the fluid packet, where τ is the latter’s proper time. Moreover, as long as the velocity v^μ is timelike, which is certainly true for fluids, let us recall it is always possible to find a (local) Lorentz transformation $\Lambda^\mu{}_\nu(t, \vec{x})$ such that

$$(1, \vec{0})^\mu \equiv v'^\mu = \Lambda^\mu{}_\nu(t, \vec{x})v^\nu(t, \vec{x}). \quad (2.2.6)$$

and the mass density-current is now

$$J'^\mu = \rho(t', \vec{x}')v'^\mu = \rho(x') \cdot \delta_0^\mu. \quad (2.2.7)$$

The local conservation law obeyed by this relativistically covariant current J^μ is now (in Cartesian coordinates)

$$\partial_\mu J^\mu = 0; \quad (2.2.8)$$

which in turn is a Lorentz invariant statement. Total mass M in a given global inertial frame at a fixed time t is

$$M \equiv \int_{\mathbb{R}^3} d^3\vec{x} J^0. \quad (2.2.9)$$

To show it is a constant, we take the time derivative, and employ eq. (2.2.8):

$$\dot{M} = \int_{\mathbb{R}^3} d^3\vec{x} \partial_0 J^0 = - \int_{\mathbb{R}^3} d^3\vec{x} \partial_i J^i. \quad (2.2.10)$$

The divergence theorem tells us that this is equal to the flux of J^i at spatial infinity. But there is no J^i at spatial infinity for physically realistic – i.e., isolated – systems.

2.3 Noether: Continuous Symmetries and Conserved Currents

A field is a substance permeating spacetime. In this section, we shall attempt to associate with it energy-momentum at every location in spacetime, by identifying the Noether’s currents associated with the symmetries of Minkowski spacetime. Specifically, the conservation of energy is due to the time translation symmetry of the system at hand. The conservation of linear momentum is due to its spatial translation symmetry; whereas the conservation of angular momentum is due to rotational symmetry. Throughout this discussion, we will assume the dynamics of the field theory is governed by some Lorentz invariant Lagrangian density that depends on the field, and on its first derivatives – but no higher.

Spacetime Translations and Stress-Energy Tensor The physical interpretation delineated here for the components of $T^{\hat{\mu}\hat{\nu}}$ is really an assertion. Let us attempt to justify it

partially, by appealing to the flat spacetime limit, where the momentum of a classical field theory may be viewed as the conserved Noether current of spacetime translation symmetry. Specifically, let us analyze the canonical scalar field theory of eq. (2.1.8) but with $g_{\mu\nu} = \eta_{\mu\nu}$.

$$\mathcal{L}(x) = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi(x)\partial_\nu\varphi(x) - V(\varphi(x)). \quad (2.3.1)$$

Since \mathcal{L} is Lorentz invariant, we may consider an infinitesimal spacetime displacement,

$$x^\mu = x'^\mu + a^\mu, \quad (2.3.2)$$

for constant but ‘small’ a^μ .

$$\mathcal{L}(x) = \mathcal{L}(x') + a^\mu\partial_{\mu'}\mathcal{L}(x') + \mathcal{O}(a^2). \quad (2.3.3)$$

On the other hand, $\partial/\partial x^\mu = \partial_\mu = \partial_{\mu'} = \partial/\partial x'^\mu$ and

$$\mathcal{L}(x' + a) = \mathcal{L}(x') + \frac{\partial\mathcal{L}}{\partial\varphi(x')}a^\nu\partial_{\nu'}\varphi(x') + \frac{\partial\mathcal{L}}{\partial\partial_{\mu'}\varphi(x')}a^\nu\partial_{\nu'}\partial_{\mu'}\varphi(x') + \mathcal{O}(a^2) \quad (2.3.4)$$

$$= \mathcal{L}(x') + a^\nu\partial_{\nu'}\varphi(x') \left\{ \frac{\partial\mathcal{L}}{\partial\varphi(x')} - \partial_{\mu'}\frac{\partial\mathcal{L}}{\partial\partial_{\mu'}\varphi(x')} \right\} + a^\nu\partial_{\mu'} \left(\partial_{\nu'}\varphi(x')\frac{\partial\mathcal{L}}{\partial\partial_{\mu'}\varphi(x')} \right) + \mathcal{O}(a^2). \quad (2.3.5)$$

Using the equations-of-motion for the scalar field

$$\frac{\partial\mathcal{L}}{\partial\varphi(x')} - \partial_{\mu'}\frac{\partial\mathcal{L}}{\partial\partial_{\mu'}\varphi(x')} = 0, \quad (2.3.6)$$

eq. (2.3.5) becomes

$$\mathcal{L}(x' + a) = \mathcal{L}(x') + a^\nu\partial_{\mu'} \left(\partial_{\nu'}\varphi(x')\frac{\partial\mathcal{L}}{\partial\partial_{\mu'}\varphi(x')} \right). \quad (2.3.7)$$

We may now equate the linear-in- a^ν terms on the right hand sides of equations (2.3.3) and (2.3.7), and find the following conservation law:

$$\partial_{\mu'} \left\{ a^\gamma \left(\partial_{\gamma'}\varphi(x')\frac{\partial\mathcal{L}}{\partial\partial_{\mu'}\varphi(x')} - \delta_\gamma^\mu\mathcal{L}(x') \right) \right\} = 0. \quad (2.3.8)$$

By setting $a^\gamma = \delta_\nu^\gamma$, for a fixed ν , we may identify the conserved quantity inside the $\{\dots\}$ as the Noether momentum p_ν due to translation symmetry along the ν -th direction.² Doing so now allows us to identify the conserved stress tensor

$$T^\mu{}_\nu = \partial_{\nu'}\varphi(x')\frac{\partial\mathcal{L}}{\partial\partial_{\mu'}\varphi(x')} - \delta_\nu^\mu\mathcal{L}(x'). \quad (2.3.9)$$

²As a simple parallel to the situation here: in classical mechanics, because the free Lagrangian $L = (1/2)\dot{\vec{x}}^2$ is space-translation invariant, $\partial L/\partial x^i = 0$, we may identify the momentum $p_i \equiv \partial L/\partial \dot{x}^i$ as the corresponding Noether charge.

Applying this to eq. (2.3.1), we obtain

$$T^\mu{}_\nu = \partial_\nu \varphi \partial^\mu \varphi - \delta_\nu^\mu \left(\frac{1}{2} (\partial\varphi)^2 - V(\varphi) \right). \quad (2.3.10)$$

This is simply the flat spacetime limit of the stress tensor in eq. (??). Unfortunately, this same Noether procedure fails to reproduce even the electromagnetic stress tensor in eq. (??); to do so, it turns out this Noether current needs to be augmented by additional terms to form the Belinfante-Rosenfeld tensor [?, ?].

Interpretation of J^μ, a^ν . Just as the local conservation of mass or electric charge leads to their appropriate currents J^μ and their divergence free properties $\partial_\mu J^\mu$, we shall see here that the conservation of energy and momentum leads us to the divergence-less energy-momentum-shear-stress tensor – or, more commonly, the energy-momentum or stress-energy tensor.

In a given inertial frame, we associate time translation symmetry with the conservation of energy. Hence, by choosing $a^\nu \partial_\nu = \partial_t$, we may associate J^μ_0 as the energy current. Whenever it is timelike, the zeroth component J^0_0 may be associated with the co-moving energy density; whereas J^{i0} is the momentum density (i.e., energy per time per area perpendicular to J^{i0}).

In the same inertial frame, we associate translation symmetry in the i th spatial direction with the conservation of the i th component of momentum. By choosing $a^\mu \partial_\nu = \partial_i$, we may associate J^μ_i with the current associated with the i th component of the (spatial) momentum. The zeroth component J^0_i is the density of the i th component of momentum; this tells us $J^{0i} \sim J^{i0}$ (up to an overall sign). Whereas J^k_i is the i th component of momentum per unit time across the spatial surface perpendicular to the k th spatial direction. In particular, when $k = i$, this would be the momentum per unit time through the surface perpendicular to the $i = k$ th direction – but this is simply the pressure (force per unit time) acting on an infinitesimal slab between x^i and $x^i + dx^i$ in the $i = k$ direction. For $i \neq k$, the J^k_i is shear: force in the i th direction per unit area perpendicular to the k th direction. Now, if the force in the k th direction per unit area perpendicular to the i th direction were not equal to the J^k_i , there will be a torque generated on the (i, k) plane on an infinitesimal area. Hence, we expect $J^{ki} = J^{ik}$.

All these considerations allow us to identify the components $J^{\alpha\beta} \equiv T^{\alpha\beta}$ as those of the energy-momentum-shear-stress (note: stress \equiv pressure) tensor, the flux of the μ th component of energy-momentum across the hypersurface orthogonal to the ν th direction.

- T^{00} is the energy density (\equiv energy per unit spatial volume).
- $T^{0i} = T^{i0}$ is the linear momentum density (\equiv energy per unit time per unit area perpendicular to the i th direction).
- $T^{ij} = T^{ji}$ for $i \neq j$ is the shear density; the flow of the i th component of momentum per unit time per unit area perpendicular to the j th surface.
- T^{ii} is the pressure/stress (\equiv force per unit area) in the i th direction.

Noether: General Case Let us suppose that a small change is induced on some real scalar field $\delta\varphi$ that leaves the Lagrangian invariant up to a total derivative.

$$\varphi \rightarrow \varphi + \delta\varphi, \quad (2.3.11)$$

$$\mathcal{L} \rightarrow \mathcal{L} + \partial_\mu K^\mu. \quad (2.3.12)$$

On the other hand, we may expand

$$\partial_\mu K^\mu = \frac{\partial \mathcal{L}}{\partial \varphi} \delta \varphi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \partial_\mu \delta \varphi \quad (2.3.13)$$

$$= \left(\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \right) \delta \varphi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \delta \varphi \right) \quad (2.3.14)$$

The first group of terms on the right hand side of the second equality is the Euler-Lagrange operation on the Lagrangian. In particular, we see that – if the EoM of the scalar is satisfied – then we may identify

$$\partial_\mu J^\mu = 0 \quad (2.3.15)$$

$$J^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \delta \varphi - K^\mu. \quad (2.3.16)$$

Noether's Currents: Ambiguities Notice we read off the Noether current from a divergence equation of the form $\partial_\mu J^\mu = 0$. That means we may add an identically conserved current to the LHS and, hence, yield a different Noether J^μ . It turns out, the most general identically conserved current takes the form $\partial_\mu \partial_\nu K^{\mu\nu}$ for arbitrary but anti-symmetric $K^{\mu\nu} = -K^{\nu\mu}$. Hence, Noether currents are always ambiguous up to this additive $\partial_\nu K^{\mu\nu}$ term. Furthermore, if J^μ is conserved, so is a constant A times it. To sum,

If, due to some continuous symmetry, J^μ is conserved when evaluated on the solutions to the Euler-Lagrange equations, so is

$$J'^\mu \equiv A \cdot (J^\mu + \partial_\nu K^{\mu\nu}) \quad (2.3.17)$$

for arbitrary constant A and anti-symmetric $K^{\mu\nu} = -K^{\nu\mu}$.

This means the stress energy tensor we ‘derived’ earlier is, likewise, ambiguous in the same manner. In a given physical situation, therefore, we need additional criteria to pin down the precise physical meanings of the components of the Noether currents.

Internal SO_D Example If a Lagrangian involves 3 scalar fields φ^I such that the former is invariant under global SO_D rotations of the latter:

$$\mathcal{L} = \frac{1}{2} \partial_\alpha \varphi^I \partial^\alpha \varphi^I - V(\varphi^I \varphi^I), \quad (2.3.18)$$

$$\varphi^I \rightarrow \widehat{R}^I_{\ J} \varphi^J, \quad (2.3.19)$$

$$\mathcal{L} \rightarrow \mathcal{L}. \quad (2.3.20)$$

Under infinitesimal rotations, we may rotate the pairs (1, 2), (1, 3) and (2, 3).

$$\varphi^I \rightarrow \varphi^I - i\theta \left(\widehat{J}^{AB} \right)_{IJ} \varphi^J \quad (2.3.21)$$

Recalling that $i(\widehat{J}^{AB})_{IJ} = \delta_{[I}^A \delta_{J]}^B$, we may recognize

$$\delta \varphi^I = \delta^{I[A} \varphi^{B]}. \quad (2.3.22)$$

The Noether currents – one for each rotation generator – are

$$J_K^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^I} \delta^{I[A} \varphi^{B]} \quad (2.3.23)$$

$$= \partial^\mu \varphi^{[A} \cdot \varphi^{B]}. \quad (2.3.24)$$

We may check explicitly that this is conserved. Firstly, the EoMs are

$$\partial^2 \varphi^I = -2\varphi^I V'(\vec{\varphi}^2). \quad (2.3.25)$$

Hence,

$$\partial_\mu J^\mu = \partial^2 \varphi^{[A} \cdot \varphi^{B]} + \partial^\mu \varphi^{[I} \cdot \partial_\mu \varphi^{B]} \quad (2.3.26)$$

$$= -2V' \cdot \varphi^{[A} \cdot \varphi^{B]} = 0. \quad (2.3.27)$$

Problem 2.2. Noether, Lorentz and Angular Momentum Above, we consider the Noether current T^μ_α , obeying $\partial_\mu T^\mu_\alpha = 0$, corresponding to spacetime translation symmetry.

Let us now consider the Noether current $J^\mu_{\alpha\beta}$, obeying $\partial_\mu J^\mu_{\alpha\beta} = 0$, from the Lorentz transformation $x^\alpha \rightarrow \Lambda^\alpha_\beta x^\beta$. (Why are there two extra indices ($\alpha\beta$) on the Noether current? Hint: How are the Lorentz generators labeled?) Show that it is possible to obtain

$$J^{\mu\alpha\beta} = x^{[\beta} T^{\alpha]\mu}, \quad (2.3.28)$$

where $T^{\mu\nu}$ is the Noether current of spacetime translations in eq. (2.3.9). Interpret the components of $J^{\mu\alpha\beta}$; i.e., what is the Noether current of spatial rotations? And of boosts? \square

Problem 2.3. Symmetric Noether Currents Explain why, if $\partial_\mu J^{\mu\alpha\beta} = 0$, where $J^{\mu\alpha\beta}$ is given by eq. (2.3.28), then $T^{\mu\nu} = T^{\nu\mu}$.

In other words, if we can obtain the Noether current of Lorentz transformations to be related to that of spacetime translations in the form of eq. (2.3.28), then the Noether current of spacetime translations must be a symmetric tensor. This symmetry property is important for interpreting $T^{\mu\nu}$ as the energy-momentum-stress tensor. \square

2.4 Hamiltonian Formulation

1D Particle Mechanics: Review In particle mechanics, from the Lagrangian $L(q, \dot{q})$, we may define the momentum conjugate to the (generalized) position q as

$$p \equiv \left(\frac{\partial L(q, \dot{q})}{\partial \dot{q}} \right)_q. \quad (2.4.1)$$

This relation between p and (q, \dot{q}) usually allows us to invert \dot{q} for the pair (q, p) ; so that every variable can now be expressed in terms of this pair – this allows the interpretation of (q, p) as *independent* “phase space” variables in what follows. The Hamiltonian itself is

$$H(q, p) \equiv p \cdot \dot{q}(q, p) - L(q, p), \quad (2.4.2)$$

$$L(q, p) \equiv L(q, \dot{q}(q, p)). \quad (2.4.3)$$

Hamilton's equations now reads

$$\dot{q} = \frac{\partial H}{\partial p} \quad \text{and} \quad \dot{p} = -\frac{\partial H}{\partial q}. \quad (2.4.4)$$

Field Theory For field theory, an analogous discussion follows. The Lagrangian \mathcal{L} depends on the field (which is analogous to the position q) and its partial derivatives. The time derivative $\dot{\varphi} \equiv \partial_0 \varphi$ is analogous to \dot{q} ; this, in turn, allows us to define the momentum conjugate to φ as

$$\Pi(x) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi(x))}. \quad (2.4.5)$$

Like the particle mechanics case, we shall assume it is possible to solve $\dot{\varphi}$ in terms of φ and Π . We then define the Hamiltonian density via the Legendre transform:

$$\mathcal{H} \equiv \Pi \cdot \partial_0 \varphi - \mathcal{L}. \quad (2.4.6)$$

We may vary this Legendre transform,

$$\delta \mathcal{H} = \delta \Pi \cdot \partial_0 \varphi + \Pi \cdot \partial_0 \delta \varphi - \frac{\partial \mathcal{L}}{\partial \varphi} \delta \varphi - \frac{\partial \mathcal{L}}{\partial \partial_0 \varphi} \partial_0 \delta \varphi - \frac{\partial \mathcal{L}}{\partial \partial_i \varphi} \partial_i \delta \varphi \quad (2.4.7)$$

$$= \delta \Pi \cdot \partial_0 \varphi + \Pi \cdot \partial_0 \delta \varphi - \frac{\partial \mathcal{L}}{\partial \varphi} \delta \varphi - \Pi \cdot \partial_0 \delta \varphi - \frac{\partial \mathcal{L}}{\partial \partial_i \varphi} \partial_i \delta \varphi \quad (2.4.8)$$

$$= \delta \Pi \cdot \partial_0 \varphi - \frac{\partial \mathcal{L}}{\partial \varphi} \delta \varphi + \partial_i \frac{\partial \mathcal{L}}{\partial \partial_i \varphi} \delta \varphi - \partial_i \left(\frac{\partial \mathcal{L}}{\partial \partial_i \varphi} \delta \varphi \right). \quad (2.4.9)$$

Applying the Euler-Lagrange equations

$$\partial_0 \frac{\partial \mathcal{L}}{\partial \partial_0 \varphi} + \partial_i \frac{\partial \mathcal{L}}{\partial \partial_i \varphi} = \frac{\partial \mathcal{L}}{\partial \varphi}, \quad (2.4.10)$$

$$\partial_i \frac{\partial \mathcal{L}}{\partial \partial_i \varphi} = \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_0 \Pi. \quad (2.4.11)$$

to eq. (2.4.9), we find that

$$\delta \mathcal{H} = \delta \Pi \cdot \partial_0 \varphi - \frac{\partial \mathcal{L}}{\partial \varphi} \delta \varphi + \left(\frac{\partial \mathcal{L}}{\partial \varphi} - \dot{\Pi} \right) \delta \varphi - \partial_i \left(\frac{\partial \mathcal{L}}{\partial \partial_i \varphi} \delta \varphi \right) \quad (2.4.12)$$

$$= \delta \Pi \cdot \dot{\varphi} - \dot{\Pi} \cdot \delta \varphi - \partial_i \left(\frac{\partial \mathcal{L}}{\partial \partial_i \varphi} \delta \varphi \right). \quad (2.4.13)$$

On the other hand, we may vary the Hamiltonian as a function of the field, its spatial gradient, and the conjugate momentum:

$$\varphi \rightarrow \varphi + \delta \varphi, \quad (2.4.14)$$

$$\Pi \rightarrow \Pi + \delta \Pi; \quad (2.4.15)$$

and discover

$$\delta \mathcal{H} = \frac{\partial \mathcal{H}}{\partial \Pi} \cdot \delta \Pi + \frac{\partial \mathcal{H}}{\partial \varphi} \cdot \delta \varphi + \frac{\partial \mathcal{H}}{\partial \partial_i \varphi} \cdot \partial_i \delta \varphi \quad (2.4.16)$$

$$= \frac{\partial \mathcal{H}}{\partial \Pi} \cdot \delta \Pi + \left(\frac{\partial \mathcal{H}}{\partial \varphi} - \partial_i \frac{\partial \mathcal{H}}{\partial \partial_i \varphi} \right) \cdot \delta \varphi + \partial_i \left(\frac{\partial \mathcal{H}}{\partial \partial_i \varphi} \cdot \delta \varphi \right). \quad (2.4.17)$$

Comparing the two variation results,

$$\delta\Pi \cdot \left(\dot{\varphi} - \frac{\partial\mathcal{H}}{\partial\Pi} \right) - \left(\dot{\Pi} + \frac{\partial\mathcal{H}}{\partial\varphi} - \partial_i \frac{\partial\mathcal{H}}{\partial\partial_i\varphi} \right) \delta\varphi = \partial_i \left\{ \left(\frac{\partial\mathcal{H}}{\partial\partial_i\varphi} + \frac{\partial\mathcal{L}}{\partial\partial_i\varphi} \right) \delta\varphi \right\}. \quad (2.4.18)$$

If we integrate both sides over space, the right hand side will be converted into a surface integral at spatial infinity, which we may argue should fall off to zero as long as $\delta\varphi$ does.

$$\int_{\mathbb{R}^D} d^D\vec{x} \left\{ \delta\Pi \cdot \left(\dot{\varphi} - \frac{\partial\mathcal{H}}{\partial\Pi} \right) - \left(\dot{\Pi} + \frac{\partial\mathcal{H}}{\partial\varphi} - \partial_i \frac{\partial\mathcal{H}}{\partial\partial_i\varphi} \right) \delta\varphi \right\} = 0 \quad (2.4.19)$$

By viewing Π and φ as independent variables, the coefficients of their variations on the left hand side must therefore be individually zero because $\delta\Pi$ and $\delta\varphi$ are arbitrary at every point in space.

$$\dot{\varphi} = \frac{\partial\mathcal{H}}{\partial\Pi} \quad (2.4.20)$$

$$\dot{\Pi} = \partial_i \frac{\partial\mathcal{H}}{\partial\partial_i\varphi} - \frac{\partial\mathcal{H}}{\partial\varphi} \quad (2.4.21)$$

This in turn implies, the right hand side of eq. (2.4.18) must be zero too. And since $\delta\varphi$ was arbitrary,

$$\frac{\partial\mathcal{H}}{\partial(\partial_i\varphi)} = -\frac{\partial\mathcal{L}}{\partial(\partial_i\varphi)}. \quad (2.4.22)$$

Example Let us work out Hamilton's equations for the canonical scalar field in eq. (2.3.1).

$$\Pi = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \dot{\varphi} \quad (2.4.23)$$

The Lagrangian is thus

$$\mathcal{L} = \frac{1}{2}\Pi^2 - \frac{1}{2}(\vec{\nabla}\varphi)^2 - V. \quad (2.4.24)$$

The Legendre transform now reads

$$\mathcal{H} = \Pi^2 - \left(\frac{1}{2}\Pi^2 - \frac{1}{2}(\vec{\nabla}\varphi)^2 - V \right) \quad (2.4.25)$$

$$= \frac{1}{2}\Pi^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + V. \quad (2.4.26)$$

Hamilton's equations are

$$\dot{\varphi} = \Pi, \quad (2.4.27)$$

$$\dot{\Pi} = \vec{\nabla}^2\varphi - V'(\varphi). \quad (2.4.28)$$

When combined, they simply yield $\ddot{\varphi} - \vec{\nabla}^2\varphi = -V'$ as before. We may also readily verify

$$\frac{\partial\mathcal{H}}{\partial(\partial_i\varphi)} = \partial_i\varphi = -\frac{\partial\mathcal{L}}{\partial(\partial_i\varphi)}. \quad (2.4.29)$$

Problem 2.4. Lagrangians from Hamiltonians Starting from the definition of momentum in eq. (2.4.5) and relationship between the Hamiltonian and the Lagrangian in eq. (2.4.6), show that Hamilton's equations in equations (2.4.20) and (2.4.21) imply the Euler-Lagrange equation $\partial_\mu(\partial\mathcal{L}/\partial(\partial_\mu\varphi)) = \partial\mathcal{L}/\partial\varphi$. \square

Problem 2.5. Hamilton's Equations for \mathbf{O}_N model Work out Hamilton's equations from the Lagrangian in eq. (2.3.18). \square

Problem 2.6. Poisson Brackets and Time Evolution Define the functional derivatives

$$\frac{\delta\varphi(t, \vec{x})}{\delta\varphi(t, \vec{y})} = \delta^{(D)}(\vec{x} - \vec{y}), \quad (2.4.30)$$

$$\frac{\delta\Pi(t, \vec{x})}{\delta\Pi(t, \vec{y})} = \delta^{(D)}(\vec{x} - \vec{y}), \quad (2.4.31)$$

$$\frac{\delta\varphi(t, \vec{x})}{\delta\Pi(t, \vec{y})} = 0 = \frac{\delta\Pi(t, \vec{x})}{\delta\varphi(t, \vec{y})}; \quad (2.4.32)$$

as well as the 'equal-time' Poisson bracket

$$\begin{aligned} & \{f(\varphi(t, \vec{x}), \Pi(t, \vec{x})), g(\varphi(t, \vec{x}'), \Pi(t, \vec{x}'))\} \\ & \equiv \int_{\mathbb{R}^D} d^D\vec{y} \left(\frac{\delta f(t, \vec{x})}{\delta\varphi(t, \vec{y})} \frac{\delta g(t, \vec{x}')}{\delta\Pi(t, \vec{y})} - \frac{\delta g(t, \vec{x}')}{\delta\varphi(t, \vec{y})} \frac{\delta f(t, \vec{x})}{\delta\Pi(t, \vec{y})} \right). \end{aligned} \quad (2.4.33)$$

Show that, for the canonical Hamiltonian in eq. (2.4.26), the Poisson bracket equations

$$\partial_t\varphi(t, \vec{x}) = \{\varphi(t, \vec{x}), H(t)\}, \quad (2.4.34)$$

$$\partial_t\Pi(t, \vec{x}) = \{\Pi(t, \vec{x}), H(t)\} \quad (2.4.35)$$

involving the total Hamiltonian

$$H(t) \equiv \int_{\mathbb{R}^D} d^D\vec{x}'' \mathcal{H}(t, \vec{x}''); \quad (2.4.36)$$

is in fact equivalent to Hamilton's equations. \square

2.5 Fourier Space(time)

One of the key insights that we need for doing (perturbative) field theory in Minkowski spacetime, is that the linear massive wave equation is really a SHO oscillator in Fourier space. That is, if we decompose

$$\varphi(t, \vec{x}) = \int_{\mathbb{R}^D} \frac{d^D\vec{k}}{(2\pi)^D} \tilde{\varphi}(t, \vec{k}) e^{i\vec{k}\cdot\vec{x}} = \int_{\mathbb{R}^D} \frac{d^D\vec{k}}{(2\pi)^D} \tilde{\varphi}(t, \vec{k}) e^{-ik_j x^j}, \quad (2.5.1)$$

we see that

$$\partial^2\varphi(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} \left(\ddot{\tilde{\varphi}} + \vec{k}^2\tilde{\varphi} \right) e^{i\vec{k}\cdot\vec{x}} \quad (2.5.2)$$

because each spatial derivative acting on the exponential amounts to the replacement rule $\partial_j \rightarrow -ik_j$ because

$$\partial_j e^{i\vec{k}\cdot\vec{x}} = \partial_j (-ik_l x^l) e^{i\vec{k}\cdot\vec{x}} \quad (2.5.3)$$

$$= (-ik_l \delta_j^l) e^{i\vec{k}\cdot\vec{x}} \quad (2.5.4)$$

$$= (-ik_j) e^{i\vec{k}\cdot\vec{x}}. \quad (2.5.5)$$

The wave equation sourced by some external J ,

$$\ddot{\varphi}(t, \vec{x}) - \vec{\nabla}^2 \varphi(t, \vec{x}) + m^2 \varphi(t, \vec{x}) = J(t, \vec{x}) \quad (2.5.6)$$

becomes the driven SHO equation

$$\ddot{\tilde{\varphi}}(t, \vec{k}) + E_{\vec{k}}^2 \tilde{\varphi}(t, \vec{k}) = \tilde{J}(t, \vec{k}), \quad (2.5.7)$$

where the SHO frequency is the energy

$$E_{\vec{k}} \equiv \sqrt{\vec{k}^2 + m^2}. \quad (2.5.8)$$

This insight is particularly important when we later quantize φ , because we may then view each Fourier mode of the scalar field as a quantum SHO. But let's first turn to two key physical features of our scalar field. First, for the linear wave equation sourced by J , let us witness that the presence or absence of the mass m determines whether φ itself is short- or long-ranged.

Classical Linear Solutions Remember that the retarded Green's function of the SHO with frequency Ω is

$$G_{\text{SHO}}(T) = \Theta(T) \frac{\sin(\Omega T)}{\Omega}, \quad (2.5.9)$$

where

$$\left(\frac{d^2}{dT^2} + \Omega^2 \right) G_{\text{SHO}}(T) = \delta(T). \quad (2.5.10)$$

The classical causality-obeying solution to eq. (2.5.7) is thus

$$\tilde{\varphi}(t, \vec{k}) = \int_{-\infty}^t \frac{\sin(E_{\vec{k}}(t-t'))}{E_{\vec{k}}} \cdot \tilde{J}(t', \vec{k}) dt', \quad (2.5.11)$$

$$\varphi(x) = \int_{\mathbb{R}^D} d^D \vec{x}' \int_{\mathbb{R}} dt' \left(\Theta(t-t') \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \frac{\sin(E_{\vec{k}}(t-t'))}{E_{\vec{k}}} \right) J(t', \vec{x}') \quad (2.5.12)$$

$$\equiv \int_{\mathbb{R}^D} d^D \vec{x}' \int_{\mathbb{R}} dt' G_{\text{ret}}(x-x') J(x'). \quad (2.5.13)$$

Problem 2.7. Long or Short Range? Massive versus Massless Consider a static point mass resting at $\vec{x} = 0$ in the $\{x^\mu\}$ inertial frame, namely

$$J(t, \vec{x}) = J_0 \delta^{(3)}(\vec{x}), \quad J_0 \text{ constant}. \quad (2.5.14)$$

Solve eq. (2.5.6). Hint: You may assume the time derivatives in eq. (2.0.9) can be neglected. Then go to Fourier \vec{k} -space. You should find

$$\tilde{\varphi}(\vec{k}) = \frac{J_0}{\vec{k}^2 + m^2}. \quad (2.5.15)$$

You should find

$$\varphi(t, r \equiv |\vec{x}|, \theta, \phi) = \frac{\exp(-mr)}{4\pi r}. \quad (2.5.16)$$

That is, φ describes a short-range force (with range $1/m$) that, when $m \rightarrow 0$, recovers the long-range Coulomb/Newtonian $1/r$ potential.

Next, consider an inertial frame $\{x'^\mu\}$ that is moving relative to the $\{x^\mu\}$ frame at velocity v along the positive x^3 axis. What is $\varphi(x')$ in the new frame? \square

Problem 2.8. Nonlinearities as Self-Coupled SHOs By going to Fourier space, consider a potential that is a polynomial of degree n in the field φ , with minimum at $\varphi = 0$,

$$V(\varphi) = \sum_{\ell=2}^n \frac{p_\ell}{\ell} \varphi^\ell. \quad (2.5.17)$$

Show that the Fourier space version of eq. (2.0.9) is:

$$\begin{aligned} \ddot{\tilde{\varphi}}(t, \vec{k}) + (k^2 + p_2)\tilde{\varphi}(t, \vec{k}) &= \tilde{J}(t, \vec{k}) \\ &- \sum_{\ell=2}^{n-1} p_{\ell+1} \prod_{s=1}^{\ell} \left(\int \frac{d^3 \vec{k}_s}{(2\pi)^3} \tilde{\varphi}(t, \vec{k}_s) \right) (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}_1 - \dots - \vec{k}_\ell). \end{aligned} \quad (2.5.18)$$

Observe, for a given \vec{k} , the non-linearities of the potential $V(\phi)$ give rise to a driving force – the second line on the right hand side – due to the field itself but from superposing over a range of Fourier modes.

Hint: Let's work out the $\ell = 2$ contribution as an example. This is comes from the cubic p_3 term in the potential:

$$V'(\varphi) = p_3 \varphi(t, \vec{x})^2 + \dots \quad (2.5.19)$$

The Fourier decomposition of φ^2 is

$$\widetilde{\varphi^2}(t, \vec{k}) = \int d^3 \vec{x} \varphi(x)^2 e^{-i\vec{k} \cdot \vec{x}} \quad (2.5.20)$$

$$= \int d^3 \vec{x} \int \frac{d^3 \vec{k}_1}{(2\pi)^3} \tilde{\varphi}(t, \vec{k}_1) e^{i\vec{k}_1 \cdot \vec{x}} \int \frac{d^3 \vec{k}_2}{(2\pi)^3} \tilde{\varphi}(t, \vec{k}_2) e^{i\vec{k}_2 \cdot \vec{x}} e^{-i\vec{k} \cdot \vec{x}} \quad (2.5.21)$$

$$= \int \frac{d^3 \vec{k}_1}{(2\pi)^3} \tilde{\varphi}(t, \vec{k}_1) \int \frac{d^3 \vec{k}_2}{(2\pi)^3} \tilde{\varphi}(t, \vec{k}_2) \int d^3 \vec{x} e^{i(\vec{k}_1 + \vec{k}_2 - \vec{k}) \cdot \vec{x}} \quad (2.5.22)$$

$$= \int \frac{d^3 \vec{k}_1}{(2\pi)^3} \tilde{\varphi}(t, \vec{k}_1) \int \frac{d^3 \vec{k}_2}{(2\pi)^3} \tilde{\varphi}(t, \vec{k}_2) (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}_1 - \vec{k}_2). \quad (2.5.23)$$

\square

Problem 2.9. Dispersion relations Consider the *massive* Klein-Gordon equation in Minkowski spacetime:

$$(\partial^2 + m^2) \varphi(t, \vec{x}) = 0, \quad (2.5.24)$$

where φ is a real scalar field. Find the general solution for φ in terms of plane waves $\exp(-ik \cdot x)$ and obtain the dispersion relation:

$$k^2 = m^2 \quad \Leftrightarrow \quad E^2 = \vec{p}^2 + m^2, \quad (2.5.25)$$

$$E \equiv k^0, \quad \vec{p} \equiv \vec{k}. \quad (2.5.26)$$

If each plane wave is associated with a particle of d -momentum k_μ , this states that it has mass m . The photon, which obeys $k^2 = 0$, has zero mass. \square

3 QFT of Massive Scalar Field

We will quantize the scalar field in this chapter. As with all quantum systems – quantum mechanics or quantum field theories – we will continue to assume that Schrodinger’s equation holds:

$$i\partial_t |\psi\rangle = H |\psi\rangle. \quad (3.0.1)$$

The key is to understand what the Hamiltonian of a quantum field theory is. We shall first do so for the non-interacting case.

3.1 Non-Interacting Case

In d dimensional Minkowski spacetime, the theory of a free massive real/Hermitian scalar field is defined by the Lagrangian density

$$\mathcal{L} = \frac{1}{2} (\partial\varphi)^2 - \frac{m^2}{2} \varphi^2 \quad (3.1.1)$$

$$= \frac{1}{2} \dot{\varphi}^2 - \frac{1}{2} \vec{\nabla}\varphi \cdot \vec{\nabla}\varphi - \frac{1}{2} m^2 \varphi^2. \quad (3.1.2)$$

The momentum conjugate to φ is defined as

$$\Pi \equiv \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \dot{\varphi}. \quad (3.1.3)$$

Its Hamiltonian density is

$$\mathcal{H} \equiv \Pi \cdot \dot{\varphi} - \mathcal{L} = \Pi^2 - \left(\frac{1}{2} \Pi^2 - \frac{1}{2} \vec{\nabla}\varphi \cdot \vec{\nabla}\varphi - \frac{1}{2} m^2 \varphi^2 \right) \quad (3.1.4)$$

$$= \frac{1}{2} \Pi^2 + \frac{1}{2} \vec{\nabla}\varphi \cdot \vec{\nabla}\varphi + \frac{1}{2} m^2 \varphi^2. \quad (3.1.5)$$

Quantization in “real” spacetime In position spacetime, and on a constant t surface, we shall identify the field at each spatial point \vec{x} as an independent quantum mechanical “system”. As such, in analogy with the quantum mechanical $[X, P] = i$ and $[X, X] = 0 = [P, P]$, we shall impose the equal-time commutation relations

$$[\varphi(t, \vec{x}), \Pi(t, \vec{x}')] = i\delta^{(d-1)}(\vec{x} - \vec{x}'), \quad (3.1.6)$$

$$[\varphi(t, \vec{x}), \varphi(t, \vec{x}')] = 0 = [\Pi(t, \vec{x}), \Pi(t, \vec{x}')] . \quad (3.1.7)$$

We reiterate, this holds when both fields are evaluated at the same time t ; we will see below that $[\varphi(x), \varphi(y)] \neq 0$ in general. Moreover, note that the second line implies

$$[\partial_{x^i}\varphi(t, \vec{x}), \varphi(t, \vec{x}')] = 0. \quad (3.1.8)$$

As we shall witness, in Fourier space, this quantization condition is intimately related to the simple harmonic oscillator (SHO) algebra, which in turn leads us to the particle interpretation.

Problem 3.1. If we decompose the operators in Fourier space,

$$\tilde{\varphi}(t, \vec{x}) = \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} \tilde{\varphi}(t, \vec{k}) e^{i\vec{k} \cdot \vec{x}}, \quad (3.1.9)$$

$$\tilde{\Pi}(t, \vec{x}) = \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} \tilde{\Pi}(t, \vec{k}) e^{i\vec{k} \cdot \vec{x}}; \quad (3.1.10)$$

show that the commutation relations in eq. (3.1.6) and (3.1.8) translates to

$$\left[\tilde{\varphi}(t, \vec{k}), \tilde{\Pi}(t, \vec{k}') \right] = (2\pi)^D i \delta^{(D)}(\vec{k} + \vec{k}'), \quad (3.1.11)$$

$$\left[\tilde{\varphi}(t, \vec{k}), \tilde{\varphi}(t, \vec{k}') \right] = 0 = \left[\tilde{\Pi}(t, \vec{k}), \tilde{\Pi}(t, \vec{k}') \right]. \quad (3.1.12)$$

□

Since the scalar field depends on spacetime, it is necessarily time dependent. We shall postulate $\varphi(x)$ is a Heisenberg picture operator. It must therefore obey the equations of motion (EoM)

$$\dot{\varphi}(t, \vec{x}) = i [H_{\text{H}}(t), \varphi(t, \vec{x})] \quad (3.1.13)$$

$$= i \int d^{d-1} \vec{y} \left[\frac{1}{2} \Pi^2(t, \vec{y}) + \frac{1}{2} \vec{\nabla} \varphi(t, \vec{y}) \cdot \vec{\nabla} \varphi(t, \vec{y}) + \frac{1}{2} m^2 \varphi^2(t, \vec{y}), \varphi(x) \right] \quad (3.1.14)$$

$$= i \int d^{d-1} \vec{y} \left[\frac{1}{2} \Pi^2(t, \vec{y}), \varphi(x) \right] \quad (3.1.15)$$

$$= \frac{i}{2} \int d^{d-1} \vec{y} (\Pi(t, \vec{y}) [\Pi(t, \vec{y}), \varphi(x)] + [\Pi(t, \vec{y}), \varphi(x)] \Pi(t, \vec{y})) \quad (3.1.16)$$

$$= \frac{i}{2} \int d^{d-1} \vec{y} (\Pi(t, \vec{y}) [\Pi(t, \vec{y}), \varphi(t, \vec{x})] + [\Pi(t, \vec{y}), \varphi(t, \vec{x})] \Pi(t, \vec{y})) \quad (3.1.17)$$

$$= i(-i) \int d^{d-1} \vec{y} \Pi(t, \vec{y}) \delta^{(d-1)}(\vec{y} - \vec{x}) = \Pi(t, \vec{x}). \quad (3.1.18)$$

Similarly,

$$\dot{\Pi}(t, \vec{x}) = i [H_{\text{H}}(t), \Pi(t, \vec{x})] \quad (3.1.19)$$

$$= i \int d^{d-1} \vec{y} \left[\frac{1}{2} \Pi^2(t, \vec{y}) + \frac{1}{2} \vec{\nabla} \varphi(t, \vec{y}) \cdot \vec{\nabla} \varphi(t, \vec{y}) + \frac{1}{2} m^2 \varphi^2(t, \vec{y}), \Pi(t, \vec{x}) \right] \quad (3.1.20)$$

$$= \frac{i}{2} \int d^{d-1} \vec{y} \left[\vec{\nabla} \varphi(t, \vec{y}) \cdot \vec{\nabla} \varphi(t, \vec{y}) + m^2 \varphi^2(t, \vec{y}), \Pi(t, \vec{x}) \right] \quad (3.1.21)$$

$$= - \int d^{d-1} \vec{y} \left(\vec{\nabla} \varphi(t, \vec{y}) \cdot \vec{\nabla}_{\vec{y}} \delta^{(d-1)}(\vec{y} - \vec{x}) + m^2 \varphi(t, \vec{y}) \delta^{(d-1)}(\vec{y} - \vec{x}) \right) \quad (3.1.22)$$

$$= \vec{\nabla}^2 \varphi(t, \vec{x}) - m^2 \varphi(t, \vec{x}). \quad (3.1.23)$$

Altogether, we have $\ddot{\varphi} = \dot{\Pi} = \vec{\nabla}^2 \varphi - m^2 \varphi$. Just like the quantum mechanical 1D SHO, the Heisenberg EoM for the φ is precisely its classical counterpart:

$$(\partial^2 + m^2) \varphi(x) = 0. \quad (3.1.24)$$

Problem 3.2. Covariant Alternative to Equal Time Commutation Relations An alternate means to canonical quantization that emphasizes the Lorentz covariance nature of quantum fields is to demand that their (anti-)commutators yield the difference between the classical retarded and advanced Green's functions. For a massless scalar field with $\mathcal{L} = (1/2)(\partial\varphi)^2$, for instance, we have

$$i[\varphi(x), \varphi(x')] = G^+(x - x') - G^-(x - x'), \quad (3.1.25)$$

$$G^\pm(z) = - \int_{-\infty \pm i0^+}^{+\infty \pm i0^+} \frac{d\omega}{2\pi} \int_{\mathbb{R}^{d-1}} \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{e^{-i\omega z^0} e^{i\vec{k}\cdot\vec{z}}}{\omega^2 - \vec{k}^2}. \quad (3.1.26)$$

Note that:

$$\partial^2 G^\pm(x) = \delta^{(4)}(x). \quad (3.1.27)$$

Show that this leads to the canonical equal time commutation relations

$$[\varphi(t, \vec{x}), \Pi(t, \vec{x}')] = [\varphi(t, \vec{x}), \dot{\varphi}(t, \vec{x}')] = i\delta^{(d-1)}(\vec{x} - \vec{x}'). \quad (3.1.28)$$

Hint: You do not need to evaluate the entire Fourier integral, but you need to apply the residue theorem to the ω integral before computing the equal time commutator. \square

Quantization in Fourier Space Let us now write the Heisenberg scalar field operator as a sum over its Fourier modes:

$$\varphi(t, \vec{x}) = \int_{\mathbb{R}^{d-1}} \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \tilde{\varphi}(t, \vec{k}) e^{i\vec{k}\cdot\vec{x}}. \quad (3.1.29)$$

The scalar EoM of eq. (3.1.24) now becomes

$$\int_{\mathbb{R}^{d-1}} \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \left(\ddot{\tilde{\varphi}} + E_{\vec{k}}^2 \tilde{\varphi} \right) e^{i\vec{k}\cdot\vec{x}} = 0, \quad E_{\vec{k}} \equiv \sqrt{\vec{k}^2 + m^2}. \quad (3.1.30)$$

This in turn means each Fourier mode of the scalar operator is a SHO, with frequency given by the energy $E_{\vec{k}}$:

$$\ddot{\tilde{\varphi}}(t, \vec{k}) + E_{\vec{k}}^2 \tilde{\varphi}(t, \vec{k}) = 0. \quad (3.1.31)$$

The solution is therefore a superposition of positive $e^{-iE_{\vec{k}}t}$ and negative $e^{+iE_{\vec{k}}t}$ modes, with the following manifestly Hermitian form:

$$\varphi(t, \vec{x}) = \int_{\mathbb{R}^D} \frac{d^D\vec{k}}{(2\pi)^D \sqrt{2E_{\vec{k}}}} \left(a_{\vec{k}} e^{-iE_{\vec{k}}t} e^{i\vec{k}\cdot\vec{x}} + a_{\vec{k}}^\dagger e^{+iE_{\vec{k}}t} e^{-i\vec{k}\cdot\vec{x}} \right). \quad (3.1.32)$$

(The $\sqrt{2E_{\vec{k}}}$ in the denominator is for convenience.) Let us now turn to work out the commutation relations of these $a_{\vec{k}}$ and $a_{\vec{k}}^\dagger$. For the canonical scalar, we have in fact already imposed their commutation relations in equations (3.1.6) and (3.1.7). Since $\Pi = \partial_t \varphi$,

$$i\delta^{(D)}(\vec{x} - \vec{x}') = i \int_{\mathbb{R}^D} \frac{d^D\vec{k}}{(2\pi)^D} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} = [\varphi(t, \vec{x}), \Pi(t, \vec{x}')]$$

$$= \int_{\vec{k}, \vec{k}'} \left[a_{\vec{k}} e^{-iE_{\vec{k}}t} e^{i\vec{k}\cdot\vec{x}} + a_{\vec{k}}^\dagger e^{+iE_{\vec{k}}t} e^{-i\vec{k}\cdot\vec{x}}, -iE_{\vec{k}} \cdot a_{\vec{k}'} e^{-iE_{\vec{k}'}t} e^{i\vec{k}'\cdot\vec{x}'} + iE_{\vec{k}} \cdot a_{\vec{k}'}^\dagger e^{+iE_{\vec{k}'}t} e^{-i\vec{k}'\cdot\vec{x}'} \right] \quad (3.1.33)$$

$$= i \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} \int_{\mathbb{R}^D} \frac{d^D \vec{k}'}{(2\pi)^D} \frac{E_{\vec{k}}}{2\sqrt{E_{\vec{k}}E_{\vec{k}'}}} \times \left\{ - [a_{\vec{k}}, a_{\vec{k}'}] e^{-iE_{\vec{k}}t} e^{i\vec{k}\cdot\vec{x}} e^{-iE_{\vec{k}'}t} e^{i\vec{k}'\cdot\vec{x}'} + [a_{\vec{k}}^\dagger, a_{\vec{k}'}^\dagger] e^{+iE_{\vec{k}}t} e^{-i\vec{k}\cdot\vec{x}} e^{+iE_{\vec{k}'}t} e^{-i\vec{k}'\cdot\vec{x}'} \right. \\ \left. + [a_{\vec{k}}, a_{\vec{k}'}^\dagger] e^{-iE_{\vec{k}}t} e^{i\vec{k}\cdot\vec{x}} e^{+iE_{\vec{k}'}t} e^{-i\vec{k}'\cdot\vec{x}'} - [a_{\vec{k}}^\dagger, a_{\vec{k}'}] e^{+iE_{\vec{k}}t} e^{-i\vec{k}\cdot\vec{x}} e^{-iE_{\vec{k}'}t} e^{i\vec{k}'\cdot\vec{x}'} \right\}; \quad (3.1.34)$$

$$0 = [\varphi(t, \vec{x}), \varphi(t, \vec{x}')] \\ = \int_{\vec{k}, \vec{k}'} \left[a_{\vec{k}} e^{-iE_{\vec{k}}t} e^{i\vec{k}\cdot\vec{x}} + a_{\vec{k}}^\dagger e^{+iE_{\vec{k}}t} e^{-i\vec{k}\cdot\vec{x}}, a_{\vec{k}'} e^{-iE_{\vec{k}'}t} e^{i\vec{k}'\cdot\vec{x}'} + a_{\vec{k}'}^\dagger e^{+iE_{\vec{k}'}t} e^{-i\vec{k}'\cdot\vec{x}'} \right] \quad (3.1.35)$$

$$= \int_{\vec{k}, \vec{k}'} \left\{ [a_{\vec{k}}, a_{\vec{k}'}] e^{-iE_{\vec{k}}t} e^{i\vec{k}\cdot\vec{x}} e^{-iE_{\vec{k}'}t} e^{i\vec{k}'\cdot\vec{x}'} + [a_{\vec{k}}^\dagger, a_{\vec{k}'}^\dagger] e^{+iE_{\vec{k}}t} e^{-i\vec{k}\cdot\vec{x}} e^{+iE_{\vec{k}'}t} e^{-i\vec{k}'\cdot\vec{x}'} \right. \\ \left. + [a_{\vec{k}}, a_{\vec{k}'}^\dagger] e^{-iE_{\vec{k}}t} e^{i\vec{k}\cdot\vec{x}} e^{+iE_{\vec{k}'}t} e^{-i\vec{k}'\cdot\vec{x}'} + [a_{\vec{k}}^\dagger, a_{\vec{k}'}] e^{+iE_{\vec{k}}t} e^{-i\vec{k}\cdot\vec{x}} e^{-iE_{\vec{k}'}t} e^{i\vec{k}'\cdot\vec{x}'} \right\}; \quad (3.1.36)$$

and

$$0 = [\Pi(t, \vec{x}), \Pi(t, \vec{x}')] \\ = \int_{\vec{k}, \vec{k}'} i^2 E_{\vec{k}} E_{\vec{k}'} \left[-a_{\vec{k}} e^{-iE_{\vec{k}}t} e^{i\vec{k}\cdot\vec{x}} + a_{\vec{k}}^\dagger e^{+iE_{\vec{k}}t} e^{-i\vec{k}\cdot\vec{x}}, -a_{\vec{k}'} e^{-iE_{\vec{k}'}t} e^{i\vec{k}'\cdot\vec{x}'} + a_{\vec{k}'}^\dagger e^{+iE_{\vec{k}'}t} e^{-i\vec{k}'\cdot\vec{x}'} \right] \quad (3.1.37)$$

$$= - \int_{\vec{k}, \vec{k}'} E_{\vec{k}} E_{\vec{k}'} \left\{ [a_{\vec{k}}, a_{\vec{k}'}] e^{-iE_{\vec{k}}t} e^{i\vec{k}\cdot\vec{x}} e^{-iE_{\vec{k}'}t} e^{i\vec{k}'\cdot\vec{x}'} + [a_{\vec{k}}^\dagger, a_{\vec{k}'}^\dagger] e^{+iE_{\vec{k}}t} e^{-i\vec{k}\cdot\vec{x}} e^{+iE_{\vec{k}'}t} e^{-i\vec{k}'\cdot\vec{x}'} \right. \\ \left. - [a_{\vec{k}}, a_{\vec{k}'}^\dagger] e^{-iE_{\vec{k}}t} e^{i\vec{k}\cdot\vec{x}} e^{+iE_{\vec{k}'}t} e^{-i\vec{k}'\cdot\vec{x}'} - [a_{\vec{k}}^\dagger, a_{\vec{k}'}] e^{+iE_{\vec{k}}t} e^{-i\vec{k}\cdot\vec{x}} e^{-iE_{\vec{k}'}t} e^{i\vec{k}'\cdot\vec{x}'} \right\}. \quad (3.1.38)$$

For the $[\varphi, \Pi]$ commutation relation, by comparing with the Fourier integral representation of the $\delta^{(D)}(\vec{x} - \vec{x}')$, the integral over \vec{k}' must yield $\delta^{(D)}(\vec{k} - \vec{k}')$ and/or $\delta^{(D)}(\vec{k} + \vec{k}')$. For instance, the $[a_{\vec{k}}, a_{\vec{k}'}] e^{i\vec{k}\cdot\vec{x} + i\vec{k}'\cdot\vec{x}'}$ must upon \vec{k}' -integration become $\delta^{(D)}(\vec{k} - \vec{k}') e^{i\vec{k}(\vec{x} - \vec{x}')$. The $[a_{\vec{k}}, a_{\vec{k}'}^\dagger] e^{i\vec{k}\cdot\vec{x} - i\vec{k}'\cdot\vec{x}'}$ term must become $\delta^{(D)}(\vec{k} - \vec{k}') e^{i\vec{k}\cdot(\vec{x} - \vec{x}')$. The other two terms are related by hermitian conjugate. In detail,

$$[a_{\vec{k}}, a_{\vec{k}'}] = \chi_{\vec{k}} \cdot \delta^{(D)}(\vec{k} + \vec{k}'), \quad (3.1.39)$$

$$[a_{\vec{k}}^\dagger, a_{\vec{k}'}^\dagger] = -\chi_{\vec{k}}^* \cdot \delta^{(D)}(\vec{k} + \vec{k}'); \quad (3.1.40)$$

and

$$[a_{\vec{k}}, a_{\vec{k}'}^\dagger] = \gamma_{\vec{k}} \cdot \delta^{(D)}(\vec{k} - \vec{k}'), \quad (3.1.41)$$

$$[a_{\vec{k}}^\dagger, a_{\vec{k}'}] = -\gamma_{\vec{k}}^* \cdot \delta^{(D)}(\vec{k} - \vec{k}'). \quad (3.1.42)$$

At this point

$$\begin{aligned}
i\delta^{(D)}(\vec{x} - \vec{x}') &= i \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} = [\varphi(t, \vec{x}), \Pi(t, \vec{x}')] \\
&= i \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} \int_{\mathbb{R}^D} \frac{d^D \vec{k}'}{(2\pi)^D} \frac{E_{\vec{k}}}{2\sqrt{E_{\vec{k}} E_{\vec{k}'}}} \\
&\quad \times \left\{ -\chi_{\vec{k}} \cdot \delta^{(D)}(\vec{k} + \vec{k}') \cdot e^{-iE_{\vec{k}} t} e^{-iE_{\vec{k}'} t} - \chi_{\vec{k}}^* \cdot \delta^{(D)}(\vec{k} + \vec{k}') \cdot e^{+iE_{\vec{k}} t} e^{+iE_{\vec{k}'} t} \right. \\
&\quad \left. + \gamma_{\vec{k}} \cdot \delta^{(D)}(\vec{k} - \vec{k}') \cdot e^{-iE_{\vec{k}} t} e^{+iE_{\vec{k}'} t} + \gamma_{\vec{k}}^* \cdot \delta^{(D)}(\vec{k} - \vec{k}') \cdot e^{+iE_{\vec{k}} t} e^{-iE_{\vec{k}'} t} \right\} \quad (3.1.43)
\end{aligned}$$

$$\begin{aligned}
&= i \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} \frac{e^{i\vec{k} \cdot (\vec{x} - \vec{x}')}}{2} \\
&\quad \times \int_{\mathbb{R}^D} \frac{d^D \vec{k}'}{(2\pi)^D} \left\{ -2\text{Re}(\chi_{\vec{k}} \cdot e^{-2iE_{\vec{k}} t}) \delta^{(D)}(\vec{k} + \vec{k}') + 2\text{Re}(\gamma_{\vec{k}}) \delta^{(D)}(\vec{k} - \vec{k}') \right\}; \quad (3.1.44)
\end{aligned}$$

$$\begin{aligned}
0 &= [\varphi(t, \vec{x}), \varphi(t, \vec{x}')] \\
&= \int_{\vec{k}, \vec{k}'} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} \left\{ \chi_{\vec{k}} \delta^{(D)}(\vec{k} + \vec{k}') \cdot e^{-iE_{\vec{k}} t} e^{-iE_{\vec{k}'} t} - \chi_{\vec{k}}^* \delta^{(D)}(\vec{k} + \vec{k}') \cdot e^{+iE_{\vec{k}} t} e^{+iE_{\vec{k}'} t} \right. \\
&\quad \left. + \gamma_{\vec{k}} \delta^{(D)}(\vec{k} - \vec{k}') \cdot e^{-iE_{\vec{k}} t} e^{+iE_{\vec{k}'} t} - \gamma_{\vec{k}}^* \delta^{(D)}(\vec{k} - \vec{k}') \cdot e^{+iE_{\vec{k}} t} e^{-iE_{\vec{k}'} t} \right\} \quad (3.1.45)
\end{aligned}$$

$$= \int_{\vec{k}, \vec{k}'} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} \left\{ 2\text{Im}(\chi_{\vec{k}} \cdot e^{-2iE_{\vec{k}} t}) \delta^{(D)}(\vec{k} + \vec{k}') + 2\text{Im}(\gamma_{\vec{k}}) \delta^{(D)}(\vec{k} - \vec{k}') \right\} \quad (3.1.46)$$

and

$$\begin{aligned}
0 &= [\Pi(t, \vec{x}), \Pi(t, \vec{x}')] \\
&= - \int_{\vec{k}, \vec{k}'} E_{\vec{k}} E_{\vec{k}'} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} \left\{ \chi_{\vec{k}} \delta^{(D)}(\vec{k} + \vec{k}') \cdot e^{-iE_{\vec{k}} t} e^{-iE_{\vec{k}'} t} - \chi_{\vec{k}}^* \delta^{(D)}(\vec{k} + \vec{k}') \cdot e^{+iE_{\vec{k}} t} e^{+iE_{\vec{k}'} t} \right. \\
&\quad \left. - \gamma_{\vec{k}} \delta^{(D)}(\vec{k} - \vec{k}') \cdot e^{-iE_{\vec{k}} t} e^{+iE_{\vec{k}'} t} + \gamma_{\vec{k}}^* \delta^{(D)}(\vec{k} - \vec{k}') \cdot e^{+iE_{\vec{k}} t} e^{-iE_{\vec{k}'} t} \right\} \quad (3.1.47)
\end{aligned}$$

$$= - \int_{\vec{k}, \vec{k}'} E_{\vec{k}} E_{\vec{k}'} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} \left\{ 2\text{Im}(\chi_{\vec{k}} \cdot e^{-2iE_{\vec{k}} t}) \cdot \delta^{(D)}(\vec{k} + \vec{k}') - 2\text{Im}(\gamma_{\vec{k}}) \delta^{(D)}(\vec{k} - \vec{k}') \right\}. \quad (3.1.48)$$

To ensure time-independence of the commutation relations, we see that $\chi_{\vec{k}} = 0$. This, in turn, implies $\gamma_{\vec{k}}$ must be real in order for the equal time $[\varphi, \varphi]$ zero. That guarantees the equal-time $[\Pi, \Pi]$ is zero as well. In fact, $\gamma_{\vec{k}} = (2\pi)^D$ in order for the \vec{k}' -integration in the equal-time $[\varphi, \Pi]$ integral to yield 1. Altogether:

$$[a_{\vec{k}}, a_{\vec{k}'}] = 0 = [a_{\vec{k}}^\dagger, a_{\vec{k}'}^\dagger]; \quad (3.1.49)$$

and

$$\left[a_{\vec{k}}, a_{\vec{k}'}^\dagger \right] = (2\pi)^D \cdot \delta^{(D)}(\vec{k} - \vec{k}'). \quad (3.1.50)$$

These relations are the generalization of the SHO's ladder operators – i.e., every single \vec{k} -mode of the massive linear scalar field *is* a quantum SHO.

Problem 3.3. Show that

$$[\varphi(t, \vec{x}), a_{\vec{q}}^\dagger] = \frac{1}{\sqrt{2E_{\vec{q}}}} e^{-iE_{\vec{q}}t} e^{i\vec{q}\cdot\vec{x}}, \quad (3.1.51)$$

$$[\varphi(t, \vec{x}), a_{\vec{q}}] = -\frac{1}{\sqrt{2E_{\vec{q}}}} e^{+iE_{\vec{q}}t} e^{-i\vec{q}\cdot\vec{x}}. \quad (3.1.52)$$

□

Hamiltonian: Fourier Decomposition We now compute the total Hamiltonian of the massive linear scalar field:

$$H(t) \equiv \int_{\mathbb{R}^D} d^D \vec{x}' \left(\frac{1}{2} \Pi^2 + \frac{1}{2} (\vec{\nabla} \varphi)^2 + \frac{m^2}{2} \varphi^2 \right) \quad (3.1.53)$$

$$= \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} \frac{1}{2} \left(|\tilde{\Pi}(t, \vec{k})|^2 + E_{\vec{k}}^2 |\tilde{\varphi}(t, \vec{k})|^2 \right). \quad (3.1.54)$$

We have

$$\varphi(t, \vec{x}) = \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} \frac{1}{\sqrt{2E_{\vec{k}}}} \left(a_{\vec{k}} e^{-iE_{\vec{k}}t} + a_{-\vec{k}}^\dagger e^{iE_{\vec{k}}t} \right) e^{i\vec{k}\cdot\vec{x}}, \quad (3.1.55)$$

$$\Pi(t, \vec{x}) = -i \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} \sqrt{\frac{E_{\vec{k}}}{2}} \left(a_{\vec{k}} e^{-iE_{\vec{k}}t} - a_{-\vec{k}}^\dagger e^{iE_{\vec{k}}t} \right) e^{i\vec{k}\cdot\vec{x}}. \quad (3.1.56)$$

Therefore,

$$|\tilde{\varphi}(t, \vec{k})|^2 = \tilde{\varphi}(t, \vec{k}) \tilde{\varphi}(t, -\vec{k}) \quad (3.1.57)$$

$$= \frac{1}{2E_{\vec{k}}} \left(a_{\vec{k}} e^{-iE_{\vec{k}}t} + a_{-\vec{k}}^\dagger e^{iE_{\vec{k}}t} \right) \left(a_{-\vec{k}} e^{-iE_{\vec{k}}t} + a_{\vec{k}}^\dagger e^{iE_{\vec{k}}t} \right) \quad (3.1.58)$$

$$= \frac{1}{2E_{\vec{k}}} \left(a_{\vec{k}} a_{-\vec{k}} e^{-2iE_{\vec{k}}t} + a_{-\vec{k}}^\dagger a_{-\vec{k}} + a_{\vec{k}} a_{\vec{k}}^\dagger + a_{-\vec{k}}^\dagger a_{\vec{k}}^\dagger e^{2iE_{\vec{k}}t} \right) \quad (3.1.59)$$

$$E_{\vec{k}}^2 |\tilde{\varphi}(t, \vec{k})|^2 = \frac{E_{\vec{k}}}{2} \left(a_{\vec{k}} a_{-\vec{k}} e^{-2iE_{\vec{k}}t} + a_{-\vec{k}}^\dagger a_{\vec{k}}^\dagger e^{2iE_{\vec{k}}t} + a_{-\vec{k}}^\dagger a_{-\vec{k}} + a_{\vec{k}}^\dagger a_{\vec{k}} + (2\pi)^D \delta^{(D)}(\vec{0}) \right), \quad (3.1.60)$$

where we have massaged $a_{\vec{k}} a_{\vec{k}}^\dagger = [a_{\vec{k}}, a_{\vec{k}}^\dagger] + a_{\vec{k}}^\dagger a_{\vec{k}}$. Similarly,

$$|\tilde{\Pi}(t, \vec{k})|^2 = \tilde{\Pi}(t, \vec{k}) \tilde{\Pi}(t, -\vec{k}) \quad (3.1.61)$$

$$= -\frac{E_{\vec{k}}}{2} \left(a_{\vec{k}} e^{-iE_{\vec{k}}t} - a_{-\vec{k}}^\dagger e^{iE_{\vec{k}}t} \right) \left(a_{-\vec{k}} e^{-iE_{\vec{k}}t} - a_{\vec{k}}^\dagger e^{iE_{\vec{k}}t} \right) \quad (3.1.62)$$

$$= -\frac{E_{\vec{k}}}{2} \left(a_{\vec{k}} a_{-\vec{k}} e^{-2iE_{\vec{k}}t} - a_{-\vec{k}}^\dagger a_{-\vec{k}} - a_{\vec{k}} a_{\vec{k}}^\dagger + a_{-\vec{k}}^\dagger a_{\vec{k}}^\dagger e^{2iE_{\vec{k}}t} \right) \quad (3.1.63)$$

$$= \frac{E_{\vec{k}}}{2} \left(-a_{\vec{k}} a_{-\vec{k}} e^{-2iE_{\vec{k}}t} - a_{-\vec{k}}^\dagger a_{\vec{k}}^\dagger e^{2iE_{\vec{k}}t} + a_{-\vec{k}}^\dagger a_{-\vec{k}} + a_{\vec{k}}^\dagger a_{\vec{k}} + (2\pi)^D \delta^{(D)}(\vec{0}) \right), \quad (3.1.64)$$

We collect

$$H = \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} \frac{E_{\vec{k}}}{2} \left(a_{-\vec{k}}^\dagger a_{-\vec{k}} + a_{\vec{k}}^\dagger a_{\vec{k}} + (2\pi)^D \delta^{(D)}(\vec{0}) \right) \quad (3.1.65)$$

$$= \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} E_{\vec{k}} \left(a_{\vec{k}}^\dagger a_{\vec{k}} + \frac{1}{2} \cdot (2\pi)^D \delta^{(D)}(\vec{0}) \right). \quad (3.1.66)$$

The momentum δ -function may be interpreted as the total volume of space because

$$(2\pi)^D \delta^{(D)}(\vec{0}) = \lim_{\vec{k} \rightarrow \vec{0}} \int_{\mathbb{R}^D} d^D \vec{x} e^{i\vec{k} \cdot \vec{x}} = \int_{\mathbb{R}^D} d^D \vec{x}. \quad (3.1.67)$$

In the 1D QM SHO case, $H = \Omega(a^\dagger a + 1/2)$. We may therefore recognize the Hamiltonian for the massive linear scalar field to be simply that of the QM SHO, but summed over all \vec{k} -modes. On physical grounds, we expect that H may be appropriately regularized to become H' :

$$H' = \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} E_{\vec{k}} \left(a_{\vec{k}}^\dagger a_{\vec{k}} \right) + E_0, \quad (3.1.68)$$

for some well-defined ground state

Problem 3.4. Show that

$$[H, a_{\vec{q}}] = -E_{\vec{q}} a_{\vec{q}} \quad (3.1.69)$$

and

$$[H, a_{\vec{q}}^\dagger] = E_{\vec{q}} a_{\vec{q}}^\dagger. \quad (3.1.70)$$

These lead us to the particle interpretation. □

Particle Interpretation If $|\mathcal{E}\rangle$ is an eigenstate of H , consider

$$H(a_{\vec{q}}|\mathcal{E}\rangle) = ([H, a_{\vec{q}}] + a_{\vec{q}}H)|\mathcal{E}\rangle \quad (3.1.71)$$

$$= (\mathcal{E} - E_{\vec{q}})a_{\vec{q}}|\mathcal{E}\rangle. \quad (3.1.72)$$

Likewise, consider

$$H(a_{\vec{q}}^\dagger|\mathcal{E}\rangle) = ([H, a_{\vec{q}}^\dagger] + a_{\vec{q}}^\dagger H)|\mathcal{E}\rangle \quad (3.1.73)$$

$$= (\mathcal{E} + E_{\vec{q}})a_{\vec{q}}^\dagger|\mathcal{E}\rangle. \quad (3.1.74)$$

Therefore the $a_{\vec{q}}$ and $a_{\vec{q}}^\dagger$ are, respectively, the lowering and raising operators – i.e., they lower and raise energy by $E_{\vec{q}}$. Now, the $E_{\vec{k}} a_{\vec{k}}^\dagger a_{\vec{k}}$ portion of H yields a non-negative positive expectation value with respect to any state; hence, it appears reasonable to expect, there must be a minimum energy (aka ground) state with respect to an appropriately regularized H' , with a well defined value E_0 ; where $H'|0\rangle = E_0|0\rangle$. This in turn means $a_{\vec{q}}|0\rangle$ must be the zero vector, for otherwise there is now a non-trivial state with lower energy, violating the minimum energy assumption.

$$a_{\vec{q}}|0\rangle = |\text{zero}\rangle \quad (3.1.75)$$

Let us now use this ground state (aka vacuum state) to define the one particle state

$$|\vec{k}\rangle \equiv \sqrt{2E_{\vec{k}}} a_{\vec{k}}^\dagger |0\rangle. \quad (3.1.76)$$

The normalization is

$$\langle \vec{k} | \vec{k}' \rangle \equiv \sqrt{2E_{\vec{k}}} \sqrt{2E_{\vec{k}'}} \langle 0 | a_{\vec{k}} a_{\vec{k}'}^\dagger | 0 \rangle \quad (3.1.77)$$

$$= \sqrt{2E_{\vec{k}}} \sqrt{2E_{\vec{k}'}} \langle 0 | \left([a_{\vec{k}}, a_{\vec{k}'}^\dagger] + a_{\vec{k}'}^\dagger a_{\vec{k}} \right) | 0 \rangle \quad (3.1.78)$$

$$= (2E_{\vec{k}})(2\pi)^D \delta^{(D)}(\vec{k} - \vec{k}'). \quad (3.1.79)$$

Despite appearances, this inner product is, in fact invariant under Lorentz transformations. For, we may obtain the same result via the following manifestly Lorentz invariant positive energy integration measure:

$$\frac{d^d k}{(2\pi)^d} \Theta(k_0) (2\pi) \delta(k_\alpha k^\alpha - m^2) = \frac{dk_0 d^D \vec{k}}{(2\pi)^D} \Theta(k_0) \left(\frac{\delta(k_0 - E_{\vec{k}})}{2E_{\vec{k}}} + \frac{\delta(k_0 + E_{\vec{k}})}{2E_{\vec{k}}} \right) \quad (3.1.80)$$

$$= \frac{dk_0 d^D \vec{k}}{(2\pi)^D} \Theta(k_0) \frac{\delta(k_0 - E_{\vec{k}})}{2E_{\vec{k}}}. \quad (3.1.81)$$

Hence we obtain the relativistically invariant result

$$\int_{\mathbb{R}^{D+1}} \frac{d^d k}{(2\pi)^D} \Theta(k_0) \delta(k^2 - m^2) \langle \vec{k}' | \vec{k} \rangle = \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D (2E_{\vec{k}})} \langle \vec{k}' | \vec{k} \rangle = 1. \quad (3.1.82)$$

Next, we turn to defining the n -particle state

$$|\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n\rangle \equiv \frac{1}{\sqrt{n!}} \sqrt{2E_{\vec{k}_1}} a_{\vec{k}_1}^\dagger \sqrt{2E_{\vec{k}_2}} a_{\vec{k}_2}^\dagger \dots \sqrt{2E_{\vec{k}_n}} a_{\vec{k}_n}^\dagger |0\rangle. \quad (3.1.83)$$

Its normalization is

$$\begin{aligned} \langle \vec{k}_1, \vec{k}_2, \dots, \vec{k}_n | \vec{k}'_1, \vec{k}'_2, \dots, \vec{k}'_n \rangle &= \frac{1}{n!} (2E_{\vec{k}_1}) \dots (2E_{\vec{k}_n}) \quad (3.1.84) \\ &\times \sum_{(\pi(1), \dots, \pi(n)) \in \text{perm.}(1, 2, \dots, n)} (2\pi)^{\delta^{(D)}}(\vec{k}_1 - \vec{k}'_{\sigma(1)}) \dots (2\pi)^{\delta^{(D)}}(\vec{k}_n - \vec{k}'_{\sigma(n)}). \end{aligned}$$

This is relativistically invariant because

$$\prod_{J=1}^n \left(\int_{\mathbb{R}^D} \frac{d^D \vec{k}_J}{(2\pi)^D (2E_{\vec{k}_J})} \right) \langle \vec{k}_1, \vec{k}_2, \dots, \vec{k}_n | \vec{k}'_1, \vec{k}'_2, \dots, \vec{k}'_n \rangle = 1. \quad (3.1.85)$$

The identity operator acting on one-particle states is

$$\mathbb{I}_{1\text{-particle}} = \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D (2E_{\vec{k}})} |\vec{k}\rangle \langle \vec{k}|. \quad (3.1.86)$$

As you will directly verify below, the set of n -particle momentum states diagonalizes the Hamiltonian H' . Hence, the collection of $|0\rangle$, 1 -, 2 -, and all $(n \geq 1)$ -particle states span the Hilbert space upon each H' acts. This space is known as the *Fock space*.

Problem 3.5. Prove eq. (3.1.84). □

Problem 3.6. Ket-Bra Form of Ladder Operator Explain why, when acting on the subspace spanned by the 0- and 1-particle states, the ladder operators can be expressed as

$$a_{\vec{k}}^\dagger = \frac{|\vec{k}\rangle \langle 0|}{\sqrt{2E_{\vec{k}}}} \quad \text{and} \quad a_{\vec{k}} = \frac{|0\rangle \langle \vec{k}|}{\sqrt{2E_{\vec{k}}}}. \quad (3.1.87)$$

Below, we shall use this result to work out how they transform under Lorentz transformations. □

Problem 3.7. Show that, if H' is the some appropriately regularized H (see eq. (3.1.68)):

$$H' |\vec{k}_1, \dots, \vec{k}_n\rangle = (E_0 + E_{\vec{k}_1} + \dots + E_{\vec{k}_n}) |\vec{k}_1, \dots, \vec{k}_n\rangle, \quad (3.1.88)$$

where the n -particle state is defined in eq. (3.1.83). □

Just as the number operator for the SHO is $a^\dagger a$, the corresponding Hermitian number operator for the scalar field theory involves the sum over all momentum.

$$N \equiv \int \frac{d^{d-1}\vec{q}}{(2\pi)^{d-1}} a_{\vec{q}}^\dagger a_{\vec{q}} \quad (3.1.89)$$

$$N |0\rangle = 0 \quad (3.1.90)$$

The commutation relation with $a_{\vec{k}}$ is

$$[N, a_{\vec{k}}] = \int \frac{d^D\vec{q}}{(2\pi)^D} \left([a_{\vec{q}}^\dagger, a_{\vec{k}}] a_{\vec{q}} + a_{\vec{q}}^\dagger [a_{\vec{q}}, a_{\vec{k}}] \right) \quad (3.1.91)$$

$$= -a_{\vec{k}}. \quad (3.1.92)$$

Taking the dagger on both sides,

$$[N, a_{\vec{k}}^\dagger] = a_{\vec{k}}^\dagger. \quad (3.1.93)$$

Therefore

$$N a_{\vec{k}}^\dagger |0\rangle = \left([N, a_{\vec{k}}^\dagger] + a_{\vec{k}}^\dagger N \right) |0\rangle \quad (3.1.94)$$

$$= a_{\vec{k}}^\dagger |0\rangle. \quad (3.1.95)$$

Problem 3.8. Particle Number Verify that

$$N |\vec{k}_1, \dots, \vec{k}_n\rangle = n |\vec{k}_1, \dots, \vec{k}_n\rangle, \quad (3.1.96)$$

where the n -particle state is defined in eq. (3.1.83).

Explain why the n -particle momentum state is orthogonal to the n' -particle momentum state whenever $n \neq n'$. □

Problem 3.9. Completeness in Fock Space

Explain why the identity operator acting

on the full Hilbert space of $(n \geq 0)$ -particle states is

$$\begin{aligned} \mathbb{I} &= |0\rangle \langle 0| + \int \frac{d^D \vec{k}}{(2\pi)^D (2E_{\vec{k}})} |\vec{k}\rangle \langle \vec{k}| + \int \frac{d^D \vec{k}}{(2\pi)^D (2E_{\vec{k}})} \int \frac{d^D \vec{k}'}{(2\pi)^D (2E_{\vec{k}'})} |\vec{k}, \vec{k}'\rangle \langle \vec{k}, \vec{k}'| \\ &\quad + \int \frac{d^D \vec{k}}{(2\pi)^D (2E_{\vec{k}})} \int \frac{d^D \vec{k}'}{(2\pi)^D (2E_{\vec{k}'})} \int \frac{d^D \vec{k}''}{(2\pi)^D (2E_{\vec{k}''})} |\vec{k}, \vec{k}', \vec{k}''\rangle \langle \vec{k}, \vec{k}', \vec{k}''| + \dots \end{aligned} \quad (3.1.97)$$

$$= |0\rangle \langle 0| + \sum_{J=1}^{+\infty} \prod_{I=1}^J \left(\int \frac{d^D \vec{k}_I}{(2\pi)^D (2E_{\vec{k}_I})} \right) |\vec{k}_1, \dots, \vec{k}_J\rangle \langle \vec{k}_1, \dots, \vec{k}_J|. \quad (3.1.98)$$

Hint: The orthogonality of the n -particle states is important here. \square **Problem 3.10. Ladder Operators in Fock Space**

Explain why the ladder operators

can be expressed as

$$a_{\vec{q}}^\dagger = \frac{1}{\sqrt{2E_{\vec{q}}}} \left(|\vec{q}\rangle \langle 0| + \sum_{J=1}^{+\infty} \prod_{I=1}^J \left(\int \frac{d^D \vec{k}_I}{(2\pi)^D (2E_{\vec{k}_I})} \right) |\vec{q}, \vec{k}_1, \dots, \vec{k}_J\rangle \langle \vec{k}_1, \dots, \vec{k}_J| \right) \quad (3.1.99)$$

and

$$a_{\vec{q}} = \frac{1}{\sqrt{2E_{\vec{q}}}} \left(|0\rangle \langle \vec{q}| + \sum_{J=1}^{+\infty} \prod_{I=1}^J \left(\int \frac{d^D \vec{k}_I}{(2\pi)^D (2E_{\vec{k}_I})} \right) |\vec{k}_1, \dots, \vec{k}_J\rangle \langle \vec{q}, \vec{k}_1, \dots, \vec{k}_J| \right). \quad (3.1.100)$$

Since the quantum scalar field φ is itself built out of a and a^\dagger , these relations provide an explicit demonstration that QFT is the quantum mechanics of variable $(n \geq 0)$ -body physics. \square If we view a state containing a particle with infinite wavelength $\vec{q} = 0$ as indistinguishable from having no particle; i.e.,

$$|\vec{q} = 0\rangle = |0\rangle, \quad (3.1.101)$$

$$|\vec{q} = 0, \vec{k}_1, \dots, \vec{k}_n\rangle = |\vec{k}_1, \dots, \vec{k}_n\rangle; \quad (3.1.102)$$

then we may notice

$$\lim_{\vec{q} \rightarrow \vec{0}} a_{\vec{q}}^\dagger = \mathbb{I} = \lim_{\vec{q} \rightarrow \vec{0}} a_{\vec{q}}. \quad (3.1.103)$$

Problem 3.11. Argue that

$$\langle 0 | \vec{k} \rangle = (2m) \cdot (2\pi)^D \delta^{(D)}(\vec{k}). \quad (3.1.104)$$

Hint: Consider $\lim_{\vec{q} \rightarrow \vec{0}} \langle \vec{q} | \sqrt{2E_{\vec{k}}} a_{\vec{k}}^\dagger | 0 \rangle$. \square **Problem 3.12.** Define the momentum operator as

$$P^\mu \equiv \int \frac{d^{d-1} \vec{k}}{(2\pi)^{d-1}} k^\mu a_{\vec{k}}^\dagger a_{\vec{k}}. \quad (3.1.105)$$

Explain why

$$P^\mu \left| \vec{k}_1 \dots \vec{k}_n \right\rangle = (k_1^\mu + \dots + k_n^\mu) \left| \vec{k}_1 \dots \vec{k}_n \right\rangle. \quad (3.1.106)$$

Up to an additive constant, can you show that P^μ is in fact the zeroth Noether current T^μ_0 derived in eq. (2.3.10)? \square

Wave Packets Now that we know how to construct the n particle momentum eigenstate, we may proceed to construct arbitrary wave packets. For instance, the general one particle wave packet is

$$|\psi\rangle \equiv \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{\tilde{\psi}(\vec{k})}{\sqrt{2E_{\vec{k}}}} \left| \vec{k} \right\rangle. \quad (3.1.107)$$

To ensure it is normalized to unity, we require

$$\langle \psi | \psi \rangle = 1 = \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \int \frac{d^{d-1}\vec{k}'}{(2\pi)^{d-1}} \frac{\tilde{\psi}(\vec{k}')^*}{\sqrt{2E_{\vec{k}'}}} \frac{\tilde{\psi}(\vec{k})}{\sqrt{2E_{\vec{k}}}} \langle \vec{k}' | \vec{k} \rangle \quad (3.1.108)$$

$$= \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} |\tilde{\psi}(\vec{k})|^2. \quad (3.1.109)$$

The general n particle wave packets state is

$$|\psi_1 \dots \psi_n\rangle \equiv \frac{1}{\sqrt{n!}} \int \frac{d^{d-1}\vec{k}_1}{(2\pi)^{d-1}} \dots \int \frac{d^{d-1}\vec{k}_n}{(2\pi)^{d-1}} \frac{\tilde{\psi}_1(\vec{k}_1)}{\sqrt{2E_{\vec{k}_1}}} \dots \frac{\tilde{\psi}_n(\vec{k}_n)}{\sqrt{2E_{\vec{k}_n}}} \left| \vec{k}_1, \dots, \vec{k}_n \right\rangle. \quad (3.1.110)$$

Time Evolution Since the Hamiltonian for our massive linear scalar field is time independent, the time evolution operator is simply the usual

$$U(t) = \exp(-iH' \cdot t), \quad (3.1.111)$$

where we have assumed the initial conditions are specified at $t = 0$. For our wave packets above, its state in the future $t > 0$ is given by

$$\begin{aligned} & |\psi_1 \dots \psi_n; t\rangle \\ &= U(t) |\psi_1 \dots \psi_n\rangle \end{aligned} \quad (3.1.112)$$

$$= \frac{1}{\sqrt{n!}} \int \frac{d^D\vec{k}_1}{(2\pi)^D} \dots \int \frac{d^D\vec{k}_n}{(2\pi)^D} \frac{\tilde{\psi}_1(\vec{k}_1)}{\sqrt{2E_{\vec{k}_1}}} \dots \frac{\tilde{\psi}_n(\vec{k}_n)}{\sqrt{2E_{\vec{k}_n}}} e^{-it(E_0 + E_{\vec{k}_1} + \dots + E_{\vec{k}_n})} \left| \vec{k}_1, \dots, \vec{k}_n \right\rangle. \quad (3.1.113)$$

Furthermore, let us recognize from eq. (3.1.68) that

$$U(t) = \exp \left[-it \left(E_0 + \int_{\mathbb{R}^D} \frac{d^D\vec{k}}{(2\pi)^D} E_{\vec{k}} a_{\vec{k}}^\dagger a_{\vec{k}} \right) \right] \quad (3.1.114)$$

$$\sim e^{-itE_0} \prod_{\vec{k}} \exp \left[-it \cdot E_{\vec{k}} a_{\vec{k}}^\dagger a_{\vec{k}} \right]. \quad (3.1.115)$$

That is, the evolution operator acts on each momentum-mode separately. This also strongly suggests that the vacuum state may be constructed by multiplying the ground state of the SHO of each and every \vec{k} -mode.

Problem 3.13. N Body Non-Relativistic Schrödinger Equation n -particle momentum eigenstate

Show that the

$$|\vec{k}_1, \dots, \vec{k}_n; t\rangle \equiv U(t) |\vec{k}_1, \dots, \vec{k}_n\rangle \quad (3.1.116)$$

obeys the n -particle non-relativistic Schrödinger equation

$$i\partial_t |\vec{k}_1, \dots, \vec{k}_n; t\rangle \approx \left(E_0 + n \cdot m + \sum_{J=1}^n \frac{\vec{k}_J^2}{2m} \right) |\vec{k}_1, \dots, \vec{k}_n; t\rangle \quad (3.1.117)$$

whenever $\vec{k}_J^2/m^2 \ll 1$. This provides an explicit and simple example of how the Hamiltonian of a quantum *field* theory incorporates the dynamics of more than one body of the same ‘species’. \square

Field Basis The time dependent scalar field $\varphi(t, \vec{x})$ was taken to be the operator in the Heisenberg picture. If we choose, for convenience, $t = 0$ to be where the Schrödinger and Heisenberg picture coincide, then

$$\varphi_S(\vec{x}) \equiv \varphi(t = 0, \vec{x}) \quad (3.1.118)$$

is a Hermitian Schrödinger (time-independent) operator, whose eigenstates $\{|\Phi\rangle\}$ form a basis analogous to the position basis $\{|\vec{x}\rangle\}$ in quantum mechanics:

$$\varphi_S(\vec{x}) |\Phi\rangle = \Phi(\vec{x}) |\Phi\rangle. \quad (3.1.119)$$

Moreover, we may write

$$\varphi_S(\vec{x}) = \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} \left(\frac{a_{\vec{k}} + a_{-\vec{k}}^\dagger}{\sqrt{2E_{\vec{k}}}} \right) e^{i\vec{k}\cdot\vec{x}} \equiv \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} \tilde{\varphi}_S(\vec{k}) e^{i\vec{k}\cdot\vec{x}} \quad (3.1.120)$$

where

$$\tilde{\varphi}_S(\vec{k})^\dagger = \tilde{\varphi}_S(-\vec{k}). \quad (3.1.121)$$

Likewise, the Schrödinger picture momentum operator is

$$\Pi_S(\vec{x}) \equiv \dot{\varphi}(t = 0, \vec{x}) = \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} \sqrt{\frac{E_{\vec{k}}}{2}} \left(-ia_{\vec{k}} + ia_{-\vec{k}}^\dagger \right) e^{i\vec{k}\cdot\vec{x}} \quad (3.1.122)$$

$$\equiv \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} \tilde{\Pi}_S(\vec{k}) e^{i\vec{k}\cdot\vec{x}}, \quad (3.1.123)$$

where

$$\tilde{\Pi}_S(\vec{k})^\dagger = \tilde{\Pi}_S(-\vec{k}). \quad (3.1.124)$$

Problem 3.14. Explain why the position spacetime equal time commutation relation translates to

$$\left[\tilde{\varphi}_S(\vec{k}), \tilde{\Pi}_S(\vec{k}')^\dagger \right] = (2\pi)^D i\delta^{(D)}(\vec{k} - \vec{k}'), \quad (3.1.125)$$

$$\left[\tilde{\varphi}_S(\vec{k})^\dagger, \tilde{\Pi}_S(\vec{k}') \right] = (2\pi)^D i\delta^{(D)}(\vec{k} - \vec{k}'); \quad (3.1.126)$$

and

$$\left[\tilde{\varphi}_S(\vec{k}), \tilde{\varphi}_S(\vec{k}')^\dagger \right] = 0 = \left[\tilde{\Pi}_S(\vec{k}), \tilde{\Pi}_S(\vec{k}')^\dagger \right]. \quad (3.1.127)$$

This final relation implies it must be possible to diagonalize either $\hat{\varphi}(\vec{k})$ or $\hat{\Pi}(\vec{k})$; though they are not simultaneously diagonalizable. \square

These equal time commutation relations in Fourier space may be implemented in, say, the *Fourier field basis*, defined by the eigenstates $\{|\tilde{\Psi}\rangle\}$ obeying

$$\tilde{\varphi}_S(\vec{k}) \left| \tilde{\Psi} \right\rangle = \tilde{\Psi}(\vec{k}) \left| \tilde{\Psi} \right\rangle. \quad (3.1.128)$$

This is a functional eigensystem equation; for instance, the eigenvalue is not a constant, but a function of momentum \vec{k} .

Suppose we postulate the following functional derivative relations – for arbitrary state $|\Phi\rangle$ –

$$\left\langle \tilde{\Psi} \left| \tilde{\Pi}_S(\vec{k})^\dagger \right| \Phi \right\rangle = -i \frac{\delta}{\delta \tilde{\Psi}(\vec{k})^*} \left\langle \tilde{\Psi} \left| \Phi \right\rangle, \quad (3.1.129)$$

$$\left\langle \tilde{\Psi} \left| \tilde{\Pi}_S(\vec{k}) \right| \Phi \right\rangle = -i \frac{\delta}{\delta \tilde{\Psi}(\vec{k})} \left\langle \tilde{\Psi} \left| \Phi \right\rangle; \quad (3.1.130)$$

Then we may verify,

$$\left\langle \tilde{\Psi} \left| \left[\tilde{\varphi}_S(\vec{k}), \tilde{\Pi}_S(\vec{k}')^\dagger \right] \right| \Phi \right\rangle \quad (3.1.131)$$

$$= \left\langle \tilde{\Psi} \left| \left(\tilde{\varphi}_S(\vec{k}) \tilde{\Pi}_S(\vec{k}')^\dagger - \tilde{\Pi}_S(\vec{k}')^\dagger \tilde{\varphi}_S(\vec{k}) \right) \right| \Phi \right\rangle \quad (3.1.132)$$

$$= \tilde{\Psi}(\vec{k})^* (-i) \frac{\delta}{\delta \tilde{\Psi}(\vec{k}')^*} \left\langle \tilde{\Psi} \left| \Phi \right\rangle - (-i) \frac{\delta}{\delta \tilde{\Psi}(\vec{k}')^*} \left(\tilde{\Psi}(\vec{k})^* \left\langle \tilde{\Psi} \left| \Phi \right\rangle \right) \right) \quad (3.1.133)$$

$$= (2\pi)^D i \delta^{(D)}(\vec{k} - \vec{k}') \cdot \left\langle \tilde{\Psi} \left| \Phi \right\rangle, \quad (3.1.134)$$

where we have employed the functional derivative

$$\frac{\delta \tilde{\Psi}(\vec{k})^*}{\delta \tilde{\Psi}(\vec{k}')^*} = (2\pi)^D \delta^{(D)}(\vec{k} - \vec{k}') = \frac{\delta \tilde{\Psi}(\vec{k})}{\delta \tilde{\Psi}(\vec{k}')}. \quad (3.1.135)$$

In this Schrödinger picture, any arbitrary but physical state must obey

$$\left\langle \tilde{\Psi} \left| i\partial_t \right| \Phi(t) \right\rangle = \left\langle \tilde{\Psi} \left| H \right| \Phi(t) \right\rangle; \quad (3.1.136)$$

leading us to the functional Schrodinger equation for the massive (real) scalar field

$$i\partial_t \left\langle \tilde{\Psi} \left| \Phi(t) \right\rangle = \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} \frac{1}{2} \left\langle \tilde{\Psi} \left| \tilde{\Pi}(\vec{k}) \cdot \tilde{\Pi}(\vec{k})^\dagger + E_k^2 \cdot \tilde{\varphi}(\vec{k}) \cdot \tilde{\varphi}(\vec{k})^\dagger \right| \Phi(t) \right\rangle \quad (3.1.137)$$

$$= \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} \frac{1}{2} \left(-\frac{\delta}{\delta \tilde{\Psi}(\vec{k})} \frac{\delta}{\delta \tilde{\Psi}(\vec{k})^*} + E_k^2 |\tilde{\Psi}(\vec{k})|^2 \right) \left\langle \tilde{\Psi} \left| \Phi(t) \right\rangle. \quad (3.1.138)$$

Problem 3.15. Translation in Field Space Let $\tilde{\Psi}$ be a classical field. Show that

$$\mathcal{T}(\tilde{\Psi})^\dagger \tilde{\varphi}_S(\vec{k}) \mathcal{T}(\tilde{\Psi}) = \tilde{\varphi}_S(\vec{k}) + \tilde{\Psi}(\vec{k}), \quad (3.1.139)$$

$$\mathcal{T}(\tilde{\Psi}) \equiv \exp \left(-i \int_{\mathbb{R}^D} \frac{d^D \vec{q}}{(2\pi)^D} \tilde{\Pi}_S(\vec{q}) \cdot \tilde{\Psi}(\vec{q}) \right). \quad (3.1.140)$$

Explain why this implies $\tilde{\Pi}_S$ generates translations in field space. \square

Problem 3.16. Vacuum Wavefunctional Verify that the *vacuum wavefunctional*, i.e., the ground state wavefunctional, of a massive scalar field theory is

$$\langle \tilde{\Psi} | \Omega \rangle = \mathcal{N} \exp \left(-\frac{1}{2} \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} E_{\vec{k}} |\tilde{\Psi}(\vec{k})|^2 \right). \quad (3.1.141)$$

You may wish to note that, since the massive scalar field is real,

$$\tilde{\Psi}(\vec{k})^* = \tilde{\Psi}(-\vec{k}) \quad (3.1.142)$$

and therefore,

$$\int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} E_{\vec{k}} |\tilde{\Psi}(\vec{k})|^2 = \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} \tilde{\Psi}(\vec{k}) E_{\vec{k}} \tilde{\Psi}(-\vec{k}) \quad (3.1.143)$$

$$= \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} \tilde{\Psi}(-\vec{k})^* E_{\vec{k}} \tilde{\Psi}(\vec{k})^*. \quad (3.1.144)$$

Can you figure out the correct normalization factor \mathcal{N} ? Hint: First write down the ground state wave function of a single Fourier mode. \square

Bose Statistics Let us observe that relativistic QFT has led us to *Bose-Einstein* statistics for spin-0 particles through the n -particle state in eq. (3.1.83). Namely, because the a^\dagger 's commute among themselves in eq. (3.1.83), we may swap the momenta in the n particle states, and arrive at the same state – i.e., these are n indistinguishable particles:

$$|\vec{k}_1 \dots \vec{k}_n \rangle = |\vec{k}_{\pi(1)} \dots \vec{k}_{\pi(n)} \rangle, \quad (3.1.145)$$

where $\pi(s)$, for $s = 1, 2, \dots, n$, simply denotes the permutation of the set $\{1, 2, \dots, n\}$. And since this indistinguishable n particle state holds for the momentum eigenstate, we may superpose them over arbitrary momenta weighted by the $\tilde{\psi}_i$'s. This tells us

$$|\psi_1 \dots \psi_n; t \rangle = |\psi_{\pi(1)} \dots \psi_{\pi(n)}; t \rangle. \quad (3.1.146)$$

Lorentz Boosts If Λ denotes a Lorentz transformation matrix, let us define the Lorentz transformed momentum vector as

$$k'_\alpha \equiv \Lambda_\alpha^\beta k_\beta. \quad (3.1.147)$$

Applying the operator

$$D_{1p}(\Lambda) \equiv \int \frac{d^D \vec{q}}{(2\pi)^D (2E_{\vec{q}})} |\vec{q}'\rangle \langle \vec{q}| \quad (3.1.148)$$

to an arbitrary 1-particle momentum state, we discover it boosts the eigenstate with momentum \vec{k} to that with momentum \vec{k}' :

$$D_{1p}(\Lambda) |\vec{k}\rangle = \int \frac{d^D \vec{q}}{(2\pi)^D (2E_{\vec{q}})} |\vec{q}'\rangle \langle \vec{q}| \vec{k}\rangle \quad (3.1.149)$$

$$= \int \frac{d^D \vec{q}}{(2\pi)^D (2E_{\vec{q}})} |\vec{q}'\rangle (2\pi)^D (2E_{\vec{q}}) \delta^{(D)}(\vec{q} - \vec{k}) \quad (3.1.150)$$

$$= |\vec{k}'\rangle. \quad (3.1.151)$$

Problem 3.17. Check that $D_{1p}(\Lambda)$ in eq. (3.1.148) is unitary. The set of $\{D_{1p}(\Lambda)\}$ forms an infinite dimensional unitary representation of the Lorentz group – can you explain why? \square

Now, we consider

$$D_{1p}(\Lambda) a_{\vec{k}}^\dagger D_{1p}(\Lambda)^\dagger \quad (3.1.152)$$

$$= \int \frac{d^D \vec{q}}{(2\pi)^D (2E_{\vec{q}})} \int \frac{d^D \vec{p}}{(2\pi)^D (2E_{\vec{p}})} |\vec{q}'\rangle \langle \vec{q}| a_{\vec{k}}^\dagger |\vec{p}\rangle \langle \vec{p}'| \quad (3.1.153)$$

$$= \int \frac{d^D \vec{q}}{(2\pi)^D (2E_{\vec{q}})} \int \frac{d^D \vec{p}}{(2\pi)^D (2E_{\vec{p}})} |\vec{q}'\rangle \frac{\langle \vec{q}| \vec{k}\rangle \langle 0| \vec{p}\rangle}{\sqrt{2E_{\vec{k}}}} \langle \vec{p}'| \quad (3.1.154)$$

$$= \int \frac{d^D \vec{q}}{(2\pi)^D (2E_{\vec{q}})} \int \frac{d^D \vec{p}}{(2\pi)^D (2E_{\vec{p}})} |\vec{q}'\rangle \frac{(2E_{\vec{q}})(2\pi)^D \delta^{(D)}[\vec{q} - \vec{k}](2m)(2\pi)^D \delta^{(D)}[\vec{p}]}{\sqrt{2E_{\vec{k}}}} \langle \vec{p}'| \quad (3.1.155)$$

$$= \frac{|\vec{k}'\rangle \langle 0|}{\sqrt{2E_{\vec{k}}}} = \frac{\sqrt{2E_{\vec{k}'}} |\vec{k}'\rangle \langle 0|}{\sqrt{2E_{\vec{k}}} \sqrt{2E_{\vec{k}'}}} = \frac{\sqrt{2E_{\vec{k}'}}}{\sqrt{2E_{\vec{k}}}} a_{\vec{k}'}^\dagger. \quad (3.1.156)$$

Taking the \dagger on both sides,

$$D_{1p}(\Lambda) a_{\vec{k}} D_{1p}(\Lambda)^\dagger = \frac{\sqrt{2E_{\vec{k}'}}}{\sqrt{2E_{\vec{k}}}} a_{\vec{k}'}. \quad (3.1.157)$$

We have used $\langle 0| \vec{k}\rangle = (2m)(2\pi)^D \delta^{(D)}(\vec{k})$.

Problem 3.18. Lorentz Transformation on Fock space What is the generalization of the Lorentz transformation in eq. (3.1.148), but acting on the entire Fock space? \square

Problem 3.19. Lorentz Transformation of Scalar Field Show that

$$D(\Lambda_\alpha^\beta) \varphi(x^\mu) D(\Lambda_\alpha^\beta)^\dagger = \varphi(x'^\mu = \Lambda^\mu{}_\nu x^\nu). \quad (3.1.158)$$

The operators on the LHS implement Lorentz transformation on momenta via eq. (3.1.147); whereas, on the RHS, the inverse Lorentz transformation is applied to the Cartesian coordinates. Hence, note the placement of the indices on the Λ ; those on the LHS are not the same as those on the RHS. This is the operator statement that the quantum scalar field does indeed transform as a scalar under Lorentz transformations. \square

Problem 3.20. Quantization Around A Classical Background Starting from the following action

$$S \equiv \int_{\mathbb{R}^{D,1}} dt d^D \vec{x} \left(\frac{1}{2} (\partial \varphi)^2 - \frac{m^2}{2} \varphi + \varphi \cdot J \right), \quad (3.1.159)$$

Show that the Hamiltonian density is

$$\mathcal{H} = \frac{1}{2} \Pi^2 + \frac{1}{2} (\vec{\nabla} \varphi)^2 + \frac{m^2}{2} \varphi^2 - \varphi \cdot J, \quad (3.1.160)$$

and that the resulting Hamilton's equations yield

$$(\partial^2 + m^2) \varphi(t, \vec{x}) = J(t, \vec{x}). \quad (3.1.161)$$

Let us attempt to quantize the fluctuations on top of this classical background. That is, starting from the action **YZ: Incomplete**. \square

Problem 3.21. QFT in a "Square Well" & Casimir Effect Consider the massless $m = 0$ free scalar field in (1+1)-dimensional Minkowski, but within a finite spatial domain of length L :

$$x^\mu \equiv (t \in \mathbb{R}, 0 \leq x \leq L). \quad (3.1.162)$$

Demand that the field vanishes at $x = 0, L$. Compute the vacuum expectation value of the energy density ρ , defined as:

$$\rho \equiv \langle \text{vac} | : T^{00} : | \text{vac} \rangle, \quad (3.1.163)$$

$$T^{00} \equiv \frac{1}{2} \dot{\varphi}^2 + \frac{1}{2} (\partial_x \varphi)^2. \quad (3.1.164)$$

What is the total energy E inside the square well, as a function of L ? What is $\partial E / \partial L$? The latter can be interpreted as an effective force experienced by the walls of the square well as a result of quantum fluctuations. (Bonus: What are the pressure and momentum densities?) \square

Problem 3.22. Cosmology We live in a universe that is well described, at large scales, by the metric

$$ds^2 = dt^2 - a(t)^2 d\vec{x} \cdot d\vec{x}; \quad (3.1.165)$$

i.e., with non-zero components $g_{00} = 1$ and $g_{ij} = -a^2 \delta_{ij}$. The associated wave operator can be constructed out of the absolute value of its determinant $|g| = a^6$ and inverse $g^{\mu\nu}$:

$$\square \varphi = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu \varphi \right) = \frac{1}{a^3} \partial_t (a^3 \dot{\varphi}) - a^{-2} \vec{\nabla}^2 \varphi. \quad (3.1.166)$$

The Lagrangian density of the massive scalar field is now

$$\mathcal{L} = \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{m^2}{2}\varphi^2 = \frac{1}{2}\dot{\varphi}^2 - \frac{a^{-2}}{2}\vec{\nabla}\varphi \cdot \vec{\nabla}\varphi - \frac{m^2}{2}\varphi^2. \quad (3.1.167)$$

Work out the conjugate momentum Π . Show that the equal time commutation relations

$$[\varphi(t, \vec{x}), \Pi(t, \vec{x}')] = i\frac{\delta^{(3)}(\vec{x} - \vec{x}')}{a(t)^3} \quad (3.1.168)$$

$$[\varphi(t, \vec{x}), \varphi(t, \vec{x}')] = 0 = [\Pi(t, \vec{x}), \Pi(t, \vec{x}')]; \quad (3.1.169)$$

leads to the Heisenberg EoM

$$(\square + m^2)\varphi = 0. \quad (3.1.170)$$

That is, just like the Minkowski case, we recover the massive scalar wave equation. \square

Problem 3.23. de Sitter Power Spectrum de Sitter spacetime in ‘flat slicing’ coordinates is described by the metric

$$ds^2 = a(\eta)^2 (d\eta^2 - d\vec{x}^2), \quad (3.1.171)$$

$$a[\eta] \equiv -(H\eta)^{-1}, \quad \eta \in (-\infty, 0). \quad (3.1.172)$$

In this problem, compute in this geometry the one point function

$$\langle \text{vac} | \tilde{\varphi}(\eta, \vec{k}) | \text{vac} \rangle \quad (3.1.173)$$

and the power spectrum of a massless scalar field

$$\langle \text{vac} | \tilde{\varphi}(\eta, \vec{k}) \tilde{\varphi}(\eta, \vec{k}') | \text{vac} \rangle. \quad (3.1.174)$$

Hints: If $\sqrt{|g|}$ is the square root of the determinant of the metric $g_{\mu\nu} = \text{diag}[1, -1, -1, -1]/(H\eta)^2$ and $g^{\mu\nu} = \text{diag}[1, -1, -1, -1](H\eta)^2$ is its inverse, the scalar field obeys the wave equation

$$\square\varphi = \frac{\partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu \varphi \right)}{\sqrt{|g|}} = 0. \quad (3.1.175)$$

The spatial translation invariance means we may Fourier decompose our scalar field in plane waves. Specifically, if we first re-scale

$$\varphi(\eta, \vec{x}) = \frac{\phi(\eta, \vec{x})}{a(\eta)}, \quad (3.1.176)$$

followed by examining a single Fourier mode

$$\phi(\eta, \vec{x}) = f(\xi) e^{i\vec{k}\cdot\vec{x}}, \quad (3.1.177)$$

where $\xi \equiv k\eta$ and $k \equiv |\vec{k}|$; show that f obeys

$$f''(\xi) + \left(1 - \frac{2}{\xi^2}\right) f(\xi) = 0. \quad (3.1.178)$$

You should find the two linearly independent solutions to be

$$f_{\pm}(\xi) \equiv \frac{e^{\pm i\xi}}{\sqrt{2}} \left(\pm i + \frac{1}{\xi}\right). \quad (3.1.179)$$

Choose your mode functions for φ such that, as $\eta \rightarrow -\infty$, the positive energy solutions approach those of Minkowski spacetime. Moreover, to quantize such a system, you should be able to verify its associated Lagrangian is

$$L_f \equiv \frac{1}{2} f'(\xi)^2 + \frac{f(\xi)f'(\xi)}{\xi} + \frac{1}{2} f(\xi)^2 \left(\frac{1}{\xi^2} - 1\right). \quad (3.1.180)$$

These vacuum fluctuations – if inflationary cosmologists are right – may be responsible for generating inhomogeneities in the very early universe, from which all of cosmic structure (galaxy clusters, etc.) were produced. In particular, comment on how the calculations apply to primordial gravitational waves. \square

Classical Solution YZ: Incomplete

Feynman Green's Function At the heart of perturbative QFT is the Feynman Green's function. The Feynman Green's function $G_F(x, x')$ is defined as the vacuum expectation value of the time-ordered product $\varphi(x)\varphi(x')$.

$$G_F(x, x') \equiv \langle \Omega | \mathbb{T} \{ \varphi(x)\varphi(x') \} | \Omega \rangle \quad (3.1.181)$$

$$\equiv \langle \Omega | \Theta(t - t') \varphi(x)\varphi(x') + \Theta(t' - t) \varphi(x')\varphi(x) | \Omega \rangle. \quad (3.1.182)$$

From this definition, note the symmetry

$$G_F(x - x') = G_F(x' - x). \quad (3.1.183)$$

The wave operator acting on G_F yields

$$\begin{aligned} & (\partial_x^2 + m^2) G_F[x, x'] \quad (3.1.184) \\ &= \langle \Omega | \delta'(t - t') \varphi(x)\varphi(x') + 2\delta(t - t') \partial_t \varphi(x)\varphi(x') + \Theta(t - t') (\partial_x^2 + m^2) \varphi(x)\varphi(x') | \Omega \rangle \\ & \quad + \langle \Omega | \delta'(t' - t) \varphi(x')\varphi(x) - 2\delta(t' - t) \varphi(x') \partial_t \varphi(x) + \Theta(t' - t) \varphi(x') (\partial_x^2 + m^2) \varphi(x) | \Omega \rangle. \end{aligned}$$

Use the Heisenberg EoM $(\partial^2 + m^2)\varphi = 0$ and the identity $z\delta'(z) = -\delta(z)$ to deduce

$$(\partial_x^2 + m^2) G_F[x, x'] = \delta(t - t') \langle \Omega | \partial_t \varphi(x)\varphi(x') - \varphi(x') \partial_t \varphi(x) | \Omega \rangle \quad (3.1.185)$$

$$= -\delta(t - t') \langle \Omega | [\varphi(t, \vec{x}'), \Pi(t, \vec{x})] | \Omega \rangle \quad (3.1.186)$$

$$= -i\delta(t - t') \delta^{(d-1)}(\vec{x} - \vec{x}'). \quad (3.1.187)$$

Similarly,

$$(\partial_{x'}^2 + m^2) G_F[x, x'] = -i\delta(t - t') \delta^{(d-1)}(\vec{x} - \vec{x}'). \quad (3.1.188)$$

Let us record that, the Feynman Green's function for the massive scalar field, $(\partial^2 + m^2)\varphi = 0$, has the following integral representation.

$$G_F(x - x') \equiv i \int \frac{d^d k}{(2\pi)^d} \frac{e^{-ik \cdot (x-x')}}{k^2 - m^2 + i\epsilon}, \quad \epsilon \equiv 0^+. \quad (3.1.189)$$

The small positive ϵ tells us the k_0 integral skirts the pole at $-E_{\vec{k}}$ by dipping into the negative imaginary plane; while skirting the pole at $+E_{\vec{k}}$ by going into the positive imaginary plane. Hence, if $t > t'$, we see that $\exp(-i(\pm i\infty)(t - t')) \rightarrow \exp((\pm\infty)(t - t'))$ tells us we need to close the contour in the lower half plane. Whereas, if $t < t'$, we see that $\exp(-i(\pm i\infty)(t - t')) \rightarrow \exp((\mp\infty)|t - t'|)$ tells us we need to close the contour in the upper half plane. Noting $k^2 - m^2 = (k_0 - E_{\vec{k}})(k_0 + E_{\vec{k}})$,

$$\begin{aligned} G_F(x - x') &= \int \frac{d^{d-1} \vec{k}}{(2\pi)^{d-1}} \frac{1}{2E_{\vec{k}}} \left(\frac{i(-i)}{+1} \Theta(t - t') e^{-iE_{\vec{k}}(t-t')} + \frac{i(+i)}{-1} \Theta(t' - t) e^{+iE_{\vec{k}}(t-t')} \right) e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} \\ &= \int \frac{d^{d-1} \vec{k}}{(2\pi)^{d-1}} \frac{1}{2E_{\vec{k}}} \left(\Theta(t - t') e^{-ik \cdot (x-x')} + \Theta(t' - t) e^{+ik \cdot (x-x')} \right), \quad k^2 = m^2. \end{aligned}$$

Within the canonical formalism, we may do a direct calculation of the following Wightman function

$$\langle 0 | \varphi(x) \varphi(x') | 0 \rangle = \int \frac{d^{d-1} \vec{k}_1}{(2\pi)^d} \frac{1}{\sqrt{2E_1}} \int \frac{d^{d-1} \vec{k}_2}{(2\pi)^d} \frac{1}{\sqrt{2E_2}} \langle 0 | a_{\vec{k}_1} a_{\vec{k}_2}^\dagger | 0 \rangle e^{-ik_1 \cdot x} e^{ik_2 \cdot x'} \quad (3.1.190)$$

$$\begin{aligned} &= \int \frac{d^{d-1} \vec{k}_1}{(2\pi)^d} \frac{1}{\sqrt{2E_1}} \int \frac{d^{d-1} \vec{k}_2}{(2\pi)^d} \frac{1}{\sqrt{2E_2}} \langle 0 | [a_{\vec{k}_1}, a_{\vec{k}_2}^\dagger] + a_{\vec{k}_2}^\dagger a_{\vec{k}_1} | 0 \rangle e^{-ik_1 \cdot x} e^{ik_2 \cdot x'} \\ &= \int \frac{d^{d-1} \vec{k}}{(2\pi)^d} \frac{1}{2E} e^{-ik \cdot (x-x')} \quad (3.1.191) \end{aligned}$$

Hence, the time ordered product itself is the Feynman Green's function

$$\langle 0 | \mathbb{T} \varphi(x) \varphi(x') | 0 \rangle = \int \frac{d^{d-1} \vec{k}}{(2\pi)^d} \frac{1}{2E_{\vec{k}}} \left(\Theta(t - t') e^{-ik \cdot (x-x')} + \Theta(t' - t) e^{+ik \cdot (x-x')} \right) \quad (3.1.192)$$

$$= G_F(x - x'). \quad (3.1.193)$$

Retarded/Advanced Green's Functions, Commutators & Micro-causality We will now argue that the retarded Green's function G^+ of the massive scalar field is given by the commutator multiplied by the causal step function, namely

$$G^+[x, x'] \equiv i\Theta[t - t'] [\varphi[x], \varphi[x']]. \quad (3.1.194)$$

Let us apply the wave operator:

$$\begin{aligned} (\partial_x^2 + m^2) G^+[x, x'] &= i\delta'(t - t') [\varphi[x], \varphi[x']] + 2i\delta(t - t') [\partial_t \varphi[x], \varphi[x']] \\ &\quad + i\Theta(t - t') [(\partial^2 + m^2) \varphi(x), \varphi(x')] \quad (3.1.195) \end{aligned}$$

Using the identity $z\delta'(z) = -\delta(z)$ and using the Heisenberg equations of motion $(\partial^2 + m^2)\varphi = 0$, we gather

$$\begin{aligned} (\partial_x^2 + m^2) G^+[x, x'] &= i\delta'(t - t') [\partial_t \varphi[t, \vec{x}], \varphi[t, \vec{x}']] \\ &= -i\delta'(t - t') [\varphi[t, \vec{x}'], \Pi[t, \vec{x}]] \end{aligned} \quad (3.1.196)$$

$$= \delta(t - t') \delta^{(d-1)}(\vec{x} - \vec{x}'). \quad (3.1.197)$$

By construction, G^+ is zero whenever $t < t'$. Furthermore it satisfies the Green's function equation. Hence, it is the retarded Green's function. A similar argument would reveal, the advanced Green's function G^- is

$$G^-[x, x'] \equiv -i\Theta[t' - t] [\varphi[x], \varphi[x']]. \quad (3.1.198)$$

We may then subtract the retarded G^+ and advanced G^- Green's functions, recognize $\Theta(t - t') + \Theta(t' - t) = 1$, and deduce

$$i [\varphi[x], \varphi[x']] = G^+[x, x'] - G^-[x, x']. \quad (3.1.199)$$

Whenever the commutator of the fields $\varphi(x)$ and $\varphi(x')$ vanish outside the light cone, i.e.,

$$[\varphi[x], \varphi[x']] = 0 \quad \text{whenever} \quad (x - x')^2 < 0, \quad (3.1.200)$$

micro-causality is said to be respected. In fact, it can be shown that the retarded and advanced Green's functions of wave operators (in generic curved spacetimes) do in fact vanish outside the light cone. Hence, the micro-causality of quantum fields is tied to the causal propagation of classical fields.

Problem 3.24. Massless Spin-1 Photons and Spin-2 Gravitons Consider a scalar-vector decomposition of the photon vector potential A_i and electric current J_i :

$$A_i = \partial_i \alpha + \alpha_i, \quad \partial_i \alpha_i = 0, \quad (3.1.201)$$

$$J_i = \partial_i \Gamma + \Gamma_i, \quad \partial_i \Gamma_i = 0. \quad (3.1.202)$$

Explain why the conservation of current $\partial^\mu J_\mu = 0$ implies

$$j^0 = \partial_i \partial_i \Gamma. \quad (3.1.203)$$

Next, recall the gauge-invariant vector potential variables

$$\Phi \equiv A_0 - \dot{\alpha} \quad \text{and} \quad A_i^T \equiv \alpha_i. \quad (3.1.204)$$

Show that the electromagnetic action

$$S_{\text{EM}} \equiv \int d^d x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu \right) \quad (3.1.205)$$

can be written in the manifestly gauge-invariant form

$$S_{\text{EM}} \equiv \int d^d x \left(\frac{1}{2} \partial_\alpha \alpha_j \partial^\alpha \alpha_j + \alpha_i \Gamma_i + \frac{1}{2} \partial_i \Phi \partial_i \Phi - \Phi J^0 \right). \quad (3.1.206)$$

(Assume it is alright to integrate-by-parts freely.) Now, let us quantize the dynamical massless spin-1 photon α_i in the (3+1)D vacuum, i.e., where $J_\mu = 0$.

$$\alpha_i(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2k}} \sum_{s=1}^2 \{ a_k^s \epsilon_i^s e^{-ik \cdot x} + (a_k^s)^\dagger (\epsilon_i^s)^* e^{+ik \cdot x} \}, \quad k \equiv |\vec{k}|. \quad (3.1.207)$$

The $\{\epsilon_i^s | s = 1, 2\}$ are the two transverse orthonormal polarization vectors of the photon: $k^i \epsilon_i^s = 0$. Whereas $k_\mu k^\mu = 0$ and the ladder operators obey the simple harmonic algebra

$$[a_{\vec{k}}^r, (a_{\vec{k}'}^s)^\dagger] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{k} - \vec{k}'). \quad (3.1.208)$$

Show that, for $x^\mu \equiv (t, \vec{x})$ and $x'^\mu \equiv (t', \vec{x}')$,

$$i [\alpha_i(x), \alpha_j(x')] = \text{sgn}(t - t') \int \frac{d^3\vec{k}}{(2\pi)^3} P_{ij}(\vec{k}) \frac{\sin(k(t - t'))}{k} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')}, \quad (3.1.209)$$

$$P_{ij}(\vec{k}) \equiv \delta_{ij} - \frac{k_i k_j}{k^2}. \quad (3.1.210)$$

Next, we turn to the massless spin-2 graviton h_{ij}^{TT} ,³ which is transverse and traceless:

$$\partial_i h_{ij}^{\text{TT}} = 0 = \delta^{ij} h_{ij}^{\text{TT}}. \quad (3.1.211)$$

Its quantum operator admits the expansion, in (3+1)D,

$$h_{ij}^{\text{TT}}(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2k}} \sum_{s=1}^2 \{ a_k^s \epsilon_{ij}^s e^{-ik \cdot x} + (a_k^s)^\dagger (\epsilon_{ij}^s)^* e^{+ik \cdot x} \}, \quad k \equiv |\vec{k}|. \quad (3.1.212)$$

The transverse traceless conditions in Fourier space reads

$$k^i \epsilon_{ij}^s = 0 = \delta^{ij} \epsilon_{ij}^s. \quad (3.1.213)$$

Imposing the simple harmonic algebra

$$[a_{\vec{k}}^r, (a_{\vec{k}'}^s)^\dagger] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{k} - \vec{k}'). \quad (3.1.214)$$

demonstrate that

$$i [h_{ij}^{\text{TT}}(x), h_{mn}^{\text{TT}}(x')] = \text{sgn}(t - t') \int \frac{d^3\vec{k}}{(2\pi)^3} P_{ijmn}(\vec{k}) \frac{\sin(k(t - t'))}{k} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')}, \quad (3.1.215)$$

$$P_{ijmn}(\vec{k}) \equiv \frac{1}{2} \left(P_{im}(\vec{k}) P_{jn}(\vec{k}) + P_{in}(\vec{k}) P_{jm}(\vec{k}) - P_{ij}(\vec{k}) P_{mn}(\vec{k}) \right). \quad (3.1.216)$$

Hints: Recall that the (Fourier space) transverse polarization vector can be obtained by projection:

$$\epsilon_i^{\text{T}} = P_{ij} \epsilon_j; \quad (3.1.217)$$

³In a weakly curved spacetime $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $|h_{\mu\nu}| \ll 1$, the h_{ij}^{TT} is the transverse-traceless part of $h_{\mu\nu}$.

and likewise for the (Fourier space) TT gravitation polarization tensor:

$$\varepsilon_{ij}^{\text{TT}} = \frac{1}{2} \left(P_{im}(\vec{k})P_{jn}(\vec{k}) + P_{in}(\vec{k})P_{jm}(\vec{k}) - P_{ij}(\vec{k})P_{mn}(\vec{k}) \right) \varepsilon_{mn}. \quad (3.1.218)$$

This implies, when one encounters a sum over orthonormal polarization vectors such as

$$\sum_s \epsilon_i^{(s)\text{T}} (\varepsilon_j^{(s)\text{T}})^* = P_{ia} P_{jb} \sum_s \epsilon_a^{(s)} (\varepsilon_b^{(s)})^*; \quad (3.1.219)$$

even though the sum only runs over the two orthonormal polarizations perpendicular to k_i , it can be extended to include $\epsilon_i^{(3)} = \widehat{k}_i$ since it is annihilated by P_{ij} anyway. This observation may then be invoked to then apply the completeness relation. Similar remarks apply for the graviton case.

3.2 In- and Out-Vacuum (Heisenberg Picture)

We will define the in-vacuum to be the Heisenberg picture ground state of the full Hamiltonian H , evaluated at the infinite past

$$|\text{in-vac}\rangle \equiv |\Omega, t \rightarrow -\infty\rangle_{\text{H}}.$$

Similarly, the out-vacuum is that evaluated at the infinite future

$$|\text{out-vac}\rangle \equiv |\Omega, t \rightarrow +\infty\rangle_{\text{H}}.$$

In Minkowski spacetime with no external time dependent fields, the Hamiltonian of a given QFT is time independent. We have, of course,

$$\begin{aligned} H|\text{in-vac}\rangle &= \Omega|\text{in-vac}\rangle, \\ H|\text{out-vac}\rangle &= \Omega|\text{out-vac}\rangle, \end{aligned}$$

where Ω is the ground state energy, a real number. This means $|\text{in-vac}\rangle$ and $|\text{out-vac}\rangle$ can only differ by a (possibly ambiguous) phase

$$\begin{aligned} |\text{in-vac}\rangle &= e^{i\Sigma} |\text{out-vac}\rangle, & e^{i\Sigma} \langle \text{in-vac} | &= \langle \text{out-vac} | \\ e^{i\Sigma} &\equiv \lim_{T \rightarrow \infty} e^{-i(2T)\Omega}. \end{aligned}$$

Therefore, the vacuum expectation value of the time-ordered product of operators

$$\mathcal{O}_1[t_1], \mathcal{O}_2[t_2], \dots, \mathcal{O}_n[t_n], \quad (3.2.1)$$

can be expressed in various ways. Firstly, it does not matter whether the expectation value is taken with respect with either the in-vacuum or out-vacuum, since they differ only by a phase.

$$\langle \text{in-vac} | \mathcal{T} \{ \mathcal{O}_1[t_1] \mathcal{O}_2[t_2] \dots \mathcal{O}_n[t_n] \} | \text{in-vac} \rangle = \langle \text{out-vac} | \mathcal{T} \{ \mathcal{O}_1[t_1] \mathcal{O}_2[t_2] \dots \mathcal{O}_n[t_n] \} | \text{out-vac} \rangle.$$

Secondly, the vacuum expectation value can also be expressed as an in-out transition amplitude,

$$\boxed{\langle \Omega' | \mathcal{T} \{ \mathcal{O}_1[t_1] \mathcal{O}_2[t_2] \dots \mathcal{O}_n[t_n] \} | \Omega' \rangle = \frac{\langle \text{out-vac} | \mathcal{T} \{ \mathcal{O}_1[t_1] \mathcal{O}_2[t_2] \dots \mathcal{O}_n[t_n] \} | \text{in-vac} \rangle}{\langle \text{out-vac} | \text{in-vac} \rangle}}, \quad (3.2.2)$$

where $\Omega' = \text{in-vac}$ or $\Omega' = \text{out-vac}$. To see this, we begin from the RHS.

$$\begin{aligned} \frac{\langle \text{out-vac} | \mathcal{T} \{ \mathcal{O}_1[t_1] \mathcal{O}_2[t_2] \dots \mathcal{O}_n[t_n] \} | \text{in-vac} \rangle}{\langle \text{out-vac} | \text{in-vac} \rangle} &= \frac{\langle \text{out-vac} | \mathcal{T} \{ \mathcal{O}_1[t_1] \mathcal{O}_2[t_2] \dots \mathcal{O}_n[t_n] \} e^{i\Sigma} | \text{out-vac} \rangle}{e^{i\Sigma} \langle \text{out-vac} | \text{out-vac} \rangle} \\ &= \frac{e^{i\Sigma} \langle \text{in-vac} | \mathcal{T} \{ \mathcal{O}_1[t_1] \mathcal{O}_2[t_2] \dots \mathcal{O}_n[t_n] \} | \text{in-vac} \rangle}{e^{i\Sigma} \langle \text{in-vac} | \text{in-vac} \rangle}, \end{aligned}$$

which yields the LHS of eq. (3.2.2) as long as the vacuum state is normalized to unity. (Actually, the $|\Omega'\rangle$ can be the vacuum at *any* time – can you see why?) This discussion is important for the upcoming discussion in the next section, because the right hand side of eq. (3.2.2) can be expressed as ratios of path integrals, allowing us to use path integrals to compute expectation values – at least when H is time independent. Up to an overall constant \mathcal{A} , we have

$$\begin{aligned} \langle \text{out-vac} | \mathcal{T} \{ \mathcal{O}_1[t_1] \mathcal{O}_2[t_2] \dots \mathcal{O}_n[t_n] \} | \text{in-vac} \rangle &= \mathcal{A} \int \mathcal{D}\varphi \exp [iS[\varphi]] \mathcal{O}_1[t_1], \mathcal{O}_2[t_2], \dots, \mathcal{O}_n[t_n], \\ \langle \text{out-vac} | \text{in-vac} \rangle &= \mathcal{A} \int \mathcal{D}\varphi \exp [iS[\varphi]]. \end{aligned} \quad (3.2.3)$$

Schwinger-Keldysh/“in-in” When H is *not* time independent, then energy eigenstates are no longer stationary states – in general we may only speak about the instantaneous eigenstates of $H[t]$ at a given time. In particular, the $|\text{in-vac}\rangle$ is no longer some phase multiplied by the $|\text{out-vac}\rangle$ and the argument above, relating the ratio of the path integrals to the in-vacuum expectation value, will not go through because time evolution is now considerably more complicated. In terms of the now highly non-trivial $U = \mathcal{T} \exp(-i \int_{-\infty}^{+\infty} H[s] ds)$,

$$\begin{aligned} |\text{in-vac}\rangle &= U |\text{out-vac}\rangle \quad (3.2.4) \\ \langle \text{out-vac} | \mathcal{T} \{ \mathcal{O}_1[t_1] \mathcal{O}_2[t_2] \dots \mathcal{O}_n[t_n] \} | \text{in-vac} \rangle &= \langle \text{in-vac} | U \mathcal{T} \{ \mathcal{O}_1[t_1] \mathcal{O}_2[t_2] \dots \mathcal{O}_n[t_n] \} | \text{in-vac} \rangle \\ \frac{\langle \text{out-vac} | \mathcal{T} \{ \mathcal{O}_1[t_1] \mathcal{O}_2[t_2] \dots \mathcal{O}_n[t_n] \} | \text{in-vac} \rangle}{\langle \text{out-vac} | \text{in-vac} \rangle} &= \frac{\langle \text{in-vac} | U \mathcal{T} \{ \mathcal{O}_1[t_1] \mathcal{O}_2[t_2] \dots \mathcal{O}_n[t_n] \} | \text{in-vac} \rangle}{\langle \text{in-vac} | U | \text{in-vac} \rangle} \\ &\neq \frac{\langle \text{in-vac} | \mathcal{T} \{ \mathcal{O}_1[t_1] \mathcal{O}_2[t_2] \dots \mathcal{O}_n[t_n] \} | \text{in-vac} \rangle}{\langle \text{in-vac} | \text{in-vac} \rangle}. \end{aligned}$$

On the other hand, the more restricted problem of computing the vacuum expectation value of some operator $O[t]$ at some time t , namely

$$\langle \text{in-vac} | O[t] | \text{in-vac} \rangle, \quad (3.2.5)$$

may still be tackled in terms of path integrals – two copies of them in fact – but the ensuing formalism is different from the above “in-out” one. It is instead known as the Schwinger-Keldysh or “in-in” formalism, named after Julian Schwinger and Leonid Keldysh.

3.3 Interactions: Correlations, Wick's Theorem & Path Integrals

One of the key objects that occur in QFT is the $n \geq 1$ point correlation function

$$\langle \Omega | \mathbb{T} \varphi(x_1) \dots \varphi(x_n) | \Omega \rangle. \quad (3.3.1)$$

We will find it easier to work in Fourier spacetime.

$$\langle \tilde{\varphi}(k_1) \dots \tilde{\varphi}(k_n) \rangle \equiv \int d^d x_1 \dots \int d^d x_n \langle \Omega | \mathbb{T} \varphi(x_1) \dots \varphi(x_n) | \Omega \rangle e^{ik_1 \cdot x_1} \dots e^{ik_n \cdot x_n}. \quad (3.3.2)$$

To begin, let us remind ourselves of the following fact from QM:

$$\int_{\vec{x}'}^{\vec{x}} \mathcal{D}\vec{q} \int \mathcal{D}\vec{p} \exp \left(i \int_{t'}^t \left\{ \vec{p} \cdot \dot{\vec{q}} - H[\vec{q}, \vec{p}] \right\} d\tau \right) A(s) B(s') C(s'') \dots \quad (3.3.3)$$

$$= {}_{\text{H}} \langle \vec{x}, t | \mathbb{T} \{ A_{\text{H}}(s) B_{\text{H}}(s') C_{\text{H}}(s'') \dots \} | \vec{x}', t' \rangle_{\text{H}}; \quad (3.3.4)$$

where on the right hand side $A_{\text{H}}, B_{\text{H}}, C_{\text{H}}, \dots$ are arbitrary Heisenberg-picture operators diagonal in the position representation; and A, B, C, \dots on the left-hand-side correspond to their complex-number counterparts; and $t > (s, s', s'', \dots) > t'$. Whereas, \mathbb{T} means the operators within the curly brackets are arranged such that the ones with later times stand on the right. For instance, if $s_2 > s_1$, then

$$\mathbb{T} \{ A_{\text{H}}(s_1) B_{\text{H}}(s_2) \} = B_{\text{H}}(s_2) A_{\text{H}}(s_1); \quad (3.3.5)$$

or, if $s'' > s > s'$, then

$$\mathbb{T} \{ A_{\text{H}}(s) B_{\text{H}}(s') C_{\text{H}}(s'') \} = C_{\text{H}}(s'') A_{\text{H}}(s) B_{\text{H}}(s'). \quad (3.3.6)$$

Free Massive Scalar We will begin with the free massive scalar, and start by showing that the two point function may be computed using the path integral

$$\langle \tilde{\varphi}(k_1) \tilde{\varphi}(k_2) \rangle = \int d^d x_1 \int d^d x_2 e^{ik_1 \cdot x_1} e^{ik_2 \cdot x_2} \frac{\int \mathcal{D}\varphi \varphi(x_1) \varphi(x_2) \exp(iS)}{\int \mathcal{D}\varphi \exp(iS)} \quad (3.3.7)$$

$$S \equiv \int d^d x \left(\frac{1}{2} (\partial\varphi)^2 - \frac{m^2}{2} \varphi^2 \right). \quad (3.3.8)$$

The two point function $\langle \tilde{\varphi}(k_1) \tilde{\varphi}(k_2) \rangle$, which is simply the Feynman Green's function in Fourier spacetime, reads

$$\langle \tilde{\varphi}(k_1) \tilde{\varphi}(k_2) \rangle = \frac{i}{k_1^2 - m^2 + i\epsilon} (2\pi)^d \delta^{(d)}(k_1 + k_2). \quad (3.3.9)$$

Check:

$$\langle 0 | \mathbb{T} \varphi(x_1) \varphi(x_2) | 0 \rangle = \int \frac{d^d k_1}{(2\pi)^d} \int \frac{d^d k_2}{(2\pi)^d} \frac{i}{k_1^2 - m^2 + i\epsilon} e^{-ik_1 \cdot x_1} e^{-ik_2 \cdot x_2} (2\pi)^d \delta^{(d)}(k_1 + k_2) \quad (3.3.10)$$

$$= \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik \cdot (x_1 - x_2)} \quad (3.3.11)$$

Let us see that the path integral would recover this result. First, convert the action into Fourier spacetime

$$S \equiv \int \frac{d^d q}{(2\pi)^d} \frac{1}{2} (q^2 - m^2) |\tilde{\varphi}(q)|^2. \quad (3.3.12)$$

The path integral measure $\int \mathcal{D}\varphi = \prod_x \int d\varphi(x)$ may be converted to Fourier spacetime.

$$\prod_x d\varphi(x) \leftrightarrow \prod_k d\tilde{\varphi}_R(k) d\tilde{\varphi}_I(k) \quad (3.3.13)$$

Hence,

$$\begin{aligned} & \int d^d x_1 \int d^d x_2 e^{ik_1 \cdot x_1} e^{ik_2 \cdot x_2} \frac{\int \mathcal{D}\varphi \varphi(x_1) \varphi(x_2) \exp(iS)}{\int \mathcal{D}\varphi \exp(iS)} \\ &= \frac{\prod_q \int d\tilde{\varphi}_R(q) \int d\tilde{\varphi}_I(q) \tilde{\varphi}(k_1) \tilde{\varphi}(k_2) \exp\left(\frac{i}{2} (q^2 - m^2 + i\epsilon) (\tilde{\varphi}_R(q)^2 + \tilde{\varphi}_I(q)^2)\right)}{\prod_{q'} \int d\tilde{\varphi}_R(q') \int d\tilde{\varphi}_I(q') \exp\left(\frac{i}{2} (q'^2 - m^2 + i\epsilon) (\tilde{\varphi}_R(q')^2 + \tilde{\varphi}_I(q')^2)\right)}. \end{aligned} \quad (3.3.14)$$

For the integration over the real and imaginary portions of the Fourier spacetime $\tilde{\varphi}$ to converge, we shall shift the $q^2 - m^2$ by $q^2 - m^2 + i\epsilon$. In more detail, each momentum slice integral goes as

$$\int_{\mathbb{R}} \exp((i/2)\alpha z^2) \dots dz, \quad \alpha \in \mathbb{R}. \quad (3.3.15)$$

The gaussian fluctuates wildly as $|z| \rightarrow \infty$ unless we shift $\alpha \rightarrow \alpha + i\epsilon$. Moreover, each momentum slice in the numerator and denominator cancels out except for the slices corresponding to $q = k_1$ and $q = k_2$. Furthermore, if $k_1 \neq \pm k_2$, then the integral over the $q = k_1$ slice and that over the $q = k_2$ slice are both odd integrands.

$$\int d\tilde{\varphi}_R \int d\tilde{\varphi}_I (\tilde{\varphi}_R + i\tilde{\varphi}_I) \exp\left(\frac{i}{2} (k_1^2 - m^2 + i\epsilon) (\tilde{\varphi}_R^2 + \tilde{\varphi}_I^2)\right) = 0 \quad (3.3.16)$$

Hence, the entire path integral is zero unless $k_1 = k_2$. When $k_1 = k_2$, the path integral becomes

$$\begin{aligned} & \int d^d x_1 \int d^d x_2 e^{ik_1 \cdot x_1} e^{ik_2 \cdot x_2} \frac{\int \mathcal{D}\varphi \varphi(x_1) \varphi(x_2) \exp(iS)}{\int \mathcal{D}\varphi \exp(iS)} \\ &= \frac{\int d\tilde{\varphi}_R \int d\tilde{\varphi}_I (\tilde{\varphi}_R^2 + i^2 \tilde{\varphi}_I^2) \exp\left(\frac{i}{2} (k_1^2 - m^2 + i\epsilon) (\tilde{\varphi}_R^2 + \tilde{\varphi}_I^2)\right)}{\int d\tilde{\varphi}_R \int d\tilde{\varphi}_I \exp\left(\frac{i}{2} (k_1^2 - m^2 + i\epsilon) (\tilde{\varphi}_R^2 + \tilde{\varphi}_I^2)\right)}, \end{aligned} \quad (3.3.17)$$

where we have thrown out those terms linear in $\tilde{\varphi}_R$ and in $\tilde{\varphi}_I$. But this integral is zero because $i^2 = -1$. Hence, the remaining possibility is when $k_1 = -k_2$. In such a situation, $\tilde{\varphi}(-k_1) = \tilde{\varphi}(k_1)^*$, $\tilde{\varphi}(k_1)\tilde{\varphi}(-k_1) = |\tilde{\varphi}(k_1)|^2$, and the path integral becomes

$$\begin{aligned} & \int d^d x_1 \int d^d x_2 e^{ik_1 \cdot x_1} e^{ik_2 \cdot x_2} \frac{\int \mathcal{D}\varphi \varphi(x_1) \varphi(x_2) \exp(iS)}{\int \mathcal{D}\varphi \exp(iS)} \\ &= \frac{\int d\tilde{\varphi}_R \int d\tilde{\varphi}_I (\tilde{\varphi}_R^2 + \tilde{\varphi}_I^2) \exp\left(\frac{i}{2} (k_1^2 - m^2 + i\epsilon) (\tilde{\varphi}_R^2 + \tilde{\varphi}_I^2)\right)}{\int d\tilde{\varphi}_R \int d\tilde{\varphi}_I \exp\left(\frac{i}{2} (k_1^2 - m^2 + i\epsilon) (\tilde{\varphi}_R^2 + \tilde{\varphi}_I^2)\right)} \end{aligned} \quad (3.3.18)$$

$$= 2 \frac{\int dz z^2 \exp\left(\frac{i}{2} (k_1^2 - m^2 + i\epsilon) z^2\right)}{\int dz \exp\left(\frac{i}{2} (k_1^2 - m^2 + i\epsilon) z^2\right)} = \frac{2i}{k_1^2 - m^2 + i\epsilon}. \quad (3.3.19)$$

Peskin & Schroeder found some way to argue away the 1/2 in the exponential; this would yield $i/(k^2 - m^2 + i\epsilon)$. Notice, too, this is why free field theories are oftentimes dubbed 'gaussian' – the exponential is in fact gaussian, because the action is quadratic in the fields.

In any case, let us arrive at this answer by completing the square instead. To this end we follow Schwinger and first introduce a linear coupling to the scalar field via

$$Z' \equiv \frac{\int \mathcal{D}\varphi \exp \left[i \int d^d x \left(\frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2 + J(x)\varphi(x) \right) \right]}{\int \mathcal{D}\varphi \exp \left[i \int d^d x \left(\frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2 \right) \right]} \quad (3.3.20)$$

Let us define

$$\tilde{\varphi}(k) \equiv \tilde{\phi}(k) + i\tilde{G}(k)\tilde{J}(k). \quad (3.3.21)$$

The Lagrangian in the numerator in Fourier spacetime now reads

$$\mathcal{L}_k[J] \equiv \frac{1}{2}(k^2 - m^2 + i\epsilon)|\tilde{\varphi}|^2 + \frac{1}{2}\tilde{J}(k)\tilde{\varphi}(-k) + \frac{1}{2}\tilde{J}(-k)\tilde{\varphi}(k) \quad (3.3.22)$$

$$\begin{aligned} &= \frac{1}{2}(k^2 - m^2 + i\epsilon) \left(|\tilde{\phi}|^2 + |\tilde{G}(k)|^2|\tilde{J}|^2 + i\tilde{G}(k)\tilde{J}(k)\tilde{\phi}(-k) - i\tilde{G}(k)^*\tilde{J}(-k)\tilde{\phi}(k) \right) \\ &+ \frac{1}{2}\tilde{J}(k)\tilde{\phi}(-k) - \frac{i}{2}\tilde{J}(k)\tilde{G}(k)^*\tilde{J}(-k) + \frac{1}{2}\tilde{J}(-k)\tilde{\phi}(k) + \frac{i}{2}\tilde{J}(-k)\tilde{G}(k)\tilde{J}(k). \end{aligned} \quad (3.3.23)$$

Focusing on the terms linear in $\tilde{\phi}$,

$$\mathcal{L}_k[J] = \frac{1}{2} \left((k^2 - m^2 + i\epsilon)i\tilde{G}(k)\tilde{J}(k) + \tilde{J}(k) \right) \tilde{\phi}(-k) + \text{c.c.} + \dots \quad (3.3.24)$$

If we choose

$$\tilde{G}(k) = \frac{i}{k^2 - m^2 + i\epsilon} \quad (3.3.25)$$

all the linear-in- $\tilde{\phi}$ terms will vanish and we obtain

$$\begin{aligned} \mathcal{L}_k[J] &= \frac{1}{2}(k^2 - m^2 + i\epsilon)|\tilde{\phi}|^2 \\ &+ \frac{1}{2}\tilde{J}(k)\frac{1}{k^2 - m^2 - i\epsilon}\tilde{J}(-k) - \frac{1}{2}\tilde{J}(k)\frac{1}{k^2 - m^2 - i\epsilon}\tilde{J}(-k) + \frac{i}{2}\tilde{J}(-k)\tilde{G}(k)\tilde{J}(k) \end{aligned} \quad (3.3.26)$$

$$= \frac{1}{2}\tilde{\phi}(k)(k^2 - m^2 + i\epsilon)\tilde{\phi}(-k) - \frac{1}{2}\tilde{J}(k)(k^2 - m^2 + i\epsilon)^{-1}\tilde{J}(-k) \quad (3.3.27)$$

The \tilde{J} term does not take part in the functional integral, while the $\tilde{\phi}$ terms simply correspond to the functional integral without \tilde{J} s. At this point, the functional integrals in the numerator and denominator cancel, and we are left with

$$Z' = \exp \left[\frac{1}{2} \int_k i\tilde{J}(k) \frac{i}{k^2 - m^2 + i\epsilon} i\tilde{J}(-k) \right] \quad (3.3.28)$$

$$= \exp \left[\frac{1}{2} \int_{x,x'} iJ(x)G_{\text{F}}(x-x')iJ(x') \right] \quad (3.3.29)$$

This tells us

$$\left. \frac{\delta^2 Z'}{\delta(iJ(z))\delta(iJ(z'))} \right|_{J=0} = G_F(x - x'). \quad (3.3.30)$$

In Fourier spacetime,

$$\left. \frac{\delta^2 Z'}{\delta(i\tilde{J}(k))\delta(i\tilde{J}(k'))} \right|_{J=0} = \frac{i}{k^2 - m^2 + i\epsilon} (2\pi)^d \delta^{(d)}(k + k'). \quad (3.3.31)$$

On the other hand, by directly differentiating the ratio of path integrals,

$$\left. \frac{\delta^2 Z'}{\delta(iJ(z))\delta(iJ(z'))} \right|_{J=0} = \frac{\int \mathcal{D}\varphi \varphi(z)\varphi(z') \exp \left[i \int d^d x \left(\frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2 \right) \right]}{\int \mathcal{D}\varphi \exp \left[i \int d^d x \left(\frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2 \right) \right]} \quad (3.3.32)$$

$$= G_F(x - x'). \quad (3.3.33)$$

Hence, we have shown that ratios of the path integrals in Z' does indeed generate the two point function.

Odd Point Functions Vanish Let us also note that all odd point functions vanish.

$$\frac{\int \mathcal{D}\varphi \varphi(z_1) \dots \varphi(z_{2n+1}) \exp \left[i \int d^d x \left(\frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2 \right) \right]}{\int \mathcal{D}\varphi \exp \left[i \int d^d x \left(\frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2 \right) \right]} = 0. \quad (3.3.34)$$

We may see this by directly differentiating Z' .

$$\sum_{s=0}^{\infty} \frac{(1/2)^s}{s!} \frac{\delta^{2n+1}}{\delta(iJ(z_1))\delta(iJ(z_2)) \dots \delta(iJ(z_{2n+1}))} \Big|_{J=0} iJ_{x_1} G_F^{x_1, x'_1} iJ_{x'_1} \dots iJ_{x_s} G_F^{x_s, x'_s} iJ_{x'_s} \quad (3.3.35)$$

But each term in the sum has even number of J s. So there must be at least one J leftover from differentiation; which will render the entire expression trivial upon setting $J = 0$.

Even Point Functions & Wick's Theorem For even point function, the result may be expressed through Wick's theorem:

The $2n$ point function is simply the sum over all fully contracted fields, where which Wick contracted pair of fields is replaced with the corresponding Feynman Green's function.

(I forgot how to do Wick contractions in L^AT_EX.) Simply differentiate.

$$\sum_{s=0}^{\infty} \frac{(1/2)^s}{s!} \frac{\delta^{2n}}{\delta(iJ(z_1))\delta(iJ(z_2)) \dots \delta(iJ(z_{2n}))} \Big|_{J=0} iJ_{x_1} G_F^{x_1, x'_1} iJ_{x'_1} \dots iJ_{x_s} G_F^{x_s, x'_s} iJ_{x'_s} \quad (3.3.36)$$

Only the $s = n$ term is relevant, because there are exactly $2n$ J s there. For every

$$\frac{\delta iJ(x_k)}{\delta iJ(z_i)} G_F^{x_k, x'_k} \frac{\delta iJ(x'_k)}{\delta iJ(z'_i)} \quad (3.3.37)$$

there is also a

$$\frac{\delta iJ(x_k)}{\delta iJ(z'_i)} G_F^{x_k, x'_i} \frac{\delta iJ(x'_i)}{\delta iJ(z_i)}. \quad (3.3.38)$$

Because there are s pairs of J s, the $(1/2)^s$ will cancel out. Moreover, since there are precisely $s!$ ways to re-arrange s Green's functions of fixed arguments, the $s!$ in the denominator will cancel out too. For example, we have the following 4 point function.

$$\langle \varphi_1 \varphi_2 \varphi_3 \varphi_4 \rangle = G_F^{1,2} G_F^{3,4} + G_F^{1,3} G_F^{2,4} + G_F^{1,4} G_F^{2,3} \quad (3.3.39)$$

To sum: for the free massive scalar theory, the n point function is completely fixed (up to combinatorics) once the two point function is known. This is analogous to the situation in statistics: when the underlying probability distribution is Gaussian, all the $n \geq 1$ moments are completely determined by the variance.

Problem 3.25. Write down the 6- and 8-point functions. □

Nonlinearities We may now proceed to construct perturbation theory. One simply breaks up the action into quadratic-in-fields piece S_2 plus the higher-than-quadratic-in-fields $S_{n \geq 3}$, followed by Taylor expanding the exponential in powers of the latter.

The general n point function with respect to the *full interacting vacuum* is now

$$\begin{aligned} & \langle \Omega | T \varphi(z_1) \dots \varphi(z_n) | \Omega \rangle \\ &= \frac{\int \mathcal{D}\varphi \varphi(z_1) \dots \varphi(z_n) \exp \left[i \int d^d x \left(\frac{1}{2} (\partial\varphi)^2 - \frac{m^2}{2} \varphi^2 \right) + i \sum_{n \geq 3} S_n[\varphi] \right]}{\int \mathcal{D}\varphi \exp \left[i \int d^d x \left(\frac{1}{2} (\partial\varphi)^2 - \frac{m^2}{2} \varphi^2 \right) + i \sum_{n \geq 3} S_n[\varphi] \right]}. \end{aligned} \quad (3.3.40)$$

We will again employ Schwinger's source trick, by considering the following generating functional.

$$Z'[J] \equiv \frac{\int \mathcal{D}\varphi \exp \left[i \int d^d x \left(\frac{1}{2} (\partial\varphi)^2 - \frac{m^2}{2} \varphi^2 + J \cdot \varphi \right) + i \sum_{n \geq 3} S_n[\varphi] \right]}{\int \mathcal{D}\varphi \exp \left[i \int d^d x \left(\frac{1}{2} (\partial\varphi)^2 - \frac{m^2}{2} \varphi^2 \right) + i \sum_{n \geq 3} S_n[\varphi] \right]} \quad (3.3.41)$$

The general n point function may again be obtained from differentiation.

$$\langle \Omega | T \varphi(z_1) \dots \varphi(z_n) | \Omega \rangle = \left. \frac{\delta^n Z''}{\delta(iJ(z_1)) \delta(iJ(z_2)) \dots \delta(iJ(z_n))} \right|_{J=0} \quad (3.3.42)$$

Now, the general path integral with external sources may be shown to be the functional determinant \mathcal{N} of the kinetic term of the action times the exponential of the 'vacuum bubbles' (related to vacuum energy) times the exponential of the sum of fully connected Feynman diagrams involving the external sources. For e.g.,

$$\begin{aligned} \int \mathcal{D}\varphi \exp \left[iS[\varphi] + i \int J \cdot \varphi d^d x \right] &= \mathcal{N} \exp \left[\sum (\text{vacuum bubbles}) \right] \\ &\times \exp \left[\sum (\text{Fully connected Feynman diagrams involving } J) \right] \end{aligned} \quad (3.3.43)$$

Therefore, when taking the ratio of path integrals, only the fully connected diagrams remain.

$$Z'[J] = \exp \left[\sum (\text{Fully connected Feynman diagrams involving } J) \right] \quad (3.3.44)$$

For example, for the free massive scalar theory, the only Feynman diagram was the one involving

$$\langle 0 | \mathbb{T} \frac{1}{2!} \int iJ_1 \varphi_1 \int iJ_2 \varphi_2 | 0 \rangle = \frac{1}{2!} \int_{1,2} iJ_1 G_F^{1,2} iJ_2. \quad (3.3.45)$$

Example: φ^4 Theory A common textbook example involves φ^4 theory, defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4}\varphi^4. \quad (3.3.46)$$

An example of a vacuum bubble would be one generated by one power of the φ^4 part of the Lagrangian.

$$\langle 0 | \mathbb{T} \frac{-i\lambda}{4} \int_x \varphi_x^4 | 0 \rangle \quad (3.3.47)$$

An example of a fully connected diagram involving 1 loop would be

$$\langle 0 | \mathbb{T} \frac{1}{2!} \int_{1,2} iJ_1 \varphi_1 iJ_2 \varphi_2 \frac{-i\lambda}{4} \int_x \varphi_x^4 | 0 \rangle; \quad (3.3.48)$$

from which one can extract the 2-point one loop function

$$\langle \Omega | \mathbb{T} \varphi_1 \varphi_2 | \Omega \rangle_{\text{one loop}}. \quad (3.3.49)$$

Summary To sum: the n -point function may be computed perturbatively as the sum of all Feynman diagrams without any disconnected vacuum bubbles:

$$\begin{aligned} & \langle \Omega | \mathbb{T} \{ \varphi(x_1) \dots \varphi(x_n) \} | \Omega \rangle \\ &= \sum_{\ell=0}^{\infty} \frac{i^\ell}{\ell!} \int_{z_1 \dots z_\ell} \langle 0 | \mathbb{T} \{ \varphi(x_1) \dots \varphi(x_n) \mathcal{L}_I[z_1] \dots \mathcal{L}_I[z_\ell] \} | 0 \rangle_{\text{No disconnected vacuum bubbles}}. \end{aligned} \quad (3.3.50)$$

\hbar Power Counting $L = P - V + 1$

Problem 3.26. 1D Euclidean φ^4 QFT Compute the 2 point Euclidean correlator:

$$\langle \varphi(x_1) \varphi(x_2) \rangle \equiv \frac{\int \mathcal{D}\varphi \varphi(x_1) \varphi(x_2) \exp(-S_E)}{\int \mathcal{D}\varphi \exp(-S_E)}, \quad (3.3.51)$$

$$S_E \equiv \int_{\mathbb{R}} dx \left(\frac{1}{2} \varphi'(x)^2 + \frac{m^2}{2} \varphi(x)^2 + \frac{\lambda}{4!} \varphi(x)^4 \right); \quad (3.3.52)$$

up to $\mathcal{O}(\lambda)$. Note that, even though this is a Euclidean QFT, the perturbation theory is similar to that in Lorentzian QFT. Hint: First solve for the *non-interacting* two point function. \square

3.4 T Matrix/Scattering Amplitudes

The T -matrix, or scattering amplitude, is the fully connected amputated Feynman diagram involving the appropriate external momentum lines. (The S matrix is the sum of amputated Feynman diagrams .) The justification of this prescription is known as the LSZ reduction formula.

$$\begin{aligned}
 & {}_s \langle \{p'_j\} | iT | \{p_i\} \rangle_s \\
 &= \sum_{\ell=1}^{\infty} \frac{i}{\ell!} \int_{z_1 \dots z_\ell} ({}_s \langle \{p'_j\} | \mathbb{T} \{ \mathcal{L}[z_1] \dots \mathcal{L}[z_\ell] \} | \{p_i\} \rangle_s)_{\text{Fully connected, amputated}}
 \end{aligned} \tag{3.4.1}$$

This formula holds provided we define Wick contractions to also include that with the external momentum eigenstates. For scalars, contraction with external momentum states simply yields unity.

4 (3+1)D Spinor Fields, U_1 Symmetry – Quantum Electrodynamics (QED)

4.1 Invariants and Covariants

Left- and Right-Handed Spinors

Tensors from Spinors in (3+1)D we may identify the following scalar/tensor objects, built out of fermion bilinears.

$$\bar{\psi}\psi \text{ (Scalar)}, \quad \bar{\psi}\gamma^\mu\psi \text{ (Vector)}, \quad \bar{\psi}\gamma^{[\mu}\gamma^{\nu]}\psi \text{ (Rank-2 tensor)}, \quad (4.1.1)$$

$$\bar{\psi}\gamma^\mu\gamma^5\psi \text{ (Pseudo-vector)}, \quad \bar{\psi}\gamma^5\psi \text{ (Pseudo-Scalar)} \quad (4.1.2)$$

The ‘pseudo-’ here means, under parity $\bar{\psi}\gamma^i\gamma^5\psi$ goes into itself; and $\bar{\psi}\gamma^5\psi$ goes into $-\bar{\psi}\gamma^5\psi$.

The $\{\gamma^\mu\}$ are the Dirac γ -matrices obeying the Clifford Algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}_{4\times 4} \quad (4.1.3)$$

and

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \{\gamma^5, \gamma^\mu\} = 0. \quad (4.1.4)$$

The commonly used Weyl (aka chiral) basis reads

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}, \quad \gamma^5 = \begin{bmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{bmatrix} \quad (4.1.5)$$

The $\{\sigma^i\}$ are the Pauli matrices; while $\sigma^0 = \mathbb{I}_{2\times 2}$. Below, we use the notation

$$\not{p} \equiv \gamma^\mu p_\mu. \quad (4.1.6)$$

Scalars from Spinors To form Lagrangians for spinor fields, we wish to construct Lorentz invariants. Since we know the Dirac equation $(i\not{\partial} - m)\psi$ is a first order equation, and since ∂_μ transforms as a 1-form, we may write down $\bar{\psi}i\not{\partial}\psi$ is our first scalar. In fact, this alone would yield the action of the EoMs for left- and right-handed massless Weyl fermions:

$$S = \int d^4x \frac{1}{2} (\bar{\psi}i\not{\partial}\psi - \partial_\mu \bar{\psi}\gamma^\mu\psi) \quad (4.1.7)$$

$$\delta_{\psi^\dagger} S = 0 \quad \Rightarrow \quad i\not{\partial}\psi = 0. \quad (4.1.8)$$

We have performed an integration-by-parts, to make the action manifestly Hermitian.

This allows us to recognize, the mass dimension of ψ is 3/2. Moreover, $\bar{\psi}\psi$ itself has mass dimension 3 and therefore its coefficient is precisely that of mass – i.e., the massive Dirac equation comes from extremizing

$$S = \int d^4x \bar{\psi}(i\not{\partial} - m)\psi \quad (4.1.9)$$

$$\delta_{\psi^\dagger} S = 0 \quad \Rightarrow \quad (i\not{\partial} - m)\psi = 0. \quad (4.1.10)$$

If we wish to allow the fermion to be electrically charged, we must write down an interaction of the form $A_\mu J^\mu$, where J^μ must be conserved on-shell (i.e., when the EoM is satisfied) and transforms as a four vector. Indeed, we may identify the electromagnetic current

$$J^\mu = e\bar{\psi}\gamma^\mu\psi. \quad (4.1.11)$$

Introducing a ‘covariant derivative’

$$D_\mu \equiv \partial_\mu - ieA_\mu, \quad (4.1.12)$$

we may now write the fermionic part of the QED action as

$$S_\psi = \int d^4x \bar{\psi}(i\not{D} - m)\psi, \quad (4.1.13)$$

$$\delta_{\psi^\dagger} S = 0 \quad \Rightarrow \quad (i\not{D} - m)\psi = 0. \quad (4.1.14)$$

U₁ Symmetry & Covariance Notice all fermion Lorentz invariants are invariant under the U₁ phase rotations

$$\psi \rightarrow e^{i\theta}\psi \quad \text{and} \quad \psi^\dagger \rightarrow \psi^\dagger e^{-i\theta} \quad (4.1.15)$$

because the ψ^\dagger and ψ occurs in pairs. This is certainly true when θ is a mere constant – i.e., when the U₁ transformation is *global* – but what happens when $\theta = \theta(x)$ is *local*, meaning it depends on the spacetime location?

To study this question, let us remember, under a gauge transformation, the vector potential itself transforms as

$$A_\mu \rightarrow A_\mu + \partial_\mu\theta(x). \quad (4.1.16)$$

Let us now consider acting the covariant derivative on the spinor field, followed by rotating ψ by a local U₁ phase rotation.

$$D_\mu\psi \rightarrow (\partial_\mu - ieA_\mu)(e^{ie\theta(x)}\psi) \quad (4.1.17)$$

$$= e^{ie\theta}(\partial_\mu + ie\partial_\mu\theta - ieA_\mu)\psi \quad (4.1.18)$$

$$= e^{ie\theta}(\partial_\mu + ie(\partial_\mu\theta - A_\mu))\psi. \quad (4.1.19)$$

We see that, if A_μ shifted by the gauge transformation in eq. (4.1.16), the covariant derivative of ψ would transform *covariantly* under local U₁ transformations. Namely, under the simultaneous transformations

$$\psi \rightarrow e^{ie\theta}\psi \quad \text{and} \quad A_\mu \rightarrow A_\mu + \partial_\mu\theta; \quad (4.1.20)$$

we have

$$D_\mu\psi \rightarrow e^{ie\theta}D_\mu\psi. \quad (4.1.21)$$

This tells us the full QED action

$$S_{\text{QED}} = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi \right) \quad (4.1.22)$$

is not only globally, but also locally U₁ ‘gauge-invariant’.

Problem 4.1. Noether current of global U_1 Work out the Noether current of global U_1 symmetry enjoyed by the fermionic action in eq. (4.1.13). You should find it to be proportional to $\bar{\psi}\gamma^\mu\psi$. In QFT, this leads to the conservation of fermion number. \square

Problem 4.2. Noether Current of Poincaré Symmetry Work out the Noether currents T^μ_α of spacetime translations for the action in eq. (4.1.13). Do the same for Lorentz transformations. Can you derive a relationship like that in eq. (2.3.28)? \square

Problem 4.3. Commutator of Covariant Derivatives Show that

$$[D_\mu, D_\nu]\psi = -ieF_{\mu\nu}\psi, \quad (4.1.23)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor. This serves as an analog to the commutator of covariant derivatives in differential geometry, which yields the Riemann curvature tensor. \square

Problem 4.4. Abelian Higgs Model We may provide a similar argument as above, to construct a local U_1 invariant complex scalar field theory coupled to electromagnetic fields. The resulting Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\varphi\overline{D^\mu\varphi} - V(|\varphi|^2) \quad (4.1.24)$$

$$V(|\varphi|^2) \equiv \frac{\lambda}{4}(|\varphi|^2 - \nu^2)^2, \quad \nu \in \mathbb{R}. \quad (4.1.25)$$

- Identify the electromagnetic current. Write down the equations of motion for both the electromagnetic fields and for the scalar field φ .
- Work out the Noether current of *global* U_1 symmetry for φ . A complex scalar field with global U_1 symmetry, just like its fermionic counterpart, has a conserved number current.
- Verify that $A_\mu = 0$ and $\varphi = \nu e^{i\chi}$ (for real constant χ) is a solution. Describe the trajectory of $\varphi = \nu e^{i\chi}$ within the potential $V(|\varphi|^2)$ versus φ field space 3D plot, as χ is varied from 0 to 2π .
- Consider inserting $\varphi = (\nu + \rho(x))e^{ie\phi(x)}$, where ρ and ϕ both depend on spacetime, into the Lagrangian. Explain how A_μ may be transformed to remove ϕ . Then work out the portion of the Lagrangian quadratic in A_μ and ρ . What are the (classical) masses of A_μ and ρ ?
- *Remarks:* We know a massless photon has 2 degrees-of-freedom; while a complex scalar should have 2. But due to local gauge-invariance, what you are showing is that the dynamics may be converted into a massive gauge boson (with 3 degrees of freedom); and one real scalar ρ . The photon acquires a mass – this is a model for the Meissner effect in superconductors.

Problem 4.5. (1+1)D Dirac Equation Coupled to a Scalar Kink In this problem we will study the Dirac equation for ψ coupled to a scalar field ϕ in (1 + 1)D flat spacetime with Cartesian coordinates $x^\mu = (t, x)$.

- To build the Dirac equation in 2D, we need to find the appropriate γ -matrices. Explain why the following choice is a valid one. Denoting the Pauli matrices by $\{\sigma^i | i = 1, 2, 3\}$, define

$$\gamma^0 \equiv \sigma^3 \quad \text{and} \quad \gamma^1 \equiv i\sigma^1. \quad (4.1.26)$$

Since Pauli matrices are 2×2 matrices, this tells us the Dirac spinor ψ is a 2-component object.

- We will now solve for the stationary state wave functions of the 2D Dirac equation coupled to a scalar field ϕ ,

$$(i\gamma^\mu \partial_\mu - g\phi)\psi = 0, \quad (4.1.27)$$

with the scalar field taking the specific form

$$\phi(x) = \nu \tanh(mx/2), \quad \nu, m \in \mathbb{R}. \quad (4.1.28)$$

Such a ϕ , known as a \mathbb{Z}_2 kink, arises as a solution to a relativistic field theory with a double well potential.

Consider the stationary state wave function ansatz

$$\psi(t, x) = \frac{e^{-iEt}}{\sqrt{2}} \begin{bmatrix} \beta_+(x) - \beta_-(x) \\ \beta_+(x) + \beta_-(x) \end{bmatrix} \quad (4.1.29)$$

Show that, upon performing the re-scaling

$$x \equiv \frac{2z}{m}, \quad g \equiv \frac{em}{2\nu}, \quad E \equiv \frac{m}{2}\bar{E}, \quad (4.1.30)$$

the following pair of Schrödinger equations are obtained:

$$-\partial_z^2 \beta_\pm(z) + (e^2 - e(e \pm 1)\text{sech}^2(z)) \beta_\pm(z) = \bar{E}^2 \beta_\pm(z). \quad (4.1.31)$$

- The fermion ψ may bind to the \mathbb{Z}_2 kink. The energy levels of these bound states are

$$\bar{E}_n^\pm = \pm \sqrt{n(2e - n)}, \quad 0 \leq n < e, \quad n = 0, 1, 2, 3, \dots \quad (4.1.32)$$

Hence, we see that every positive energy solution has a negative energy counterpart – except for the zero energy mode $\bar{E}_0 = 0$. Solve the $\beta_\pm(z)$ for this zero mode, and write down the corresponding Dirac spinor $\psi_0 \equiv \psi(t, x; n = 0)$.

Bonus Can you solve the β_\pm (and, hence, ψ_n) for all the bound state wave functions? (Warning: It's hard work, involving the hypergeometric function ${}_2F_1$.)

Problem 4.6. (2+1)D Dirac Equation

- Show that $\gamma^0 \equiv \sigma^3$, $\gamma^1 \equiv i\sigma^1$ and $\gamma^2 \equiv i\sigma^2$ satisfies the Clifford Algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}$.

5 Quantum Field Theory of Fermions

5.1 Canonical Quantization & Fermi-Dirac Statistics

We start with the free Dirac Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi. \quad (5.1.1)$$

The conjugate momentum to ψ is

$$\Pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = i\psi^\dagger. \quad (5.1.2)$$

The Hamiltonian is

$$\mathcal{H} = i\psi^\dagger \partial_0 \psi - \psi^\dagger i\partial_0 \psi - \bar{\psi}(i\vec{\gamma} \cdot \vec{\nabla} - m)\psi \quad (5.1.3)$$

$$= -\bar{\psi}(i\vec{\gamma} \cdot \vec{\nabla} - m)\psi. \quad (5.1.4)$$

Start with the mode expansion:

$$\psi(x) = \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \sum_s (a_k^s u_k^s e^{-ik \cdot x} + (b_k^s)^\dagger v_k^s e^{+ik \cdot x}) \quad (5.1.5)$$

where

$$u_p^s = \begin{bmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{bmatrix} \quad v_p^s = \begin{bmatrix} \sqrt{p \cdot \sigma} \eta^s \\ -\sqrt{p \cdot \bar{\sigma}} \eta^s \end{bmatrix}. \quad (5.1.6)$$

They satisfy the Dirac equation.

$$(\not{p} - m) u_p^r = 0 = (\not{p} + m) v_p^r \quad (5.1.7)$$

$$\bar{u}_p^r (\not{p} - m) = 0 = \bar{v}_p^r (\not{p} + m) \quad (5.1.8)$$

Commutation Relations Let us impose equal time commutation relations.

$$[\psi_\alpha(t, \vec{x}), \Pi_\beta(t, \vec{x}')] = i\delta^{(3)}(\vec{x} - \vec{x}')\delta_{\alpha\beta} \quad (5.1.9)$$

Starting from the LHS,

$$\begin{aligned} & [\psi(t, \vec{x}), \Pi(t, \vec{x}')] = i [\psi(t, \vec{x}), \psi^\dagger(t, \vec{x}')] \\ &= i \int_{\vec{k}, \vec{k}'} \sum_{s, s'} \frac{1}{\sqrt{2E_k} \sqrt{2E_{k'}}} \left[u_k^s a_k^s e^{-ik \cdot x} + v_k^s (b_k^s)^\dagger e^{ik \cdot x}, (u_{k'}^{s'})^\dagger (a_{k'}^{s'})^\dagger e^{ik' \cdot x'} + (v_{k'}^{s'})^\dagger (b_{k'}^{s'}) e^{-ik' \cdot x'} \right] \\ &= i \int_{\vec{k}, \vec{k}'} \sum_{s, s'} \frac{1}{\sqrt{2E_k} \sqrt{2E_{k'}}} \left(u_k^s (\bar{u}_{k'}^{s'})^\dagger e^{-ik \cdot x + ik' \cdot x'} \left[a_k^s, (a_{k'}^{s'})^\dagger \right] - v_k^s (\bar{v}_{k'}^{s'}) e^{ik \cdot x - ik' \cdot x'} \left[b_{k'}^{s'}, (b_k^s)^\dagger \right] \right) \gamma^0 \end{aligned}$$

where we have assumed the a s and b commute. By imposing

$$\left[a_k^s, (a_{k'}^{s'})^\dagger \right] = \delta_{s'}^s (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') = - \left[b_{k'}^{s'}, (b_k^s)^\dagger \right], \quad (5.1.10)$$

followed by the computations

$$\sum_{s=1,2} u_p^r \bar{u}_p^s = \sum_s \begin{bmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{bmatrix} \begin{bmatrix} (\xi^s)^\dagger \sqrt{p \cdot \bar{\sigma}} \\ (\xi^s)^\dagger \sqrt{p \cdot \sigma} \end{bmatrix}^T \quad (5.1.11)$$

$$= \begin{bmatrix} m & p \cdot \sigma \\ p \cdot \bar{\sigma} & m \end{bmatrix} = \not{p} + m. \quad (5.1.12)$$

$$\sum_{s=1,2} v_p^r \bar{v}_p^s = \sum_s \begin{bmatrix} \sqrt{p \cdot \sigma} \eta^s \\ -\sqrt{p \cdot \bar{\sigma}} \eta^s \end{bmatrix} \begin{bmatrix} -(\eta^s)^\dagger \sqrt{p \cdot \bar{\sigma}} \\ (\eta^s)^\dagger \sqrt{p \cdot \sigma} \end{bmatrix}^T \quad (5.1.13)$$

$$= \begin{bmatrix} -m & p \cdot \sigma \\ p \cdot \bar{\sigma} & -m \end{bmatrix} = \not{p} - m; \quad (5.1.14)$$

the LHS of the equal time commutator reads

$$[\psi(t, \vec{x}), \Pi(t, \vec{x}')] \quad (5.1.15)$$

$$= i \int_{\vec{k}} \frac{1}{2E_k} \left((k + m) e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} + (k - m) e^{-i\vec{k} \cdot (\vec{x} - \vec{x}')} \right) \gamma^0$$

$$= \frac{i}{2} \int_{\vec{k}} \frac{1}{E_k} \left((\gamma^0 E_k + \gamma^j k_j + m) e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} + (\gamma^0 E_k - \gamma^j k_j - m) e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} \right) \gamma^0 \quad (5.1.16)$$

$$= i \delta^{(3)}(\vec{x} - \vec{x}'). \quad (5.1.17)$$

All seems well! But let us now compute the Hamiltonian.

$$H(t) \equiv \int_{\mathbb{R}^3} \mathcal{H}(t, \vec{x}) d^3 \vec{x} \quad (5.1.18)$$

$$= - \int_{\mathbb{R}^3} \bar{\psi} \left(i \gamma^0 \partial_0 + i \vec{\gamma} \cdot \vec{\nabla} - m - i \gamma^0 \partial_0 \right) \psi = i \int_{\mathbb{R}^3} \psi^\dagger \partial_0 \psi d^3 \vec{x} \quad (5.1.19)$$

$$= \int_{\vec{x}, \vec{k}, \vec{k}'} \sum_{s, s'} \frac{1}{\sqrt{2E_k} \sqrt{2E_{k'}}} \left((a_k^s)^\dagger (u_k^s)^\dagger e^{+ik \cdot x} + (v_k^s)^\dagger (b_k^s)^\dagger e^{-ik \cdot x} \right) \times E_{k'} \left(a_{k'}^{s'} u_{k'}^{s'} e^{-ik' \cdot x} - v_{k'}^{s'} (b_{k'}^{s'})^\dagger e^{ik' \cdot x} \right). \quad (5.1.20)$$

Let us note that

$$(u_{\vec{p}}^s)^\dagger v_{-\vec{p}}^r = (\xi^s)^\dagger \sqrt{p \cdot \sigma} \sqrt{p \cdot \bar{\sigma}} \eta^r + (\xi^s)^\dagger \sqrt{p \cdot \bar{\sigma}} (-) \sqrt{p \cdot \sigma} \eta^r \quad (5.1.21)$$

$$= (\xi^s)^\dagger \eta^r (m - m) = 0. \quad (5.1.22)$$

Likewise

$$(v_{\vec{p}}^s)^\dagger u_{-\vec{p}}^r = (\eta^s)^\dagger \sqrt{p \cdot \sigma} \sqrt{p \cdot \bar{\sigma}} \xi^r - (\eta^s)^\dagger \sqrt{p \cdot \bar{\sigma}} \sqrt{p \cdot \sigma} \xi^r \quad (5.1.23)$$

$$= (m - m) (\eta^s)^\dagger \xi^r = 0. \quad (5.1.24)$$

Therefore,

$$H(t) = \frac{1}{2} \int_{\vec{k}} \sum_{s, s'} \left((a_k^s)^\dagger a_{k'}^{s'} (u_k^s)^\dagger u_{k'}^{s'} - (v_k^s)^\dagger v_{k'}^{s'} (b_k^s)^\dagger (b_{k'}^{s'})^\dagger \right). \quad (5.1.25)$$

We have

$$(u_p^r)^\dagger u_p^s = (\xi^r)^\dagger p \cdot \sigma \xi^s + (\xi^r)^\dagger p \cdot \bar{\sigma} \xi^s \quad (5.1.26)$$

$$= 2E_p \delta_s^r \quad (5.1.27)$$

and

$$(v_p^r)^\dagger v_p^s = (\eta^r)^\dagger p \cdot \sigma \eta^s + (\eta^r)^\dagger p \cdot \bar{\sigma} \eta^s \quad (5.1.28)$$

$$= 2E_p \delta_s^r. \quad (5.1.29)$$

We have arrived at

$$H = \int \frac{d^3 \vec{k}}{(2\pi)^3} \sum_s E_k ((a_k^s)^\dagger a_k^s - (b_k^s)(b_k^s)^\dagger). \quad (5.1.30)$$

If we define

$$N_a \equiv \int \frac{d^3 \vec{k}}{(2\pi)^3} \tilde{N}_a(\vec{k}) \equiv \int \frac{d^3 \vec{k}}{(2\pi)^3} \sum_s (a_k^s)^\dagger a_k^s \quad (5.1.31)$$

and

$$N_b \equiv \int \frac{d^3 \vec{k}}{(2\pi)^3} \tilde{N}_b(\vec{k}) \equiv \int \frac{d^3 \vec{k}}{(2\pi)^3} \sum_s b_k^s (b_k^s)^\dagger; \quad (5.1.32)$$

let us note that

$$[N_a, (a_p^r)^\dagger] = \int \frac{d^3 \vec{k}}{(2\pi)^3} \sum_s (a_k^s)^\dagger [a_k^s, (a_p^r)^\dagger] \quad (5.1.33)$$

$$= (a_p^r)^\dagger \quad (5.1.34)$$

whereas

$$[N_b, (b_p^r)] = \int \frac{d^3 \vec{k}}{(2\pi)^3} \sum_s b_k^s [(b_k^s)^\dagger, b_p^r] \quad (5.1.35)$$

$$= b_p^r. \quad (5.1.36)$$

In words: N_a is the number operator for particles created by $(a_p^r)^\dagger$ and destroyed by a_p^r ; while N_b is the number operator for particles created by b_p^r and destroyed by $(b_p^r)^\dagger$. Furthermore, we may now write the Hamiltonian as

$$H = \int \frac{d^3 \vec{k}}{(2\pi)^3} E_k (\tilde{N}_a(\vec{k}) - \tilde{N}_b(\vec{k})) \quad (5.1.37)$$

The Hamiltonian has no lower bound because of the $-$ sign in front of \tilde{N}_b !

Anti-Commutation Relations We now solve this unbounded-from-below problem with the Hamiltonian by imposing instead the anti-commutation relations

$$\{\psi(t, \vec{x}), \Pi(t, \vec{x}')\} = i\delta^{(3)}(\vec{x} - \vec{x}') \quad (5.1.38)$$

$$\{\psi(t, \vec{x}), \psi(t, \vec{x}')^\dagger\} = \delta^{(3)}(\vec{x} - \vec{x}'). \quad (5.1.39)$$

We will assume ψ and ψ^\dagger anti-commutes with itself

$$\{\psi(t, \vec{x}), \psi(t, \vec{x}')\} = 0 = \{\psi(t, \vec{x})^\dagger, \psi(t, \vec{x}')^\dagger\}. \quad (5.1.40)$$

As we will soon witness, these amount to imposing anti-commutation relations for the ladder operators

$$\{a_k^r, (a_{k'}^s)^\dagger\} = \delta_s^r (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') = \{b_k^r, (b_{k'}^s)^\dagger\} \quad (5.1.41)$$

$$\{a_k^r, a_{k'}^s\} = 0 = \{b_k^r, b_{k'}^s\}. \quad (5.1.42)$$

Starting from the LHS,

$$\begin{aligned} & \{\psi(t, \vec{x}), \psi^\dagger(t, \vec{x}')\} \\ &= \int_{\vec{k}, \vec{k}'} \sum_{s, s'} \frac{1}{\sqrt{2E_k} \sqrt{2E_{k'}}} \left\{ u_k^s a_k^s e^{-ik \cdot x} + v_k^s (b_k^s)^\dagger e^{ik \cdot x}, (u_{k'}^{s'})^\dagger (a_{k'}^{s'})^\dagger e^{ik' \cdot x'} + (v_{k'}^{s'})^\dagger (b_{k'}^{s'}) e^{-ik' \cdot x'} \right\} \\ &= \int_{\vec{k}, \vec{k}'} \sum_{s, s'} \frac{1}{\sqrt{2E_k} \sqrt{2E_{k'}}} \left(u_k^s (\bar{u}_{k'}^{s'})^\dagger e^{-ik \cdot x + ik' \cdot x'} \left\{ a_k^s, (a_{k'}^{s'})^\dagger \right\} + v_k^s (\bar{v}_{k'}^{s'}) e^{ik \cdot x - ik' \cdot x'} \left\{ b_{k'}^{s'}, (b_k^s)^\dagger \right\} \right) \gamma^0 \\ &= \int_{\vec{k}} \frac{1}{2E_k} \left((k + m) e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} + (k - m) e^{-i\vec{k} \cdot (\vec{x} - \vec{x}')} \right) \gamma^0 \end{aligned} \quad (5.1.43)$$

$$= \frac{1}{2} \int_{\vec{k}} \frac{1}{E_k} \left((\gamma^0 E_k + \gamma^j k_j + m) e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} + (\gamma^0 E_k - \gamma^j k_j - m) e^{-i\vec{k} \cdot (\vec{x} - \vec{x}')} \right) \gamma^0 \quad (5.1.44)$$

$$= \delta^{(3)}(\vec{x} - \vec{x}'). \quad (5.1.45)$$

Next, let us compute the Hamiltonian. Everything goes through as before.

$$H(t) \equiv \int_{\mathbb{R}^3} \mathcal{H}(t, \vec{x}) d^3 \vec{x} \quad (5.1.46)$$

$$= - \int_{\mathbb{R}^3} \bar{\psi} \left(i\gamma^0 \partial_0 + i\vec{\gamma} \cdot \vec{\nabla} - m - i\gamma^0 \partial_0 \right) \psi = i \int_{\mathbb{R}^3} \psi^\dagger \partial_0 \psi d^3 \vec{x} \quad (5.1.47)$$

$$\begin{aligned} &= \int_{\vec{x}, \vec{k}, \vec{k}'} \sum_{s, s'} \frac{1}{\sqrt{2E_k} \sqrt{2E_{k'}}} \left((a_k^s)^\dagger (u_k^s)^\dagger e^{+ik \cdot x} + (v_k^s)^\dagger (b_k^s) e^{-ik \cdot x} \right) \\ & \quad \times E_{k'} \left(a_{k'}^{s'} u_{k'}^{s'} e^{-ik' \cdot x} - v_{k'}^{s'} (b_{k'}^{s'})^\dagger e^{ik' \cdot x} \right). \end{aligned} \quad (5.1.48)$$

Hence,

$$H = \frac{1}{2} \int_{\vec{k}} \sum_{s, s'} \left((a_k^s)^\dagger a_k^{s'} (u_k^s)^\dagger u_k^{s'} - (v_k^s)^\dagger v_k^{s'} (b_k^s) (b_k^{s'})^\dagger \right) \quad (5.1.49)$$

$$= \int \frac{d^3 \vec{k}}{(2\pi)^3} \sum_s E_k \left((a_k^s)^\dagger a_k^s - (b_k^s) (b_k^s)^\dagger \right). \quad (5.1.50)$$

But now, we may say $(b_k^s)(b_k^s)^\dagger = \{(b_k^s), (b_k^s)^\dagger\} - (b_k^s)^\dagger(b_k^s)$ and

$$H = \int \frac{d^3\vec{k}}{(2\pi)^3} \sum_s E_k ((a_k^s)^\dagger a_k^s + (b_k^s)^\dagger(b_k^s)) + \text{constant}. \quad (5.1.51)$$

If we (re-)define

$$N_a \equiv \int \frac{d^3\vec{k}}{(2\pi)^3} \tilde{N}_a(\vec{k}) \equiv \int \frac{d^3\vec{k}}{(2\pi)^3} \sum_s (a_k^s)^\dagger a_k^s \quad (5.1.52)$$

and

$$N_b \equiv \int \frac{d^3\vec{k}}{(2\pi)^3} \tilde{N}_b(\vec{k}) \equiv \int \frac{d^3\vec{k}}{(2\pi)^3} \sum_s (b_k^s)^\dagger b_k^s; \quad (5.1.53)$$

let us note that

$$\{N_a, (a_p^r)^\dagger\} = \int \frac{d^3\vec{k}}{(2\pi)^3} \sum_s (a_k^s)^\dagger \{a_k^s, (a_p^r)^\dagger\} \quad (5.1.54)$$

$$= (a_p^r)^\dagger \quad (5.1.55)$$

whereas

$$\{N_b, (b_p^r)^\dagger\} = \int \frac{d^3\vec{k}}{(2\pi)^3} \sum_s (b_k^s)^\dagger \{b_k^s, (b_p^r)^\dagger\} \quad (5.1.56)$$

$$= (b_p^r)^\dagger. \quad (5.1.57)$$

If we define the vacuum to be such that

$$a_p^s |0\rangle = 0 = b_p^s |0\rangle, \quad (5.1.58)$$

then we see that

$$N_a (a_p^s)^\dagger |0\rangle = (\{N_a, (a_p^s)^\dagger\} - (a_p^s)^\dagger N_a) |0\rangle \quad (5.1.59)$$

$$= (a_p^s)^\dagger |0\rangle \quad (5.1.60)$$

and

$$N_b (b_p^s)^\dagger |0\rangle = (\{N_b, (b_p^s)^\dagger\} - (b_p^s)^\dagger N_b) |0\rangle \quad (5.1.61)$$

$$= (b_p^s)^\dagger |0\rangle. \quad (5.1.62)$$

In words: $N_a = \int_{\vec{k}} \sum_s (a_k^s)^\dagger a_k^s$ is the number operator for *fermions* created by $(a_p^r)^\dagger$ and destroyed by a_p^r ; while $N_b = \int_{\vec{k}} \sum_s (b_k^s)^\dagger b_k^s$ is the number operator for *anti-fermions* created by b_p^r and destroyed by $(b_p^r)^\dagger$.

Furthermore, up to an additive constant, we may now write the Hamiltonian as

$$H = \int \frac{d^3\vec{k}}{(2\pi)^3} E_k (\tilde{N}_a(\vec{k}) + \tilde{N}_b(\vec{k})). \quad (5.1.63)$$

It is now bounded from below!

Fermi-Dirac Statistics An important consequence of the anti-commutation relations is that of Fermi-Dirac statistics: no more than one fermion or anti-fermion can occupy the same state, since

$$(a_p^s)^\dagger (a_p^s)^\dagger |0\rangle = \frac{1}{2} \{ (a_p^s)^\dagger, (a_p^s)^\dagger \} |0\rangle = 0 \quad (5.1.64)$$

$$(b_p^s)^\dagger (b_p^s)^\dagger |0\rangle = \frac{1}{2} \{ (b_p^s)^\dagger, (b_p^s)^\dagger \} |0\rangle = 0. \quad (5.1.65)$$

Feynman Rules The contraction of ψ (which contains a) with an incoming momentum/fermion is

$$\psi(a_p^s)^\dagger |0\rangle \quad \rightarrow \quad u_p^s. \quad (5.1.66)$$

(The convention is: the fermion flow is going *inwards* for an *incoming* fermion.) The contraction of ψ (which contains b^\dagger) with an outgoing momentum/anti-fermion is

$$\langle 0 | b_p^s \psi \quad \rightarrow \quad v_p^s. \quad (5.1.67)$$

(The convention is: the fermion flow is going *inwards* for an *outgoing* anti-fermion.) The contraction of $\bar{\psi}$ (which contains a^\dagger) with an incoming momentum/anti-fermion is

$$\bar{\psi}(b_p^s)^\dagger |0\rangle \quad \rightarrow \quad \bar{v}_p^s. \quad (5.1.68)$$

(The convention is: the fermion flow is going *outwards* for an *incoming* anti-fermion.) The contraction of $\bar{\psi}$ (which contains a^\dagger) with an outgoing momentum/fermion is

$$\langle 0 | a_p^s \bar{\psi} \quad \rightarrow \quad \bar{u}_p^s. \quad (5.1.69)$$

(The convention is: the fermion flow is going *outwards* for an *outgoing* fermion.) Also, since the order of the creation operators matter, we shall employ the convention

$$|p_1, s_1; p_2, s_2; \dots; p_n, s_n\rangle \equiv \sqrt{2E_{p_1}} \dots \sqrt{2E_{p_n}} (a_{p_1}^{s_1})^\dagger \dots (a_{p_n}^{s_n})^\dagger |0\rangle; \quad (5.1.70)$$

which in turn implies the convention

$$\langle p_1, s_1; p_2, s_2; \dots; p_n, s_n | = \sqrt{2E_{p_1}} \dots \sqrt{2E_{p_n}} \langle 0 | (a_{p_n}^{s_n})^\dagger \dots (a_{p_1}^{s_1})^\dagger. \quad (5.1.71)$$

5.2 Fermionic Path Integrals

Feynman Green's Function Because of the anti-commuting character of fermionic fields, the non-interacting vacuum expectation value of the time ordered products is now

$$\langle 0 | \mathbb{T} \psi(x) \bar{\psi}(y) | 0 \rangle = \langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle \quad \text{when } x^0 > y^0 \quad (5.2.1)$$

$$= - \langle 0 | \bar{\psi}(y) \psi(x) | 0 \rangle \quad \text{when } x^0 < y^0. \quad (5.2.2)$$

By a direct calculation, one may show that this is equal to the Feynman Green's function

$$G_F(x-y) \equiv \int \frac{d^4k}{(2\pi)^4} \frac{i(\not{k} + m)}{k^2 - m^2 + i\epsilon} e^{-ik \cdot (x-y)} \quad (5.2.3)$$

$$= \langle 0 | \mathbb{T} \psi(x) \bar{\psi}(y) | 0 \rangle. \quad (5.2.4)$$

We may also see that

$$(i\not{\partial}_x - m)G_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i(\not{k} - m)(\not{k} + m)}{k^2 - m^2 + i\epsilon} e^{-ik \cdot (x-y)} \quad (5.2.5)$$

$$= i \int \frac{d^4k}{(2\pi)^4} \frac{\not{k}\not{k} - m^2}{k^2 - m^2 + i\epsilon} e^{-ik \cdot (x-y)} \quad (5.2.6)$$

$$= i\delta^{(4)}(x-y). \quad (5.2.7)$$

Note: $\not{k}^2 = (1/2) \{\gamma^\mu, \gamma^\nu\} k_\mu k_\nu = k^2$.

To introduce a path integral for fermions, we first have to introduce Grassmann numbers. The basic rule is that Grassmann numbers commute: $\theta\eta = -\eta\theta$, so for instance $\theta^2 = 0$. Also, for complex numbers A and B and Grassmann number α , Berezin defined

$$\int d\alpha (A + B\alpha) \equiv B. \quad (5.2.8)$$

The corresponding Fermionic path integral that generates all n point functions is

$$Z'[\bar{J}, J] \equiv \frac{\int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left[i \int_x (\bar{\psi}(i\not{\partial} - m)\psi + \bar{J}\psi + \bar{\psi}J) \right]}{\int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left[i \int_x \bar{\psi}(i\not{\partial} - m)\psi \right]} \quad (5.2.9)$$

Let us write the action in Fourier spacetime.

$$S \equiv \int_k \left(\tilde{\bar{\psi}}(-k)(\not{k} - m)\tilde{\psi}(k) + \tilde{J}(-k)\tilde{\psi}(k) + \tilde{\bar{\psi}}(-k)\tilde{J}(k) \right). \quad (5.2.10)$$

If we shift

$$\tilde{\psi}(k) \rightarrow \tilde{\psi}(k) + i\tilde{G}_F(k)\tilde{J}(k) \quad (5.2.11)$$

$$\tilde{\bar{\psi}}(k) \rightarrow \tilde{\bar{\psi}}(k) - i\tilde{J}(k)^\dagger \tilde{G}_F(k)^\dagger \gamma^0; \quad (5.2.12)$$

the action becomes

$$\begin{aligned} S \rightarrow \int_k \left(\tilde{\bar{\psi}}(-k)(\not{k} - m)\tilde{\psi}(k) + i\tilde{\bar{\psi}}(-k)(\not{k} - m)\tilde{G}_F(k)\tilde{J}(k) \right. \\ - i\tilde{J}(-k)^\dagger \tilde{G}_F(-k)^\dagger \gamma^0 (\not{k} - m)\tilde{\psi}(k) + \tilde{J}(-k)^\dagger \tilde{G}_F(-k)^\dagger \gamma^0 (\not{k} - m)\tilde{G}_F(k)\tilde{J}(k) \\ + \tilde{J}(-k)\tilde{\psi}(k) + i\tilde{J}(-k)\tilde{G}_F(k)\tilde{J}(k) \\ \left. + \tilde{\bar{\psi}}(-k)\tilde{J}(k) - i\tilde{J}(-k)^\dagger \tilde{G}_F(-k)^\dagger \gamma^0 \tilde{J}(k) \right). \quad (5.2.13) \end{aligned}$$

To ‘complete-the-square’ we like the 2nd term to cancel $\tilde{J}(-k)\tilde{\psi}(k)$ while the 3rd term to cancel $\tilde{\psi}(-k)\tilde{J}(k)$. Let us choose

$$(\not{k} - m)i\tilde{G}_F(k) = -1. \quad (5.2.14)$$

Since this is a matrix equation, the solution is unique.

$$\tilde{G}_F(k) = \frac{i(\not{k} + m)}{k^2 - m^2 + i\epsilon}. \quad (5.2.15)$$

Check: $(\not{k} - m)i^2(\not{k} + m) = i^2(\not{k} + m)(\not{k} - m) = -(\not{k}\not{k} + m \cdot \not{k} - m \cdot \not{k} - m^2) = -(k^2 - m^2)$.
Moreover,

$$\begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}^\dagger \begin{bmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix} = \begin{bmatrix} \bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{bmatrix} \quad (5.2.16)$$

$$= \begin{bmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix} \quad (5.2.17)$$

$$(\gamma^\mu)^\dagger \gamma^0 = \gamma^0 \gamma^\mu. \quad (5.2.18)$$

This in turn means

$$-i\tilde{G}_F(-k)^\dagger \gamma^0 (\not{k} - m) = -\gamma^0 i\tilde{G}_F(-k) (\not{k} - m) \quad (5.2.19)$$

$$= +\gamma^0. \quad (5.2.20)$$

YZ: Incomplete!

5.3 Appendix

We may verify the following results.

$$\bar{u}_p^r u_p^s = (\xi^r)^\dagger \sqrt{p \cdot \bar{\sigma}} \sqrt{p \cdot \sigma} \xi^s + (\xi^r)^\dagger \sqrt{p \cdot \sigma} \sqrt{p \cdot \bar{\sigma}} \xi^s \quad (5.3.1)$$

$$= 2m\delta_s^r. \quad (5.3.2)$$

$$\bar{v}_p^r v_p^s = -(\eta^r)^\dagger \sqrt{p \cdot \bar{\sigma}} \sqrt{p \cdot \sigma} \eta^s + (\eta^r)^\dagger \sqrt{p \cdot \sigma} (-) \sqrt{p \cdot \bar{\sigma}} \eta^s \quad (5.3.3)$$

$$= -2m\delta_s^r. \quad (5.3.4)$$

Problem 5.1. Lepton/Quark Decays Consider the following Lagrangian density describing 2 massive and 2 massless fermions ($\psi_{1,2}$ and $\nu_{1,2}$ respectively); and their 4-fermion interaction:

$$\begin{aligned} \mathcal{L} &= \sum_{I=1}^2 \{ \bar{\psi}_I (i\not{\partial} - m_I) \psi_I + \bar{\nu}_I i\not{\partial} \nu_I \} \\ &+ \frac{g^2}{M^2} \bar{\psi}_1 \gamma^\mu (c_1 - c_2 \gamma^5) \nu_1 \bar{\nu}_2 \gamma_\mu (c_3 - c_4 \gamma^5) \psi_2 + \text{h.c.}, \quad c_{1,2,3,4} \in \mathbb{R}. \end{aligned} \quad (5.3.5)$$

Assuming $m_1 > m_2$, compute – to leading order in g^2 – the decay rate of the process

$$\psi_1 \rightarrow \nu_1 + \psi_2 + \bar{\nu}_2 \quad (5.3.6)$$

due to the interaction in eq. (5.3.5). Muon and quark decays are special cases of this process. \square

Problem 5.2. Gaussian & Grassmannian Integrals Compute the integrals involving real variables x^i and symmetric matrix A_{ij} :

Compute the integrals involving

$$\langle x^{i_1} x^{i_2} \rangle \equiv \int_{\mathbb{R}} \exp(i x^i A_{ij} x^j) x^{i_1} x^{i_2} d^D \vec{x}, \quad (5.3.7)$$

$$\langle x^{i_1} x^{i_2} x^{i_3} \rangle \equiv \int_{\mathbb{R}} \exp(i x^i A_{ij} x^j) x^{i_1} x^{i_2} x^{i_3} d^D \vec{x}, \quad (5.3.8)$$

$$\langle x^{i_1} x^{i_2} x^{i_3} x^{i_4} \rangle \equiv \int_{\mathbb{R}} \exp(i x^i A_{ij} x^j) x^{i_1} x^{i_2} x^{i_3} x^{i_4} d^D \vec{x}. \quad (5.3.9)$$

Explain how to obtain the general even-point function $\langle x^{i_1} \dots x^{i_{2n}} \rangle$ and odd-point function $\langle x^{i_1} \dots x^{i_{2n+1}} \rangle$.

Next, compute the integrals involving complex Grassmann variables θ^i and Hermitian matrix H_{ij} :

$$\langle \theta^{i_1} \overline{\theta^{i_2}} \rangle \equiv \int_{\mathbb{R}} \exp(i \overline{\theta^i} H_{ij} \theta^j) \theta^{i_1} \overline{\theta^{i_2}} d^D \vec{\theta} d^D \vec{\theta}^*, \quad (5.3.10)$$

$$\langle \theta^{i_1} \overline{\theta^{i_2}} \theta^{i_3} \overline{\theta^{i_4}} \rangle \equiv \int_{\mathbb{R}} \exp(i \overline{\theta^i} H_{ij} \theta^j) \theta^{i_1} \overline{\theta^{i_2}} \theta^{i_3} \overline{\theta^{i_4}} d^D \vec{\theta} d^D \vec{\theta}^*. \quad (5.3.11)$$

Also evaluate the integrals involving only θ or only its complex conjugate $\overline{\theta} \equiv \theta^*$, namely $\langle \theta^{i_1} \dots \theta^{i_N} \rangle$ and $\langle \overline{\theta^{i_1}} \dots \overline{\theta^{i_N}} \rangle$. Can you explain how to get the general function $\langle \theta^{i_1} \overline{\theta^{j_1}} \dots \theta^{i_n} \overline{\theta^{j_n}} \rangle$? This and the above real variable case is the discrete analog of \square

Problem 5.3. Fractional Fermion Number on Scalar Kinks in (1+1)D In this problem we will explore, in (1+1)D with Cartesian coordinates $x^\mu \equiv (t, x)$, how the number operator of a fermion ψ can acquire a fractional eigenvalue in the presence of a background static ‘kink’ solution to the scalar field ϕ . We start with the total Lagrangian

$$\mathcal{L} \equiv \mathcal{L}_\phi + \mathcal{L}_\psi, \quad (5.3.12)$$

$$\mathcal{L}_\phi \equiv \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\lambda}{4} (\phi^2 - \eta^2)^2, \quad (5.3.13)$$

$$\mathcal{L}_\psi \equiv \overline{\psi} (i \not{\partial} - g \phi) \psi. \quad (5.3.14)$$

Note that the Dirac spinor is a 2-component object in $(1+1)D$. Explain why the following choice is a valid one for the γ^μ matrices. Denoting the Pauli matrices by $\{\sigma^i | i = 1, 2, 3\}$, define

$$\gamma^0 \equiv \sigma^3 \quad \text{and} \quad \gamma^1 \equiv i \sigma^1. \quad (5.3.15)$$

Scalar Solutions First focus on the scalar sector. Verify that the following are static solutions. You may be able to actually *derive* them; try plotting the potential.

$$\phi_V = \pm \eta \quad (5.3.16)$$

$$\phi_K = \pm \eta \tanh \left(\eta \sqrt{\lambda/2} \cdot x \right). \quad (5.3.17)$$

Fermion Solutions Next, solve the Dirac equation

$$(i \not{\partial} - g \phi_V) \psi = 0. \quad (5.3.18)$$

And, derive the *zero energy* solutions to

$$(i\cancel{\partial} - g\phi_K) \psi = 0. \quad (5.3.19)$$

Fractional Fermion Number Quantize the Dirac fermion in eq. (5.3.19) and explain why the existence of zero energy solutions on a kink background ϕ_K leads to half-integer eigenvalues for the fermion number operator. \square

6 Classical Electromagnetism

6.1 Minkowski: d -dimensions

In this section we will discuss in some detail Minkowski spacetime electromagnetism to illustrate both its Lorentz and gauge symmetries. It will also provide us the opportunity to introduce the action principle, which is key formulating both classical and quantum field theories.

Maxwell & Lorentz We begin with Maxwell's equations in the following Lorentz covariant form, written in Cartesian coordinates $\{x^\mu\}$ so that $g_{\mu\nu} = \eta_{\mu\nu}$:

$$\partial_\mu F^{\mu\nu} = J^\nu, \quad \partial_{[\mu} F_{\alpha\beta]} = 0, \quad F_{\mu\nu} = -F_{\nu\mu}. \quad (6.1.1)$$

The $J^\mu \equiv \rho v^\mu$ is the electromagnetic current. Assuming J^μ is timelike, v^μ is its d -proper velocity with $v^2 \equiv v^\mu v_\mu = 1$; and $\rho \equiv J^\mu v_\mu$ is the electric charge in the (local) rest frame where $v^\mu = \delta_0^\mu$. Defined this way, ρ is a Lorentz scalar and J^μ is a Lorentz vector since v^μ is a Lorentz vector. It is then reasonable to suppose $F_{\mu\nu}$ is a rank-2 Lorentz tensor. Specifically, let two inertial frames $\{x^\mu\}$ and $\{x'^\mu\}$ be related via the Lorentz transformation

$$x^\mu = \Lambda^\mu_\alpha x'^\alpha, \quad \Lambda^\mu_\alpha \Lambda^\nu_\beta \eta_{\mu\nu} = \eta_{\alpha\beta}. \quad (6.1.2)$$

The same rules would apply to any vector v^μ ; specifically,

$$v^\mu(x^\alpha = \Lambda^\alpha_\beta x'^\beta) = \Lambda^\mu_\nu v'^\nu(x'), \quad (6.1.3)$$

$$v'^\mu(x') = (\Lambda^{-1})^\mu_\nu v^\nu(x^\alpha = \Lambda^\alpha_\beta x'^\beta). \quad (6.1.4)$$

(Note: $v^\mu(x)$ is the μ -th component of the vector in the $\{x^\alpha\}$ frame; while $v'^\mu(x')$ is the μ -th component of the vector in the $\{x'^\alpha\}$ frame.) Then the Faraday tensor transforms as

$$F_{\alpha'\beta'}(x') = F_{\mu\nu}(x(x')) = \Lambda \cdot x' \Lambda^\mu_\alpha \Lambda^\nu_\beta \quad (6.1.5)$$

Its derivatives are also Lorentz covariant, for keeping in mind eq. (6.1.2),

$$\partial_{x'^\lambda} F_{\alpha'\beta'}(x') = \frac{\partial x^\sigma}{\partial x'^\lambda} \partial_\sigma F_{\mu\nu}(x(x')) = \Lambda \cdot x' \Lambda^\mu_\alpha \Lambda^\nu_\beta \quad (6.1.6)$$

$$= \Lambda^\sigma_\lambda \partial_\sigma F_{\mu\nu}(x(x')) \Lambda^\mu_\alpha \Lambda^\nu_\beta. \quad (6.1.7)$$

This immediately tells us $\partial_\mu F^{\mu\nu} = \eta^{\mu\alpha} \partial_\mu F_{\alpha\beta} \eta^{\beta\nu}$ in eq. (6.1.1) is a Lorentz vector.

(3+1)D Lorentz Force Law & $F^{\mu\nu}$ vs (\vec{E}, \vec{B}) The electromagnetic force on a point charge q of mass m must take a Lorentz covariant form $mdz^\mu/d\tau^2 = f^\mu$, where $z^\mu(\tau)$ is its

worldline trajectory and τ is the associated proper time. The force f^μ itself must be built out of $F^{\mu\nu}$ and the proper velocity, if we are to recover the $\vec{E} + \vec{v} \times \vec{B}$ in the non-relativistic limit. A good guess would be to assert

$$m \frac{d^2 z^\mu}{d\tau^2} = q F^\mu{}_\nu v^\nu. \quad (6.1.8)$$

Taking the non-relativistic limit, and focusing only on the spatial components,

$$m \frac{d^2 z^i}{dt^2} = q \left(F^i{}_0 + F^i{}_j \frac{dz^j}{dt} \right); \quad (6.1.9)$$

allowing us to identify the electric field as

$$E^i = F^{i0} \quad (6.1.10)$$

and the magnetic field as

$$\epsilon^{ijk} B^k = -F^{ij} \quad \Leftrightarrow \quad B^k = \frac{1}{2} \epsilon^{ijk} F^{ij}, \quad \epsilon^{123} \equiv 1. \quad (6.1.11)$$

Notice: we are able to convert the rank 2 object F^{ij} into a spatial vector B^k and vice versa, because the Levi-Civita symbol has 3 indices in 3D space. In spatial dimension D , the magnetic field would be associated with the $(D^2 - D)/2 = D(D-1)/2$ number of independent components of F^{ij} . For e.g., in 2D space, the magnetic field is a scalar $B = (1/2)\epsilon^{ij}F^{ij}$; in 4D space, we have $4 \cdot 3/2 = 6$ components for the magnetic field.

Problem 6.1. 4D Maxwell's Equations in term of (\vec{E}, \vec{B}) Let us check that eq. (6.1.1) does in fact reproduce Maxwell's equations in terms of electric E^i and magnetic B^i fields in 4D. Given a Lorentzian inertial frame, define

$$F^{i0} \equiv E^i \quad \text{and} \quad F^{ij} \equiv \epsilon^{ijk} B^k; \quad (6.1.12)$$

with $\epsilon^{123} \equiv -1$. Show that the $\partial_\mu F^{\mu\nu} = J^\nu$ from eq. (6.1.1) translates to

$$\vec{\nabla} \cdot \vec{E} = J^0 \quad \text{and} \quad \vec{\nabla} \times \vec{B} - \partial_t \vec{E} = \vec{J}. \quad (6.1.13)$$

(The over-arrow refers to the spatial components; for instance $\vec{B} = (B^1, B^2, B^3)$.) The $\partial_{[\alpha} F_{\mu\nu]} = 0$ from eq. (6.1.1) translates to

$$\partial_t \vec{B} + \vec{\nabla} \times \vec{E} = 0 \quad \text{and} \quad \vec{\nabla} \cdot \vec{B} = 0. \quad (6.1.14)$$

Hint: Note that $(\vec{\nabla} \times \vec{A})^i = -\epsilon^{ijk} \partial_j A^k$, for any Cartesian vector \vec{A} . Also, when you compute $\partial_{[i} F_{jk]}$, you simply need to set $\{i, j, k\}$ to be any distinct permutation of $\{1, 2, 3\}$. (Why?)

Next, verify the Lorentz invariant relations, with $\epsilon^{0123} \equiv -1$:

$$F_{\mu\nu} F^{\mu\nu} = -2 \left(\vec{E}^2 - \vec{B}^2 \right), \quad \vec{E}^2 \equiv E^i E^i, \quad \vec{B}^2 \equiv B^i B^i, \quad (6.1.15)$$

$$\epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} = 4 \partial_\mu \left(\epsilon^{\mu\nu\alpha\beta} A_\nu \partial_\alpha A_\beta \right) = 8 \vec{E} \cdot \vec{B}. \quad (6.1.16)$$

How does $F_{\mu\nu}F^{\mu\nu}$ transform under time reversal, $t \equiv x^0 \rightarrow -t$? How does it transform under parity flips, $x^i \rightarrow -x^i$ (for a fixed i)? Answer the same questions for $\tilde{F}^{\mu\nu}F_{\mu\nu}$, where the dual of $F_{\mu\nu}$ is

$$\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\tilde{\epsilon}^{\mu\nu\alpha\beta}F_{\alpha\beta}. \quad (6.1.17)$$

$d \neq 4$ Can you comment what the analog of the magnetic field ought to be in spacetime dimensions different from 4 – is it still a ‘vector’? – and what is the lowest dimension that the magnetic field still exists? How many components does the electric field have in 1+1 dimensions? \square

Problem 6.2. 4D Lorentz boosts of electric and magnetic fields Consider a Lorentz boost along the 1–direction, where

$$x^\mu = \Lambda^\mu{}_\nu x'^\nu, \quad (6.1.18)$$

and

$$\Lambda^\mu{}_\nu = \begin{bmatrix} \frac{1}{\sqrt{1-v^2}} & -\frac{v}{\sqrt{1-v^2}} & 0 & 0 \\ -\frac{v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad 0 \leq |v| < 1. \quad (6.1.19)$$

Write down the relationship between the electric $\vec{E}(x) = (E^1, E^2, E^3)$ and magnetic $\vec{B}(x) = (B^1, B^2, B^3)$ fields in the $\{x^\mu\}$ frame and the $\vec{E}'(x') = (E^{1'}, E^{2'}, E^{3'})$ and magnetic $\vec{B}'(x') = (B^{1'}, B^{2'}, B^{3'})$ in the $\{x'^\mu\}$ frame. Hint: Use the Lorentz transformation rules for $F^{\mu\nu}$; and assume $E^i \equiv F^{i0}$ and $F^{ij} \equiv \epsilon^{ijk}B^k$ (with $\epsilon^{123} \equiv -1$) holds in all inertial frames. \square

Current conservation Taking the divergence of $\partial_\mu F^{\mu\nu} = J^\nu$ yields the conservation of the electric current as a consistency condition. For, by the antisymmetry $F_{\mu\nu} = -F_{\nu\mu}$, $\partial_\nu \partial_\mu F^{\mu\nu} = (1/2)\partial_\nu \partial_\mu F^{\mu\nu} - (1/2)\partial_\mu \partial_\nu F^{\nu\mu} = 0$.

$$\partial_\mu J^\mu = 0. \quad (6.1.20)$$

Problem 6.3. Total charge is constant in all inertial frames Even though we defined ρ in the $J^\mu = \rho v^\mu$ as the charge density in the local rest frame of the electric current itself, we may also define the charge density $\tilde{J}^0 \equiv u_\mu J^\mu$ in the rest frame of an arbitrary family of inertial time-like observers whose worldlines’ tangent vector is $u^\mu \partial_\mu = \partial_\tau$. (In other words, in their frame, the spacetime metric is $ds^2 = (d\tau)^2 - d\vec{x} \cdot d\vec{x}$.) Show that total charge is independent of the Lorentz frame by demonstrating that

$$Q \equiv \int_{\mathbb{R}^D} d^D \Sigma_\mu J^\mu, \quad d^D \Sigma_\mu \equiv d^D \vec{x} u_\mu, \quad D \equiv d - 1, \quad (6.1.21)$$

is a constant. \square

Vector Potential & Gauge Symmetry The other Maxwell equation (cf eq. (6.1.1)) leads us to introduce a vector potential A_μ . For $\partial_{[\mu}F_{\alpha\beta]} = 0 \Leftrightarrow dF = 0$ tells us, by the Poincaré lemma, that

$$F = dA \quad \Leftrightarrow \quad F_{\mu\nu} = \partial_{[\mu}A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (6.1.22)$$

Notice the dynamics in eq. (6.1.1) is not altered if we add to A_μ any object L_μ that obeys $dL = 0$, because that does not alter the Faraday tensor: $F = d(A + L) = F + dL = F$. Now, $dL = 0$ means, again by the Poincaré lemma, that $L_\mu = \partial_\mu L$, where L on the right hand side is a scalar. *Gauge symmetry*, in the context of electromagnetism, is the statement that the following replacement involving the gauge potential

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu L(x) \quad (6.1.23)$$

leaves the dynamics encoded in Maxwell's equations (6.1.1) unchanged.

The use of the gauge potential A_μ makes the $dF = 0$ portion of the dynamics in eq. (6.1.1) redundant; and what remains is the vector equation

$$\partial_\mu F^{\mu\nu} = \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = J^\nu. \quad (6.1.24)$$

The symmetry under the gauge transformation of eq. (6.1.23) means that solutions to eq. (6.1.24) cannot be unique – in particular, since A_μ and $A_\mu + \partial_\mu L$ are simultaneously solutions, there really is an infinity of solutions parametrized by the arbitrary function L . In this same vein, by going to Fourier space, namely

$$A_\mu(x) \equiv \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} \tilde{A}_\mu(k) e^{-ik_\mu x^\mu} \quad \text{and} \quad J_\mu(x) \equiv \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} \tilde{J}_\mu(k) e^{-ik_\mu x^\mu}, \quad (6.1.25)$$

we may see that the differential operator in eq. (6.1.24) cannot be inverted because it has a zero eigenvalue. Firstly, the Fourier version of eq. (6.1.24) reads

$$-K^{\mu\nu} \tilde{A}_\mu = \tilde{J}^\nu, \quad (6.1.26)$$

$$K^{\mu\nu} \equiv k_\sigma k^\sigma \eta^{\mu\nu} - k^\nu k^\mu. \quad (6.1.27)$$

If K^{-1} exists, the solution in Fourier space would be (schematically) $\tilde{A} = -K^{-1}\tilde{J}$. However, since $K^{\mu\nu} = K^{\nu\mu}$ is a real symmetric matrix, it must be diagonalizable via an orthogonal transformation, with $\det K^{\mu\nu}$ equal to the product of its eigenvalues. That $\det K^{\mu\nu} = 0$ and therefore K^{-1} does not exist can now be seen by observing that k_μ is in fact its null eigenvector:

$$K^{\mu\nu} k_\mu = (k_\sigma k^\sigma) k^\nu - k^\nu k^\mu k_\mu = 0. \quad (6.1.28)$$

Problem 6.4. Can you explain why eq. (6.1.28) amounts to the statement that $F_{\mu\nu}$ is invariant under the gauge transformation of eq. (6.1.23)? Hint: Consider eq. (6.1.23) in Fourier space.

Lorenz gauge To make $K^{\mu\nu}$ invertible, one *fixes a gauge*. A common choice is the Lorenz gauge; in Fourier spacetime:

$$k^\mu \tilde{A}_\mu = 0. \quad (6.1.29)$$

In ‘position’/real spacetime, this reads instead

$$\partial^\mu A_\mu = 0 \quad (\text{Lorenz gauge}). \quad (6.1.30)$$

With the constraint in eq. (6.1.29), Maxwell’s equations in eq. (6.1.26) becomes

$$-\left(k_\sigma k^\sigma \tilde{A}^\nu - k^\nu (k^\mu \tilde{A}_\mu)\right) = -k_\sigma k^\sigma \tilde{A}^\nu = \tilde{J}^\nu. \quad (6.1.31)$$

Now, Maxwell’s equations have become invertible:

$$\tilde{A}_\mu(k) = \frac{\tilde{J}_\mu(k)}{-k^2}, \quad k^2 \equiv k_\sigma k^\sigma, \quad (\text{Lorenz gauge}). \quad (6.1.32)$$

In position/real spacetime, eq. (6.1.31) is equivalent to

$$\partial^2 A^\nu(x) = J^\nu(x) \quad \partial^2 \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu. \quad (6.1.33)$$

Problem 6.5. Lorentz covariance Suppose Λ^α_μ is a Lorentz transformation; let two inertial frames $\{x^\mu\}$ and $\{x'^\mu\}$ be related via

$$x^\mu = \Lambda^\mu_\alpha x'^\alpha + a^\mu, \quad (6.1.34)$$

where a^μ is a constant vector. Suppose we solved the Lorenz gauge Maxwell’s equations in the $\{x^\mu\}$ frame, namely

$$\frac{\partial A^\mu(x)}{\partial x^\mu} = 0, \quad \eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} A_\alpha(x) = J_\alpha(x). \quad (6.1.35)$$

Explain how to solve $A_{\alpha'}(x')$, the solution in the $\{x'^\mu\}$ frame. \square

Problem 6.6. Poincaré transformations in Fourier spacetime Given a 1–form $A_\mu(x)$ in Minkowski spacetime – not necessarily the vector potential – let us define its Fourier transform as

$$A_\mu(x) = \int_{\mathbb{R}^{d-1,1}} \frac{d^d k}{(2\pi)^d} \tilde{A}_\mu(k) e^{-ik \cdot x}, \quad k \cdot x \equiv k_\sigma x^\sigma. \quad (6.1.36)$$

Show that, under a Poincaré transformation,

$$x^\mu = \Lambda^\mu_\nu x'^\nu + a^\mu, \quad (6.1.37)$$

the Fourier coefficient in the $\{x'\}$ frame

$$A_{\mu'}(x') = \int_{\mathbb{R}^{d-1,1}} \frac{d^d k'}{(2\pi)^d} \tilde{A}_{\mu'}(k') e^{-ik' \cdot x'}, \quad k' \cdot x' \equiv k'_\sigma x'^\sigma, \quad (6.1.38)$$

is related to the one in the $\{x\}$ frame (cf. (6.1.36)) through

$$\tilde{A}_{\mu'}(k') = \tilde{A}_\alpha(k) e^{-ik \cdot a} \Lambda^\alpha_\mu \Big|_{k_\alpha = k'_\mu (\Lambda^{-1})^\mu_\alpha}. \quad (6.1.39)$$

Because the equation obeyed by the Lorenz gauge vector potential, namely eq. (6.1.33), is Lorentz covariant – its Fourier coefficients must transform according to eq. (6.1.39). \square

⁴In the Lorenz gauge, we have d Minkowski scalar wave equations, one for each Cartesian component. We may express its position spacetime solution by inverting the Fourier transform in eq. (6.1.32):

$$A_\mu(x) = \int_{\mathbb{R}^{d-1,1}} d^d x' G_d^+(x-x') J_{\mu'}(x'), \quad (6.1.40)$$

$$G_d^+(x-x') \equiv \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} \frac{e^{-ik \cdot (x-x')}}{-k^2}. \quad (6.1.41)$$

Because A_μ is not gauge-invariant, its physical interpretation can be ambiguous. Classically it is the electromagnetic fields $F_{\mu\nu}$ that exert forces on charges/currents, so we need its solution. In fact, we may take the curl of eq. (6.1.33) to see that

$$\partial^2 F_{\mu\nu} = \partial_{[\mu} J_{\nu]}; \quad (6.1.42)$$

this means, using the same Green's function in eq. (6.1.41):

$$F_{\mu\nu}(x) = \int_{\mathbb{R}^{d-1,1}} d^d x' G_d^+(x-x') \partial_{[\mu'} J_{\nu']}(x'). \quad (6.1.43)$$

We may verify that equations (6.1.40) and (6.1.41) solve eq. (6.1.33) readily:

$$\begin{aligned} \partial_x^2 G_d^+(x-x') &= \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} \frac{\partial_\sigma \partial^\sigma e^{-ik \cdot (x-x')}}{-k^2} \\ &= \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} \frac{\partial_\sigma (-ik^\rho \delta_\rho^\sigma e^{-ik \cdot (x-x')})}{-k^2} \\ &= \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} \frac{\partial_\sigma (-ik^\sigma e^{-ik \cdot (x-x')})}{-k^2} \\ &= \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} \frac{(-ik_\sigma)(-ik^\sigma) e^{-ik \cdot (x-x')}}{-k^2} \\ &= \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} e^{-ik \cdot (x-x')} = \delta^{(d)}(x-x'); \end{aligned} \quad (6.1.44)$$

with a similar calculation showing $\partial_{x'}^2 G_d^+(x-x') = \delta^{(d)}(x-x')$. To sum,

$$\partial_x^2 G_d^+(x-x') = \partial_{x'}^2 G_d^+(x-x') = \delta^{(d)}(x-x'); \quad (6.1.45)$$

Moreover, comparing each Cartesian component of the wave equation in eq. (6.1.33) with the one obeyed by the Green's function in eq. (6.1.45), we may identify the source J of the Green's function itself to be a unit strength spacetime point source at some fixed location x' . It is often useful to think of x as the spacetime location of some observer; so $x^0 - x'^0 \equiv t - t'$ is the time elapsed while $|\vec{x} - \vec{x}'|$ is the observer-source spatial distance. Altogether, we may now view the solution in eq. (6.1.40) as the sum of the field generated by all spacetime point sources, weighted by the physical electric current $J_\mu(x')$.

⁴Eq. (6.1.33) is valid in any dimension $d \geq 3$. In 2D, the $dF = 0$ portion of Maxwell's equations is trivial – i.e., any F would satisfy it – because there cannot be three distinct indices in $\partial_{[\mu} F_{\alpha\beta]} = 0$.

We now may verify directly that eq. (6.1.40) is indeed a solution to eq. (6.1.33).

$$\begin{aligned}\partial_x^2 A_\mu(x) &= \partial_x^2 \left(\int_{\mathbb{R}^{d-1,1}} d^d x' G_d^+(x-x') J_{\mu'}(x') \right) = \int_{\mathbb{R}^{d-1,1}} d^d x' \delta^{(d)}(x-x') J_{\mu'}(x') \\ &= J_\mu(x).\end{aligned}\tag{6.1.46}$$

Lorenz gauge: Existence That we have managed to solve Maxwell's equations using the Lorenz gauge, likely convinces the practical physicist that the Lorenz gauge itself surely exists. However, it is certainly possible to provide a general argument. For suppose $\partial^\mu A_\mu$ were not zero, then all one has to show is that we may perform a gauge transformation (cf. (6.1.23)) that would render the new gauge potential $A'_\mu \equiv A_\mu - \partial_\mu L$ satisfy

$$\partial^\mu A'_\mu = \partial^\mu A_\mu - \partial^2 L = 0.\tag{6.1.47}$$

But all that means is, we have to solve $\partial^2 L = \partial^\mu A_\mu$; and since the Green's function $1/\partial^2$ exists, we have proved the assertion.

Lorenz gauge and current conservation You may have noticed, by taking the divergence of both sides of eq. (6.1.33),

$$\partial^2 (\partial^\sigma A_\sigma) = \partial^\sigma J_\sigma.\tag{6.1.48}$$

This teaches us the consistency of the Lorenz gauge is intimately tied to the conservation of the electric current $\partial^\sigma J_\sigma = 0$. Another way to see this, is to take the time derivative of the divergence of the vector potential, followed by subtracting and adding the spatial Laplacian of A_0 so that $\partial^2 A_0 = J_0$ may be employed:

$$\begin{aligned}\partial^\sigma \dot{A}_\sigma &= \ddot{A}_0 + \partial^i \dot{A}_i = \partial^0 \partial_0 A_0 + \partial^i \partial_i A_0 + \partial^i \partial_0 A_i - \partial^i \partial_i A_0 \\ &= \partial^2 A_0 - \partial^i (\partial_i A_0 - \partial_0 A_i) \\ \partial_0 (\partial^\sigma A_\sigma) &= J_0 - \partial^i F_{i0}.\end{aligned}\tag{6.1.49}$$

Notice the right hand side of the last line is zero if the $\nu = 0$ component of $\partial_\mu F^{\mu\nu} = J^\nu$ is obeyed – and if the latter is obeyed the ‘left-hand-side’ of Lorenz gauge condition $\partial_\mu A^\mu$ is a time independent quantity.

6.2 Gauge Invariant Variables for Electromagnetic Vector Potential

Although the vector potential A_μ itself is not a gauge invariant object, we will now exploit the spatial translation symmetry of Minkowski spacetime to seek a gauge-invariant set of partial differential equations involving a ‘‘scalar-vector’’ decomposition of A_μ . There are at least two reasons for doing so.

- We will witness how, for a given inertial frame, the only portion of the vector potential A_μ that obeys a wave equation is its gauge-invariant ‘‘transverse’’ spatial portion. (Even though every component of A_μ in the Lorenz gauge (cf. eq. (6.1.33)) obeys the wave equation, remember such a statement is not gauge-invariant.) We shall also identify a gauge-invariant scalar potential sourced by charge density.

- This will be a warm-up to an analogous analysis for gravitation linearized about a Minkowski “background” spacetime.

Scalar-Vector Decomposition The scalar-vector decomposition is the statement that the spatial components of the vector potential may be expressed as a gradient of a scalar α plus a transverse vector α_i :

$$A_i = \partial_i \alpha + \alpha_i, \quad (6.2.1)$$

where by “transverse” we mean

$$\partial_i \alpha_i = 0. \quad (6.2.2)$$

To demonstrate the generality of eq. (6.2.1) we shall first write A_i in Fourier space

$$A_i(t, \vec{x}) = \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} \tilde{A}_i(t, \vec{k}) e^{i\vec{k} \cdot \vec{x}}, \quad (6.2.3)$$

where $\vec{k} \cdot \vec{x} \equiv \delta_{ij} k^i x^j = -k_j x^j$. Every spatial derivative ∂_j acting on $A_i(t, \vec{x})$ becomes in Fourier space a $-ik_j$, since

$$\begin{aligned} \partial_j A_i &= \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} \partial_j (i\delta_{ab} k^a x^b) \tilde{A}_i(t, \vec{k}) e^{i\vec{k} \cdot \vec{x}} \\ &= \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} (i\delta_{ab} k^a \delta_j^b) \tilde{A}_i(t, \vec{k}) e^{i\vec{k} \cdot \vec{x}} \\ &= \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} i k^j \tilde{A}_i(t, \vec{k}) e^{i\vec{k} \cdot \vec{x}} \\ &= \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} (-ik_j) \tilde{A}_i(t, \vec{k}) e^{i\vec{k} \cdot \vec{x}}. \end{aligned} \quad (6.2.4)$$

As such, the transverse property of $\alpha_i(t, \vec{x})$ would in Fourier space become

$$-ik_i \tilde{\alpha}_i(t, \vec{k}) = 0. \quad (6.2.5)$$

At this point we simply write down

$$\tilde{A}_i(t, \vec{k}) = \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2} \right) \tilde{A}_j(t, \vec{k}) + \frac{k_i k_j}{\vec{k}^2} \tilde{A}_j(t, \vec{k}). \quad (6.2.6)$$

This is mere tautology, of course. However, we may now check that the first term on the left hand side of eq. (6.2.6) is transverse:

$$-ik_i \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2} \right) \tilde{A}_j(t, \vec{k}) = -i \left(k_j - \frac{\vec{k}^2 k_j}{\vec{k}^2} \right) \tilde{A}_j(t, \vec{k}) = 0. \quad (6.2.7)$$

The second term on the right hand side of eq. (6.2.6) is a gradient because it is

$$-ik_i \left(\frac{ik_j}{k^2} \tilde{A}_j \right). \quad (6.2.8)$$

To sum, we have identified the α and α_i terms of eq. (6.2.1) as

$$\alpha(t, \vec{x}) = \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} \frac{ik_j}{k^2} \tilde{A}_j(t, \vec{k}) e^{i\vec{k} \cdot \vec{x}}, \quad (6.2.9)$$

and the transverse portion as

$$\begin{aligned} \alpha_i(t, \vec{x}) &= \int_{\mathbb{R}^D} \frac{d^D \vec{k}}{(2\pi)^D} P_{ij}(\vec{k}) \tilde{A}_j(t, \vec{k}) e^{i\vec{k} \cdot \vec{x}}, \\ P_{ij}(\vec{k}) &\equiv \delta_{ij} - \frac{k_i k_j}{k^2}. \end{aligned} \quad (6.2.10)$$

Notice it is really the projector P_{ij} that is “transverse”; i.e.

$$k_i P_{ij}(\vec{k}) = 0. \quad (6.2.11)$$

Let us also note that this scalar-vector decomposition is unique, in that – if we have the Fourier-space equation

$$-ik_i \tilde{\alpha} + \tilde{\alpha}_i = -ik_i \tilde{\beta} + \tilde{\beta}_i, \quad (6.2.12)$$

where $k_i \tilde{\alpha}_i = k_i \tilde{\beta}_i = 0$, then

$$\tilde{\alpha} = \tilde{\beta} \quad \text{and} \quad \tilde{\alpha}_i = \tilde{\beta}_i. \quad (6.2.13)$$

For, we may first “dot” both sides of eq. (6.2.12) with \vec{k} and see that – for $\vec{k} \neq \vec{0}$,

$$\vec{k}^2 \tilde{\alpha} = \vec{k}^2 \tilde{\beta} \quad \Leftrightarrow \quad \tilde{\alpha} = \tilde{\beta}. \quad (6.2.14)$$

Plugging this result back into eq. (6.2.12), we also conclude $\tilde{\alpha}_i = \tilde{\beta}_i$.

Now, this scalar-vector decomposition is really just a mathematical fact, and may even be performed in a curved space – as long as the latter is infinite – since it depends on the existence of the Fourier transform and not on the metric structure. (A finite space would call for a discrete Fourier-like series of sorts.) However, to determine its usefulness, we would need to insert it into the partial differential equations obeyed by A_i , where the metric structure does matter. As we now turn to examine, because of the spatial translation symmetry of Minkowski spacetime, Maxwell’s equations themselves admit a scalar-vector decomposition. This, in turn, would lead to PDEs for the gauge-invariant portions of A_μ .

Problem 6.7. Gauge transformations We first define

$$\Phi \equiv A_0 - \dot{\alpha}, \quad (6.2.15)$$

$$A_i^\Gamma \equiv \alpha_i. \quad (6.2.16)$$

By first doing a scalar-vector decomposition of the gauge transformation rules $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$, show that Φ and A_i^T are gauge-invariant. (Recall the uniqueness discussion above.) Proceed to show that

$$F_{0i} = \dot{\alpha}_i - \partial_i \Phi \quad (6.2.17)$$

$$F_{ij} = \partial_{[i} \alpha_{j]}. \quad (6.2.18)$$

Electric current We also need to perform a scalar-vector decomposition of the electric current

$$J_\mu \equiv (\rho_E, \partial_i \mathcal{J} + \mathcal{J}_i). \quad (6.2.19)$$

Show that its conservation, namely $\partial^\mu J_\mu$, leads to

$$\dot{\rho}_E = \vec{\nabla}^2 \mathcal{J}. \quad (6.2.20)$$

Notice the transverse portion of the current does not appear in the conservation law. \square

Maxwell's Equations At this point, we are ready to write down Maxwell's equations $\partial^\mu F_{\mu\nu} = J_\nu$. From eq. (6.2.17), the $\nu = 0$ component is

$$-\partial_i F_{i0} = \partial_i (\dot{\alpha}_i - \partial_i \Phi) = -\vec{\nabla}^2 \Phi = \rho_E. \quad (6.2.21)$$

The $\nu = i$ component of $\partial^\mu F_{\mu\nu} = J_\nu$, according to eq. (6.2.17) and (6.2.18),

$$\partial_0 F_{0i} - \partial_j F_{ji} = \partial_i \mathcal{J} + \mathcal{J}_i \quad (6.2.22)$$

$$\ddot{\alpha}_i - \partial_i \dot{\Phi} - \partial_j (\partial_j \alpha_i - \partial_i \alpha_j) = \partial_i \mathcal{J} + \mathcal{J}_i \quad (6.2.23)$$

$$\partial^2 \alpha_i - \partial_i \dot{\Phi} = \mathcal{J}_i + \partial_i \mathcal{J}. \quad (6.2.24)$$

As already advertised, we see that the spatial components of Maxwell's equations does admit a scalar-vector decomposition. By the uniqueness argument above, we may read off the “transverse-vector” portion to be

$$\partial^2 \alpha_i = \mathcal{J}_i. \quad (6.2.25)$$

and the “scalar” portion to be

$$-\dot{\Phi} = \mathcal{J}. \quad (6.2.26)$$

We have gotten 3 (groups of) equations – (6.2.21), (6.2.25), (6.2.26) – for 2 sets of variables (Φ, α_i) . Let us argue that eq. (6.2.26) is actually redundant. Taking into account eq. (6.2.20), we may take a time derivative of both sides of eq. (6.2.21),

$$-\vec{\nabla}^2 \dot{\Phi} = \dot{\rho}_E = \vec{\nabla}^2 \mathcal{J}. \quad (6.2.27)$$

For the physically realistic case of isolated electric currents, where we may assume implies both $\dot{\Phi} \rightarrow 0$ and $\mathcal{J} \rightarrow 0$ as the observer- J_i distance goes to infinity, the solution to this above Poisson equation is then unique. This hands us eq. (6.2.26).

Gauge-Invariant Formalism To sum: for physically realistic situations in Minkowski spacetime, if we perform a scalar-vector decomposition of the photon vector potential A_μ through eq. (6.2.1) and that of the current J_μ through eq. (6.2.19), we find a gauge-invariant Poisson equation

$$-\vec{\nabla}^2\Phi = \rho_E \quad (6.2.28)$$

for the scalar potential Φ sourced by the electric charge density ρ_E ; as well as a gauge-invariant wave equation

$$\partial^2\alpha_i = \mathcal{J}_i; \quad (6.2.29)$$

for the transverse photon α_i sourced by the transverse portion of the electric current \mathcal{J}_i . The gauge-invariant scalar Φ and photon $A_i^T \equiv \alpha_i$ are defined in equations (6.2.15) and (6.2.16).

These illuminate the theoretical structure of electromagnetism.⁵

Vacuum solution & Spin/Helicity-1 As we now turn to study, the vacuum solutions to the transverse-vector portion of A_μ may be identified with massless spin-1 photons in (3+1)-dimensions. By vacuum we mean the absence of any electric current, namely $J_\mu = 0$. For a given inertial frame, eq. (6.2.28) tells us $\Phi = 0$ (assuming $\Phi \rightarrow 0$ at spatial infinity); and eq. (6.2.29) translates to the vacuum wave equation

$$\partial^2\alpha_i = 0. \quad (6.2.30)$$

In Fourier space we may immediately write down

$$\alpha_i(t, \vec{x}) = \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^{d-1}} \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2} \right) \left\{ \epsilon_j(\vec{k}) e^{-ik \cdot x} + \text{c.c.} \right\}, \quad (6.2.31)$$

$$e^{-ik \cdot x} = e^{-i|\vec{k}|t + i\vec{k} \cdot \vec{x}}. \quad (6.2.32)$$

Here, the ϵ_j is an arbitrary \vec{k} -dependent $(d-1)$ -component object and ‘c.c.’ is the complex conjugate of the preceding term.

Let us examine a single \vec{k} -mode and suppose k_i points in the positive 3-axis, so that

$$k_\mu = k(1, 0, 0, -1) \quad \text{and} \quad k^\mu = k(1, 0, 0, 1). \quad (6.2.33)$$

This means the plane wave itself becomes

$$\exp(-ik_\mu x^\mu) = \exp(-ik(t - x^3)); \quad (6.2.34)$$

i.e., it indeed describes propagation in the positive 3-direction. The polarization vector may then be decomposed as follows:

$$\epsilon_j = \kappa \cdot \epsilon_j^\parallel + a_+ \epsilon_j^+ + a_- \epsilon_j^-; \quad (6.2.35)$$

⁵(1+1)D remark: The one constraint $\partial_i \alpha_i = 0$ obeyed by the spin-1 photon α_i means it has really $D-1 = d-2$ independent components, since in Fourier space $k_i \tilde{\alpha}_i = 0$ implies (for $\vec{k} \neq 0$) the $\{\tilde{\alpha}_i\}$ are linearly dependent. In particular, in (1+1)D $k_1 \tilde{\alpha}_1 = 0$ and as long as $k_1 \neq 0$, the spin-1 photon itself is trivial: $\tilde{\alpha}_1 = 0$.

where the κ and a 's are (scalar) complex amplitudes; while the basis vectors ϵ^\pm are

$$\epsilon_j^\parallel \equiv (0, 0, 1)^T, \quad (6.2.36)$$

$$\epsilon_\mu^\pm \equiv \frac{1}{\sqrt{2}} (0, \mp 1, i, 0)^T. \quad (6.2.37)$$

Problem 6.8. Show that for $k_\mu = k(1, 0, 0, -1)$ and referring to eq. (6.2.35),

$$\left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) \epsilon_j = a_+ \epsilon^+_i + a_- \epsilon^-_i. \quad (6.2.38)$$

That is, in a given inertial frame, the projector in eq. (6.2.31) selects only the 2D space of polarization vectors perpendicular to \vec{k} . \square

Now, under the following rotation on the (1, 2)-plane orthogonal to \vec{k} , namely

$$\widehat{R}(\theta)_{ij} \doteq \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (6.2.39)$$

the spatial polarizations in eq. (6.2.37) transform as

$$\widehat{R}(\theta)_{ij} \epsilon^\pm_j = (e^{-i\theta J})_{ij} \epsilon^\pm_j = e^{\pm i\theta} \epsilon^\pm_j. \quad (6.2.40)$$

These ϵ^\pm are the spin-1 modes because they are eigenvectors of the (Hermitian) generator of rotations on the 2D (x^1, x^2) plane.

Problem 6.9. Verify eq. (6.2.40). \square

(3+1)D Spin-1 Waves To sum, given an inertial frame, the electromagnetic vector potential A_μ in vacuum is given by the following superposition of spin-1 waves:

$$\alpha_j(t, \vec{x}) = \text{Re} \int_{\mathbb{R}^3} \frac{d^3 \vec{k}}{(2\pi)^3} \left(a_+ \epsilon^+_j(\vec{k}) + a_- \epsilon^-_j(\vec{k}) \right) e^{-ik \cdot x}, \quad (6.2.41)$$

where ϵ^\pm_μ are spatial polarization tensors orthogonal to the k_i ; and, under a rotation by an angle θ around the plane perpendicular to k_i transforms as $\epsilon^\pm \rightarrow \exp(\pm i\theta) \epsilon^\pm$.

Problem 6.10. Circularly polarized light from 4D spin-1 Consider a single spin-1 vacuum plane wave (cf. (6.2.37)) propagating along the 3-axis, with $k_\mu = k(1, 0, 0, -1)$:

$$\alpha_j^\pm(t, x, y, z) \equiv \text{Re} \left\{ a_\pm \epsilon^\pm_j e^{-ik(t-z)} \right\}, \quad a_\pm \in \mathbb{R}. \quad (6.2.42)$$

Compute the electric field ${}_\pm E^i = F^{i0}$ and show that these plane waves give rise to circularly polarized light, i.e., for either a fixed time t or spatial location z – the electric field direction rotates in a circular fashion:

$${}_\pm E^i = -\frac{ka_\pm}{\sqrt{2}} \left(\pm \sin(k(t-z)) \widehat{x}^i + \cos(k(t-z)) \widehat{y}^i \right), \quad (6.2.43)$$

where \widehat{x} and \widehat{y} are the unit vectors in the 1- and 2-directions:

$$\widehat{x}^i \doteq (1, 0, 0) \quad \text{and} \quad \widehat{y}^i \doteq (0, 1, 0). \quad (6.2.44)$$

\square

6.3 4 dimensions

4D Maxwell We now focus on the physically most relevant case of $(3 + 1)D$. In 4D, the wave operator ∂^2 has the following inverse – i.e., retarded Green’s function – that obeys causality:

$$G_4^+(x - x') \equiv \frac{\delta(t - t' - |\vec{x} - \vec{x}'|)}{4\pi|\vec{x} - \vec{x}'|}, \quad x^\mu = (t, \vec{x}), \quad x'^\mu = (t', \vec{x}'), \quad (6.3.1)$$

$$\partial_x^2 G_4^+(x - x') = \partial_{x'}^2 G_4^+(x - x') = \delta^{(4)}(x - x'), \quad (6.3.2)$$

$$\partial_x^2 \equiv \eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \quad \partial_{x'}^2 \equiv \eta^{\mu\nu} \frac{\partial}{\partial x'^\mu} \frac{\partial}{\partial x'^\nu}. \quad (6.3.3)$$

To see that G_4^+ obeys causality, that it respects the principle that cause precedes effect, one merely needs to focus on the δ -function in eq. (6.3.1). It is non-zero only when the time elapsed $t - t'$ is precisely equal to the observer-source distance $|\vec{x} - \vec{x}'|$. That is, if the source is located at a spatial distance $R = |\vec{x} - \vec{x}'|$ away from the observer, and if the source emitted an instantaneous flash at time t' , then the observer would see a signal at time R later (i.e., at $t = t' + R$). In other words, the retarded Green’s function propagates signals on the *forward* light cone of the source.⁶

Problem 6.11. Analogy: Driven Simple Harmonic Oscillator Suppose we only Fourier-transformed the spatial coordinates in the Lorenz gauge Maxwell eq. (6.1.33). Show that this leads to

$$\ddot{\tilde{A}}_\mu(t, \vec{k}) + k^2 \tilde{A}_\mu(t, \vec{k}) = \tilde{J}_\mu(t, \vec{k}), \quad k \equiv |\vec{k}|. \quad (6.3.4)$$

⁷Compare this to the simple harmonic oscillator (in flat space), with Cartesian coordinate vector $\vec{q}(t)$, mass m , spring constant σ , and driven by an external force \vec{f} :

$$m\ddot{\vec{q}} + \sigma\vec{q} = \vec{f}, \quad (6.3.5)$$

where each over-dot corresponds to a time derivative. Identify k^2 and \tilde{J} in eq. (6.3.4) with the appropriate quantities in eq. (6.3.5). Even though the Lorenz gauge Maxwell equations are fully relativistic, notice the analogy with the non-relativistic driven harmonic oscillator! In particular, when the electric current is not present (i.e., $J_\mu = 0$), the ‘mixed-space’ equations of (6.3.4) are in fact a collection of free simple harmonic oscillators.

Now, how does one solve eq. (6.3.5)? Explain why the inverse of $(d/dt)^2 + k^2$ is

$$G_{\text{SHO}}(t - t', k) = - \int_{\mathbb{R}} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega^2 - k^2}. \quad (6.3.6)$$

That is, verify that this equation satisfies

$$\left(\frac{d^2}{dt^2} + k^2 \right) G_{\text{SHO}}(t - t', k) = \left(\frac{d^2}{dt'^2} + k^2 \right) G_{\text{SHO}}(t - t', k) = \delta(t - t'). \quad (6.3.7)$$

⁶The advanced Green’s function $G_4^-(x - x') = \delta(t - t' + |\vec{x} - \vec{x}'|)/(4\pi|\vec{x} - \vec{x}'|)$ also solves eq. (6.1.45), but propagates signals on the past light cone: $t = t' - R$.

⁷This equation actually holds in all dimensions $d \geq 3$.

If one tries to integrate ω over the real line in eq. (6.3.6), one runs into trouble – explain the issue. In other words, eq. (6.3.6) is actually ambiguous as it stands.

Now evaluate the Green's function G_{SHO}^+ in eq. (6.3.6) using the contour running just slightly above the real line, i.e., $\omega \in (-\infty + i0^+, +\infty + i0^+)$. You should find

$$G_{\text{SHO}}^+(t - t', k) = \Theta(t - t') \frac{\sin(k(t - t'))}{k}. \quad (6.3.8)$$

Here, Θ is the step function

$$\Theta(x) = 1, \quad \text{if } x > 0, \quad (6.3.9)$$

$$= 0, \quad \text{if } x < 0. \quad (6.3.10)$$

Hence, the mixed-space Maxwell's equations have the solution

$$\tilde{A}_\mu(t, \vec{k}) = \int_{-\infty}^t dt' G_{\text{SHO}}^+(t - t', k) \tilde{J}_\mu(t', \vec{k}). \quad (6.3.11)$$

By performing an inverse-Fourier transform, namely

$$A_\mu(x) = \int_{\mathbb{R}^{3,1}} d^4x' G_4^+(x - x') J_{\mu'}(x'), \quad (6.3.12)$$

arrive at the expression in eq. (6.3.1) □

Redshift For each Lorenz-gauge plane wave in an inertial frame $\{x^\mu = (t, \vec{x})\}$,

$$\epsilon^\pm_\mu(k) \exp(-ik \cdot x) = \epsilon^\pm_\mu(k) \exp(-ik_j x^j) \exp(-i\omega t), \quad \omega \equiv |\vec{k}|, \quad (6.3.13)$$

we may read off its frequency ω as the coefficient of the time coordinate t . Quantum mechanics tells us ω is also the energy of the associated photon. Suppose a different Lorenz inertial frame $\{x'\}$ is related to the previous through the Lorenz transformation Λ^α_μ : $x^\alpha = \Lambda^\alpha_\mu x'^\mu$. Because the phase in the plane wave solution of eq. (6.3.13) is a scalar, in the $\{x'\}$ Lorenz frame

$$-ik_\alpha x^\alpha = -ik_\alpha \Lambda^\alpha_\mu x'^\mu = -i(k_\alpha \Lambda^\alpha_0) t' - i(k_\alpha \Lambda^\alpha_i) x'^i. \quad (6.3.14)$$

The frequency ω' and hence the photon's energy in this $\{x'\}$ frame is therefore

$$\omega' = k_\alpha \Lambda^\alpha_0 = \omega \left(\Lambda^0_0 + \hat{k}_i \Lambda^i_0 \right) \quad (6.3.15)$$

$$\hat{k}_i \equiv k_i / |\vec{k}| = k_i / \omega. \quad (6.3.16)$$

There is a slightly different way to express this redshift result that would help us generalize the analysis to curved spacetime, at least in the high frequency 'JWKB' limit. To extract the frequency directly from the phase $S \equiv k \cdot x$, we may take its time derivative using the unit norm vector $u \equiv \partial_t = \partial_0$ that we may associate with the worldlines of observers at rest in the $\{x\}$ frame:

$$u^\mu \partial_\mu S = \partial_0(k_\alpha x^\alpha) = \omega. \quad (6.3.17)$$

The observers at rest in the $\{x'\}$ frame have $u' \equiv \partial_{t'} = \partial_{0'}$ as their timelike unit norm tangent vector. (Note: $x^\alpha = \Lambda^\alpha_\mu x'^\mu \Leftrightarrow \partial_{\mu'} = \Lambda^\alpha_\mu \partial_\alpha$.) The energy of the photon is then

$$\begin{aligned} u'^\alpha \partial_{\alpha'} S &= \partial_{t'} S = \Lambda^\alpha_0 \partial_\alpha(k \cdot x) \\ &= \Lambda^\alpha_0 k_\alpha = \omega \left(\Lambda^0_0 + \hat{k}_i \Lambda^i_0 \right). \end{aligned} \quad (6.3.18)$$

Problem 6.12. Consider a single photon with wave vector $k_\mu = \omega(1, \hat{n}_i)$ (where $\hat{n}_i \hat{n}_j \delta^{ij} = 1$) in some inertial frame $\{x^\mu\}$. Let a family of inertial observers be moving with constant velocity $v^\mu \equiv (1, v^i)$ with respect to the frame $\{x^\mu\}$. What is the photon's frequency ω' in their frame? Compute the redshift formula for ω'/ω . Comment on the redshift result when v^i is (anti)parallel to \hat{n}_i and when v^i is perpendicular to \hat{n}_i .

7 (Semi-)Classical Action Principles & Symmetries

7.1 Dirac Spinors in (3+1)D

The Lagrangian for the Dirac equation coupled to an electromagnetic vector potential is

$$\mathcal{L} = \bar{\psi}(i\mathcal{D} - m)\psi. \quad (7.1.1)$$

Here, the covariant derivative is

$$D_\mu \equiv \partial_\mu - ieA_\mu \quad (7.1.2)$$

and $\mathcal{D} \equiv \gamma^\mu D_\mu$. The mass is m and $\bar{\psi} \equiv \psi^\dagger \gamma^0$ and $\{\gamma^\mu\}$ are the Dirac matrices obeying the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. Furthermore, in the chiral basis, which may be expressed in terms of the Pauli matrices $\sigma^\mu = (\mathbb{I}, \sigma^i)$ and $\bar{\sigma}^\mu = (\mathbb{I}, -\sigma^i)$ as

$$\gamma^\mu \equiv \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}; \quad (7.1.3)$$

we may split the 4-component spinor into left and right handed Weyl spinors $\psi = (\psi_L \ \psi_R)^T$ and recognize

$$\mathcal{L} = \psi_L^\dagger \bar{\sigma}^\mu D_\mu \psi_L + \psi_R^\dagger \sigma^\mu D_\mu \psi_R - m \left(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L \right). \quad (7.1.4)$$

Those familiar with the $SL_{2,\mathbb{C}}$ transformations would recognize this to be manifestly invariant under the Lorentz group. In particular, the mass term couples the left- and right-handed Weyl spinors.

Because ψ is complex, we may vary ψ and ψ^\dagger independently – for e.g., the complex number $z = x + iy$ may be varied by altering x and y independently or z and z^* independently. Doing so would yield

$$\int d^4x' \delta_\psi \mathcal{L} = \int d^4x' \psi^\dagger \gamma^0 (i\mathcal{D} - m) \delta\psi \quad (7.1.5)$$

$$= \int d^4x' \{ (-i\partial_\mu + eA_\mu) \psi^\dagger \} \gamma^0 \gamma^\mu \delta\psi - m \psi^\dagger \gamma^0 \delta\psi + \text{surface term}. \quad (7.1.6)$$

Because

$$\gamma^0 \gamma^\mu = \begin{bmatrix} \bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{bmatrix}, \quad (7.1.7)$$

this means $\gamma^0 \gamma^\mu$ is hermitian:

$$\gamma^0 \gamma^\mu = (\gamma^\mu)^\dagger \gamma^0. \quad (7.1.8)$$

And

$$\int d^4x' \delta_\psi \mathcal{L} = \int d^4x' \{ \gamma^\mu (i\partial_\mu + eA_\mu) \psi - m\psi \}^\dagger \gamma^0 \delta\psi + \text{surface term}. \quad (7.1.9)$$

This leads us to the Dirac equation

$$(i\mathcal{D} - m)\psi = 0. \quad (7.1.10)$$

We may also vary ψ^\dagger to deduce this same equation more readily.

$$\int d^4x' \delta\psi^\dagger \mathcal{L} = \int d^4x' \delta\psi^\dagger \gamma^0 (i\mathcal{D} - m)\psi. \quad (7.1.11)$$

Gauge Invariance Note that this Dirac Lagrangian is invariant under the simultaneous replacements

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu\theta(x), \quad (7.1.12)$$

$$\psi \rightarrow e^{ie\theta}\psi. \quad (7.1.13)$$

This guarantees the gauge-invariance of the Dirac equation $(i\mathcal{D} - m)\psi = 0$.

7.2 Noether's Theorem & Internal Symmetries

If a Lagrangian is invariant, up to an additive total divergence, under a transformation of the fields and/or spacetime coordinates – then there exists a corresponding conserved current when it is evaluated on the equations-of-motion. This is known as Noether's theorem.

Dirac Lagrangians and U_1 Symmetry Let us notice, under a global U_1 phase rotation

$$\psi \rightarrow e^{i\theta}\psi \quad \text{and} \quad \psi^\dagger \rightarrow \psi^\dagger e^{-i\theta}; \quad (7.2.1)$$

the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(i\mathcal{D} - m)\psi \quad (7.2.2)$$

is invariant. We may identify

$$\delta\psi = i\theta\psi \quad \text{and} \quad \delta\psi^\dagger = -i\theta\psi^\dagger. \quad (7.2.3)$$

Varying the Lagrangian directly,

$$0 = -i\theta\bar{\psi}(i\mathcal{D} - m)\psi + i^2\theta\partial_\mu(\psi^\dagger\gamma^0\gamma^\mu\psi) + i\{(-i\partial_\mu + eA_\mu)\psi^\dagger\}\gamma^0\gamma^\mu\psi + i\theta\psi^\dagger\gamma^0(-m)\psi. \quad (7.2.4)$$

Assuming the EoM holds, $(i\mathcal{D} - m)\psi = 0$,

$$0 = i^2\theta\partial_\mu(\psi^\dagger\gamma^0\gamma^\mu\psi) + i\{(\gamma^\mu(i\partial_\mu + eA_\mu) - m)\psi\}^\dagger\gamma^0\psi \quad (7.2.5)$$

$$= i^2\theta\partial_\mu(\bar{\psi}\gamma^\mu\psi). \quad (7.2.6)$$

We have the Noether current of U_1 symmetry:

$$\partial_\mu J^\mu = 0 \quad \text{and} \quad J^\mu = \bar{\psi}\gamma^\mu\psi. \quad (7.2.7)$$

Talk about divergence of current using Dirac equation. Also chiral current. $\partial_\mu J^{\mu 5} = 2im\bar{\psi}\gamma^5\psi$. $\psi \rightarrow e^{i\alpha\gamma^5}\psi$.

7.3 Noether's Theorem & Energy-Momentum-Shear-Stress

Free Photons We turn to the free photon, satisfying the free Maxwell equation

$$\partial_\mu F^{\mu\nu} = 0 \quad \text{with} \quad F_{\mu\nu} \equiv \partial_{[\mu} A_{\nu]}, \quad (7.3.1)$$

$$0 = \partial_{[\alpha} F_{\beta\gamma]}; \quad (7.3.2)$$

which in turn arises from the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (7.3.3)$$

Here, there is a subtlety. While $F_{\mu\nu}$ is a Lorentz rank-2 tensor; the transformation properties of A_μ depends on the choice of gauge. If a non-covariant gauge is chosen, there is no reason to claim A_μ is a rank-1 Lorentz tensor. To this end, for technical convenience, we shall choose the Lorentz covariant Lorenz gauge

$$\partial^\mu A_\mu = 0. \quad (7.3.4)$$

Translations Under spacetime translations, the vector potential then transforms as follows.

$$A_\mu(x) \rightarrow A_\mu(x+a) = A_\mu(x) + a^\nu \partial_\nu A_\mu(x) \quad (7.3.5)$$

$$\partial_\mu A_\nu(x) \rightarrow \partial_\mu A_\nu(x+a) = \partial_\mu A_\nu(x) + a^\sigma \partial_\sigma \partial_\mu A_\nu(x). \quad (7.3.6)$$

The Lagrangian density transforms as

$$\mathcal{L}(x) \rightarrow \mathcal{L}(x) + a^\sigma \partial_\sigma \mathcal{L}(x) \quad (7.3.7)$$

$$\rightarrow \mathcal{L} + \partial_\mu (a^\sigma \delta_\sigma^\mu \mathcal{L}). \quad (7.3.8)$$

Whereas, if we had expanded it via its fields,

$$\mathcal{L} \rightarrow \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \partial_\lambda A_\tau} a^\sigma \partial_\sigma \partial_\lambda A_\tau(x) \quad (7.3.9)$$

$$= \mathcal{L} + \partial_\lambda \left(\frac{\partial \mathcal{L}}{\partial \partial_\lambda A_\tau} a^\sigma \partial_\sigma A_\tau(x) \right) - \partial_\lambda \left(\frac{\partial \mathcal{L}}{\partial \partial_\lambda A_\tau} \right) a^\sigma \partial_\sigma A_\tau(x). \quad (7.3.10)$$

The Noether current evaluated on the EoM $\partial_\lambda (\partial \mathcal{L} / \partial (\partial_\lambda A_\tau)) = 0$ is

$$J^\mu_\nu = \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\tau} \partial_\nu A_\tau(x) - \delta^\mu_\nu \mathcal{L} \quad (7.3.11)$$

$$= -F^{\mu\tau} \partial_\nu A_\tau + \frac{1}{4} \delta^\mu_\nu F_{\lambda\tau} F^{\lambda\tau}. \quad (7.3.12)$$

Notice this is not symmetric nor gauge-invariant!⁸ Let's backtrack a little.

$$a^\sigma \partial_\sigma \mathcal{L} = \delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\tau)} a^\sigma \partial_\sigma \partial_\lambda A_\tau \quad (7.3.13)$$

$$= -a^\sigma F^{\lambda\tau} \partial_\sigma \partial_\lambda A_\tau = -\frac{1}{2} a^\sigma F^{\lambda\tau} \partial_\sigma F_{\lambda\tau}. \quad (7.3.14)$$

Let us use the Bianchi identity $\partial_{[\sigma} F_{\lambda\tau]} = 0$ to deduce

$$\partial_\sigma F_{\lambda\tau} = \partial_\lambda F_{\sigma\tau} + \partial_\tau F_{\lambda\sigma} = -\partial_{[\lambda} F_{\tau]\sigma}. \quad (7.3.15)$$

Inserting this back into the first order variation of the Lagrangian,

$$a^\sigma \partial_\mu (\delta_\sigma^\mu \mathcal{L}) = \frac{1}{2} a^\sigma F^{\lambda\tau} \partial_{[\lambda} F_{\tau]\sigma} \quad (7.3.16)$$

$$= a^\sigma F^{\lambda\tau} \partial_\lambda F_{\tau\sigma} = -a^\sigma F^{\lambda\tau} \partial_\lambda F_{\sigma\tau} \quad (7.3.17)$$

$$= -a^\nu \partial_\mu (F^{\mu\sigma} F_{\nu\sigma}). \quad (7.3.18)$$

Our gauge invariant and symmetric Noether current is therefore

$$J^{\mu\nu} = -F^{\mu\sigma} F^\nu{}_\sigma + \frac{1}{4} \eta^{\mu\nu} F^{\sigma\rho} F_{\sigma\rho} \equiv T^{\mu\nu} \quad (7.3.19)$$

$$0 = \partial_\mu J^{\mu\nu}. \quad (7.3.20)$$

Lorentz transformations We now turn to the Noether current of Lorentz transformations. For *fixed* (α, β) , we have

$$\partial_{\mu'} A_{\nu'}(x') = \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial x^\rho}{\partial x'^\nu} \partial_\sigma A_\rho(x = \Lambda x') \quad (7.3.21)$$

$$\begin{aligned} &= \partial_\mu A_\nu(x \rightarrow x') + i\omega \left(\widehat{J}^{\alpha\beta} \right)_\mu^\sigma \partial_\sigma A_\nu(x \rightarrow x') + i\omega \left(\widehat{J}^{\alpha\beta} \right)_\nu^\sigma \partial_\mu A_\sigma(x \rightarrow x') \\ &\quad - i\omega \left(\widehat{J}^{\alpha\beta} \right)_\rho^\sigma x'^\rho \partial_{\sigma'} \partial_\mu A_\nu(x \rightarrow x'). \end{aligned} \quad (7.3.22)$$

The Lagrangian, being a Lorentz scalar, transforms as

$$\mathcal{L}(x = \Lambda x') = \mathcal{L}(x \rightarrow x') - i\omega \left(\widehat{J}^{\alpha\beta} \right)_\rho^\sigma x'^\rho \partial_{\sigma'} \mathcal{L}(x \rightarrow x'). \quad (7.3.23)$$

This means, under $x \rightarrow \Lambda x$, we have

$$x^{[\beta} \partial^{\alpha]} \mathcal{L} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \left(\delta_\mu^{[\alpha} \partial^{\beta]} A_\nu + \delta_\nu^{[\alpha} \partial_\mu A^{\beta]} + x^{[\beta} \partial^{\alpha]} \partial_\mu A_\nu \right) \quad (7.3.24)$$

$$= -F^{\mu\nu} \left(\delta_\mu^{[\alpha} \partial^{\beta]} A_\nu + \delta_\nu^{[\alpha} \partial_\mu A^{\beta]} + x^{[\beta} \partial^{\alpha]} \partial_\mu A_\nu \right). \quad (7.3.25)$$

⁸Several sources, including the *An Introduction to Quantum Field Theory* textbook by Peskin and Schroeder and the *Classical Field Theory* book by Davison Soper, stops here. The latter goes on to assert the electromagnetic Noether current generated by spacetime translations is not gauge-invariant nor symmetric because of 'internal spin'. It is true that the Noether current is not unique; a $\partial_\sigma K^{\mu\sigma\nu}$ may always be added to $J^{\mu\nu}$, while still maintaining $\partial_\mu J^{\mu\nu} = 0$, if K is anti-symmetric in the first two indices: $K^{\mu\sigma\nu} = -K^{\sigma\mu\nu}$. However, we will soon witness, if the index anti-symmetries and Bianchi identities are properly employed, a symmetric and gauge invariant free Maxwell stress tensor may be obtained.

Using the explicit expression $F^{\mu\nu} = \partial^{[\mu} A^{\nu]}$, and via a direct calculation, we may verify the first two terms on the last line vanish.

$$\partial_\mu (x^{[\beta} \eta^{\alpha]\mu} \mathcal{L}) = -F^{\mu\nu} x^{[\beta} \partial^{\alpha]} \partial_\mu A_\nu = -\frac{1}{2} F^{\mu\nu} x^{[\beta} \partial^{\alpha]} F_{\mu\nu} \quad (7.3.26)$$

$$= \frac{1}{2} F^{\mu\nu} x^{[\beta} \partial_{[\mu} F_{\nu]}^{\alpha]} = -F^{\mu\nu} x^{[\beta} \partial_\mu F_{\nu]}^{\alpha]}. \quad (7.3.27)$$

Bianchi was employed in the second line. Recalling the free photon EoM,

$$\partial_\mu (x^{[\beta} \eta^{\alpha]\mu} \mathcal{L}) = -\partial_\mu (F^{\mu\nu} x^{[\beta} F_{\nu]}^{\alpha]}) + F^{\mu\nu} \delta_\mu^{[\beta} F_{\nu]}^{\alpha]} \quad (7.3.28)$$

$$= \partial_\mu (-x^{[\beta} F_{\nu]}^{\alpha]} F^{\mu\nu}) + F^{\mu\nu} F_{\mu\nu}^{\alpha\beta}. \quad (7.3.29)$$

The second term on the final line is zero, because we are anti-symmetrizing the indices of a symmetric tensor. Comparison with the stress tensor of the free electromagnetic field, we obtain the angular momentum Noether tensor

$$M^{\mu\alpha\beta} = x^{[\alpha} T^{\beta]\mu} = x^{[\alpha} \left(-F^{\beta]\sigma} F^\mu_\sigma + \frac{1}{4} \eta^{\beta]\mu} F_{\sigma\rho} F^{\sigma\rho} \right) \quad (7.3.30)$$

$$0 = \partial_\mu M^{\mu\alpha\beta}. \quad (7.3.31)$$

Higher Spin Fields Let us now move on to Lorentz transform non-scalar fields.

$$\Phi_I(x) \rightarrow \Phi_I(x) + \frac{1}{2} \omega^{\mu\nu} x_{[\mu} \partial_{\nu]} \Phi_I(x) - \frac{i}{2} \omega^{\mu\nu} (J_{\mu\nu})_I^J \Phi_J(x) \quad (7.3.32)$$

$$\partial_\alpha \Phi_I(x) \rightarrow \partial_\alpha \Phi_I(x) + \omega_\alpha^{\beta} \partial_\beta \Phi_I(x) + \frac{1}{2} \omega^{\mu\nu} x_{[\mu} \partial_{\nu]} \partial_\alpha \Phi_I(x) - \frac{i}{2} \omega^{\mu\nu} (J_{\mu\nu})_I^J \partial_\alpha \Phi_J(x). \quad (7.3.33)$$

YZ: To be continued.

Problem 7.1. Massive Dirac field Use the Noether method to derive – up to an additive identically-divergence-free current – the energy-momentum-shear-stress tensor of the Dirac field. Assume its Lagrangian is

$$\mathcal{L} = -\frac{i}{2} (\partial_\mu \bar{\psi}) \gamma^\mu \psi + \frac{i}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi. \quad (7.3.34)$$

Bonus: Also work out the angular momentum Noether tensor $M^{\mu\alpha\beta}$ by considering an infinitesimal Lorentz transformation. \square

8 QFT of Photons

The path integral quantization of photons involves integrating over the action

$$S_A \equiv \int \left(-\frac{1}{2} F^2 \right) d^4x \quad (8.0.1)$$

$$= -\frac{1}{4} \int_k \left(k_\mu \tilde{A}_\nu(k) - k_\nu \tilde{A}_\mu(k) \right) \left(k^\mu \tilde{A}^\nu(k)^* - k^\nu \tilde{A}^\mu(k)^* \right) \quad (8.0.2)$$

$$= -\frac{1}{2} \int_k \tilde{A}_\alpha(k) (k^2 \eta^{\alpha\beta} - k^\alpha k^\beta) \tilde{A}_\beta(-k) \quad (8.0.3)$$

However, because k_α is the null eigenvector of $k^2\eta^{\alpha\beta} - k^\alpha k^\beta$, the path integral involving photons is like integrating the gaussian

$$\int d^D \vec{x} \exp(-x^i M_{ij} x^j), \quad (8.0.4)$$

where the real symmetric M_{ij} has N positive eigenvalues $\{\lambda_I\}$; but $D - N$ eigenvalues equal to zero. That means

$$\int d^D \vec{x} \exp(-x^i M_{ij} x^j) = \left(\prod_{I=1}^N \int dz_I \exp(-(z_I)^2 \lambda_I) \right) \left(\prod_{J=1}^{D-b} \int dz'_J \right). \quad (8.0.5)$$

We see the integral is wildly divergent because of the $D - N$ zeroes. To factor out this divergence in the path integral, we employ a procedure due to Fadeev and Popov. Consider

$$\int \prod_{\mu} \mathcal{D}A_{\mu} O[A] \exp \left[-\frac{i}{4} \int_x F_{\mu\nu} F^{\mu\nu} d^4x \right], \quad (8.0.6)$$

where we assume $O[A]$ is a gauge invariant object built out of the vector potential A_{μ} . We insert an identity via

$$1 = \int \mathcal{D}\lambda \delta[\partial \cdot A - \omega - \partial^2 \lambda] \det \frac{\delta(\partial \cdot A - \omega - \partial^2 \lambda)}{\delta \lambda}, \quad (8.0.7)$$

where the δ -function is a functional one; and ω is an arbitrary function.

$$\int \prod_{\mu} \mathcal{D}A_{\mu} O[A] \exp \left[-\frac{i}{4} \int_x F_{\mu\nu} F^{\mu\nu} d^4x \right] \quad (8.0.8)$$

$$= \det[-\partial^2] \int \mathcal{D}\lambda \int \prod_{\mu} \mathcal{D}A_{\mu} O[A] \exp \left[-\frac{i}{4} \int_x F_{\mu\nu} F^{\mu\nu} d^4x \right] \delta[\partial \cdot A - \omega - \partial^2 \lambda] \quad (8.0.9)$$

But everything except the functional δ -function is gauge-invariant. So we may shift $A \rightarrow A + \partial\lambda$ and deduce

$$\int \prod_{\mu} \mathcal{D}A_{\mu} O[A] \exp \left[-\frac{i}{4} \int_x F_{\mu\nu} F^{\mu\nu} d^4x \right] \quad (8.0.10)$$

$$= \det[-\partial^2] \int \mathcal{D}\lambda \int \prod_{\mu} \mathcal{D}A_{\mu} O[A] \exp \left[-\frac{i}{4} \int_x F_{\mu\nu} F^{\mu\nu} d^4x \right] \delta[\partial \cdot A - \omega]. \quad (8.0.11)$$

One more trick. Let us multiply throughout by $\exp(-i \int_x \omega(x)^2 / (2\xi))$, followed by integrating over ω . On the LHS, we obtain a ξ dependent quantity $\mathcal{N}(\xi) = \int \mathcal{D}\omega \exp(-i \int_x \omega(x)^2 / (2\xi))$. On the right hand side,

$$\det[-\partial^2] \int \mathcal{D}\lambda \int \prod_{\mu} \mathcal{D}A_{\mu} O[A] \exp \left[-\frac{i}{4} \int_x F_{\mu\nu} F^{\mu\nu} d^4x \right] \int \mathcal{D}\omega \delta[\partial \cdot A - \omega] e^{-i \int_x \frac{\omega(x)^2}{2\xi}} \quad (8.0.12)$$

$$= \det[-\partial^2] \int \mathcal{D}\lambda \int \prod_{\mu} \mathcal{D}A_{\mu} O[A] \exp \left[-\frac{i}{4} \int_x F_{\mu\nu} F^{\mu\nu} d^4x \right] e^{-i \int_x \frac{(\partial A)^2}{2\xi}}. \quad (8.0.13)$$

The exponent now reads

$$\int_x -\frac{1}{4}F^2 - (\partial \cdot A)^2(2\xi)^{-1} = \int_k \frac{1}{2}\widehat{A}_\alpha(k) (-k^2\eta^{\alpha\beta} + k^\alpha k^\beta (1 - \xi^{-1})) \widetilde{A}_\beta(-k). \quad (8.0.14)$$

Hence, we gather at this point,

$$\begin{aligned} \mathcal{N}(\xi) \int \prod_\mu \mathcal{D}A_\mu O[A] \exp \left[-\frac{i}{4} \int_x F_{\mu\nu} F^{\mu\nu} d^4x \right] \\ = \det [-\partial^2] \int \mathcal{D}\lambda \int \prod_\mu \mathcal{D}A_\mu O[A] \exp \left[\frac{i}{2} \int_k \widehat{A}_\alpha(k) (-k^2\eta^{\alpha\beta} - k^\alpha k^\beta (1 - \xi^{-1})) \widetilde{A}_\beta(-k) \right]. \end{aligned} \quad (8.0.15)$$

The non-interacting vacuum expectation value of the gauge-invariant operator $O[A]$ is therefore

$$\langle 0 | \mathcal{T} O[A] | 0 \rangle = \frac{\int \prod_\mu \mathcal{D}A_\mu O[A] \exp [iS_0[A]]}{\int \prod_\mu \mathcal{D}A_\mu \exp [iS_0[A]]}, \quad (8.0.16)$$

$$S_0[A] \equiv \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \widehat{A}_\alpha(k) (-k^2\eta^{\alpha\beta} + k^\alpha k^\beta (1 - \xi^{-1})) \widetilde{A}_\beta(-k) \quad (8.0.17)$$

We see the Fadeev-Popov gauge-fixing procedure has eliminated all the divergences related to gauge-freedom. Moreover, upon comparison with the “completing-the-square” steps taken for the scalar case, we see the Feynman photon Green’s function is defined by

$$(-k^2\eta^{\alpha\beta} + k^\alpha k^\beta (1 - \xi^{-1})) \widetilde{G}_{\beta\gamma} = i\delta_\gamma^\alpha. \quad (8.0.18)$$

It’s convenient to choose the usual Lorenz gauge via

$$\xi = 1. \quad (8.0.19)$$

Then,

$$G_F^{\alpha\beta}(x - x') = \int_k \frac{-i\eta^{\alpha\beta}}{k^2 + i\epsilon} e^{-ik \cdot (x - x')}. \quad (8.0.20)$$

Feynman Rules The contraction of a photon A_μ with an external line with incoming momentum/photon is given by

$$\epsilon_\mu(p) \quad (8.0.21)$$

whereas the contraction of a photon A_μ with an external line with outgoing momentum/photon is given by

$$\epsilon_\mu(p)^*. \quad (8.0.22)$$

9 Scattering & Decay Processes

9.1 1 Scalar + 1 Fermion: Yukawa Theory

In this section we shall consider the theory defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m_\varphi^2\varphi^2 + \bar{\psi}(i\not{\partial} - m_\psi)\psi + g\varphi\bar{\psi}\psi. \quad (9.1.1)$$

This theory has a single interaction vertex: scalar-fermion-fermion.

Fermion-Antifermion Annihilation A pair of fermion-antifermion may annihilate and produce a pair of scalars. At leading order,

$$\begin{aligned} & \left\langle p_1, p_2 \left| \frac{(ig)^2}{2} \int_{x,y} \varphi_x \bar{\psi}_x \psi_x \varphi_y \bar{\psi}_y \psi_y \right| p_\psi, s; p_{\bar{\psi}}, \bar{s} \right\rangle_c \\ &= -ig^2 \left(\bar{v}_{p_{\bar{\psi}}}^{\bar{s}} \frac{\not{k} + m_\psi}{k^2 - m_\psi^2 + i\epsilon} u_{p_\psi}^s + (p_1 \leftrightarrow p_2) \right) (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_\psi - p_{\bar{\psi}}). \end{aligned} \quad (9.1.2)$$

Fermion-Fermion Scattering Let us now consider scattering 2 fermions into another 2 fermions.

$$\begin{aligned} & \left\langle p'_1, s'_1; p'_2, s'_2 \left| \frac{(ig)^2}{2} \int_{x,y} \varphi_x \bar{\psi}_x \psi_x \varphi_y \bar{\psi}_y \psi_y \right| p_1, s_1; p_2, s_2 \right\rangle_c \\ &= ig^2 \left(\frac{(\bar{u}_{p'_2}^{s'_2} u_{p_2}^{s_2})(\bar{u}_{p'_1}^{s'_1} u_{p_1}^{s_1})}{k^2 - m_\varphi^2 + i\epsilon} - \frac{(\bar{u}_{p'_1}^{s'_1} u_{p_2}^{s_2})(\bar{u}_{p'_2}^{s'_2} u_{p_1}^{s_1})}{k^2 - m_\varphi^2 + i\epsilon} \right) (2\pi)^4 \delta^{(4)}(p'_1 + p'_2 - p_1 - p_2). \end{aligned} \quad (9.1.3)$$

Decay of Scalar If the scalar is more than twice as massive as the fermion, $m_\varphi > 2m_\psi$, it can decay into a fermion-antifermion pair. To leading order, we need to study the transition amplitude

$$\left\langle p_\psi, s; p_{\bar{\psi}}, \bar{s} \left| ig \int_x \varphi_x \bar{\psi}_x \psi_x \right| p_\varphi \right\rangle_c = ig(2\pi)^4 \delta^{(4)}(p_\varphi - p_\psi - p_{\bar{\psi}}) \bar{u}_{p_\psi}^s v_{p_{\bar{\psi}}}^{\bar{s}}; \quad (9.1.4)$$

where s is the spin of the fermion and \bar{s} is that of antifermion. The leading order amplitude is (up to overall $-$ signs)

$$\mathcal{M}(\varphi \rightarrow \psi + \bar{\psi}) = g \bar{u}_{p_\psi}^s v_{p_{\bar{\psi}}}^{\bar{s}}. \quad (9.1.5)$$

9.2 Quantum Electrodynamics (QED)

In this section we shall consider Quantum Electrodynamics (QED) defined by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{\text{I}} \{ \bar{\psi}_{\text{I}}(i\not{\partial} - m_{\text{I}})\psi_{\text{I}} - e_{\text{I}}\bar{\psi}_{\text{I}}\not{A}\psi_{\text{I}} \}. \quad (9.2.1)$$

In what follows, it is common to define

$$\alpha \equiv \frac{e^2}{4\pi} \approx \frac{1}{137}. \quad (9.2.2)$$

For the I-th species of fermion, the theory has a single interaction vertex: photon-fermion-fermion. In the Lorenz gauge the photon propagator is

$$\langle \tilde{A}_\mu(p) \tilde{A}_\nu(p') \rangle = \frac{-i\eta_{\mu\nu}}{p^2 + i\epsilon} (2\pi)^4 \delta^{(4)}(p + p'). \quad (9.2.3)$$

The fermion propagator is

$$\langle \tilde{\psi}_I(p) \tilde{\psi}_I(p') \rangle = \frac{i(\not{p} + m)}{p^2 - m_I^2 + i\epsilon} (2\pi)^4 \delta^{(4)}(p + p'). \quad (9.2.4)$$

Electron-Positron to Muon-Antimuon The annihilation of electron-positron can produce a muon-antimuon pair. The amplitude at leading order is given by

$$(ie)^2 \left\langle p_\mu, s_\mu; p_{\bar{\mu}}, s_{\bar{\mu}} \left| \text{T} \int_{x,y} \bar{\mu}_x \not{A}_x \mu_x \bar{e}_y \not{A}_y e_y \right| p_e, p_{\bar{e}} \right\rangle_c \quad (9.2.5)$$

$$= -e^2 \bar{u}_{p_\mu}^{s_\mu} \gamma^\alpha v_{p_{\bar{\mu}}}^{s_{\bar{\mu}}} \frac{-i\eta_{\alpha\beta}}{k^2 + i\epsilon} \bar{v}_{p_{\bar{e}}}^{s_{\bar{e}}} \gamma^\beta u_{p_e}^{s_e} (2\pi)^4 \delta^{(4)}(p_e + p_{\bar{e}} - p_\mu - p_{\bar{\mu}}) \quad (9.2.6)$$

$$i\mathcal{M} = \frac{ie^2}{k^2 + i\epsilon} \bar{u}_{p_\mu}^{s_\mu} \gamma^\alpha v_{p_{\bar{\mu}}}^{s_{\bar{\mu}}} \bar{v}_{p_{\bar{e}}}^{s_{\bar{e}}} \gamma^\beta u_{p_e}^{s_e}. \quad (9.2.7)$$

Let's square the amplitude and average of the initial spins and sum over the final spins. We'll see it simplifies the end result.

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{4k^4} \sum_{\text{spins}} \bar{v}_{p_{\bar{\mu}}}^{s_{\bar{\mu}}} \gamma^\beta u_{p_\mu}^{s_\mu} \bar{u}_{p_\mu}^{s_\mu} \gamma^\alpha v_{p_{\bar{\mu}}}^{s_{\bar{\mu}}} \bar{v}_{p_{\bar{e}}}^{s_{\bar{e}}} \gamma^\alpha u_{p_e}^{s_e} \bar{u}_{p_e}^{s_e} \gamma^\beta v_{p_{\bar{e}}}^{s_{\bar{e}}} \quad (9.2.8)$$

$$= \frac{e^4}{4k^4} \text{Tr} \left[(\not{p}_{\bar{\mu}} - m_\mu) \gamma^\beta (\not{p}_\mu + m_\mu) \gamma^\alpha \right] \text{Tr} \left[(\not{p}_{\bar{e}} - m_e) \gamma^\alpha (\not{p}_e + m_e) \gamma^\beta \right] \quad (9.2.9)$$

Let's examine

$$\text{Tr} \left[(\not{p}_{\bar{\mu}} - m_\mu) \gamma^\beta (\not{p}_\mu + m_\mu) \gamma^\alpha \right] \quad (9.2.10)$$

$$= \text{Tr} \left[\not{p}_{\bar{\mu}} \gamma^\beta (\not{p}_\mu + m_\mu) \gamma^\alpha \right] - m_\mu \left(\text{Tr} \left[\gamma^\beta \not{p}_\mu \gamma^\alpha \right] + m_\mu \text{Tr} \left[\gamma^\beta \gamma^\alpha \right] \right) \quad (9.2.11)$$

$$= \text{Tr} \left[\not{p}_{\bar{\mu}} \gamma^\beta \not{p}_\mu \gamma^\alpha \right] + m_\mu \text{Tr} \left[\not{p}_{\bar{\mu}} \gamma^\beta \gamma^\alpha \right] - m_\mu^2 \text{Tr} \left[\gamma^\beta \gamma^\alpha \right] \quad (9.2.12)$$

$$= \text{Tr} \left[\not{p}_{\bar{\mu}} \gamma^\beta \not{p}_\mu \gamma^\alpha \right] - m_\mu^2 \text{Tr} \left[\gamma^\beta \gamma^\alpha \right] \quad (9.2.13)$$

$$= 4 \left(p_{\bar{\mu}}^{\{\beta} p_\mu^{\alpha\}} - (p_{\bar{\mu}} \cdot p_\mu) \eta^{\alpha\beta} - m_\mu^2 \eta^{\alpha\beta} \right) \quad (9.2.14)$$

Next we have

$$\text{Tr} \left[(\not{p}_{\bar{e}} - m_e) \gamma^\alpha (\not{p}_e + m_e) \gamma^\beta \right] = 4 \left(p_{\bar{e}}^{\{\beta} p_e^{\alpha\}} - (p_{\bar{e}} \cdot p_e) \eta^{\alpha\beta} - m_e^2 \eta^{\alpha\beta} \right). \quad (9.2.15)$$

We'll follow Peskin and Schröder and simplify the analysis even further by assuming the scattering is occurring at high enough center-of-mass energy $E \gg m_e$ that the electron mass can be neglected. This leads us to

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{k^4} \left(m_\mu^2 (p_e \cdot p_{\bar{e}}) + (p_e \cdot p_\mu) (p_{\bar{e}} \cdot p_{\bar{\mu}}) + (p_{\bar{e}} \cdot p_\mu) (p_e \cdot p_{\bar{\mu}}) \right) \quad (9.2.16)$$

Furthermore, let us work in the COM frame, and take (remember $m_e = 0$)

$$p_e = E(1, \hat{z}), \quad (9.2.17)$$

$$p_{\bar{e}} = E(1, -\hat{z}); \quad (9.2.18)$$

and

$$p_\mu = (E, \vec{k}), \quad (9.2.19)$$

$$p_{\bar{\mu}} = (E, -\vec{k}), \quad E^2 = k^2 + m_\mu^2. \quad (9.2.20)$$

Now, $(p_\mu + p_{\bar{\mu}})^2 = (p_e + p_{\bar{e}})^2 = (2E)^2 = k^2 = 2m_\mu^2 + 2(p_\mu \cdot p_{\bar{\mu}})$.

$$p_e \cdot p_{\bar{e}} = 2E^2, \quad k^4 = 16E^4 \quad (9.2.21)$$

$$p_e \cdot p_{\bar{\mu}} = E^2 + E\vec{k} \cdot \hat{z} \quad (9.2.22)$$

$$p_e \cdot p_\mu = E^2 - E\vec{k} \cdot \hat{z} \quad (9.2.23)$$

$$p_{\bar{e}} \cdot p_{\bar{\mu}} = E^2 - E\vec{k} \cdot \hat{z} \quad (9.2.24)$$

$$p_{\bar{e}} \cdot p_\mu = E^2 + E\vec{k} \cdot \hat{z} \quad (9.2.25)$$

Hence, we have

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{16E^4} \left(m_\mu^2(2E^2) + E^2(E - \vec{k} \cdot \hat{z})^2 + E^2(E + \vec{k} \cdot \hat{z})^2 \right) \quad (9.2.26)$$

$$= \frac{e^4}{2E^4} E^2 \left(2m_\mu^2 + 2E^2 + 2(E^2 - m_\mu^2)(\hat{k} \cdot \hat{z})^2 \right) \quad (9.2.27)$$

$$= e^4 \left(1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2} \right) (\hat{k} \cdot \hat{z})^2 \right) \quad (9.2.28)$$

The differential cross section reads

$$d^6\sigma = |\mathcal{M}|^2 \frac{S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2}} \prod_{i=1}^n \left(\frac{d^3\vec{p}_i}{(2\pi)^3 2E_i'} \right) (2\pi)^4 \delta^{(4)} \left(p_1 + p_2 - \sum_{i=1}^n p_n' \right) \quad (9.2.29)$$

$$= \frac{e^4}{8E^2} \left(1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2} \right) (\hat{k} \cdot \hat{z})^2 \right) \frac{d^3\vec{p}_\mu}{(2\pi)^3 2E_\mu} \frac{d^3\vec{p}_{\bar{\mu}}}{(2\pi)^3 2E_{\bar{\mu}}} \quad (9.2.30)$$

$$\times (2\pi)^4 \delta(2E - E_\mu - E_{\bar{\mu}}) \delta^{(3)}(\vec{p}_\mu + \vec{p}_{\bar{\mu}}) \quad (9.2.31)$$

$$d^3\sigma = \frac{e^4}{16E^2} \left(1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2} \right) (\hat{k} \cdot \hat{z})^2 \right) \frac{d^2\hat{k} dE_\mu |\vec{k}|}{(2\pi)^3 4E_\mu} (2\pi) \delta(E - E_\mu) \quad (9.2.32)$$

$$d^2\sigma = \frac{e^4}{16E^2} \left(1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2} \right) (\hat{k} \cdot \hat{z})^2 \right) \frac{d^2\hat{k} \sqrt{E^2 - m_\mu^2}}{(2\pi)^2 (4E)} \quad (9.2.33)$$

Note: $k^2 = E^2 - m_\mu^2 \Rightarrow dk \cdot k = E dE \Rightarrow dk \cdot k^2 / E = k dE$.

$$\frac{d^2\sigma}{d\Omega} = \frac{\alpha^2}{16E^2} \left(1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2} \right) (\hat{k} \cdot \hat{z})^2 \right) \sqrt{1 - \frac{m_\mu^2}{E^2}} \quad (9.2.34)$$

where we have used $\alpha^2 = e^4/(16\pi^2)$. The total cross section is

$$\sigma = 4\pi \frac{\alpha^2}{16E^2} \left(1 + \frac{m_\mu^2}{E^2} + \frac{1}{3} \left(1 - \frac{m_\mu^2}{E^2} \right) \right) \sqrt{1 - \frac{m_\mu^2}{E^2}} \quad (9.2.35)$$

$$= \frac{4\pi}{3} \frac{\alpha^2}{4E^2} \left(1 - \frac{1}{2} \frac{m_\mu^2}{E^2} \right) \sqrt{1 - \frac{m_\mu^2}{E^2}}. \quad (9.2.36)$$

9.3 4-Fermi Theory

The 4-Fermi theory is defined by the Lagrangian density

$$\mathcal{L} = \sum_{I=1}^2 \{ \bar{\psi}_I (i\not{\partial} - m_I) \psi_I + \bar{\nu}_I i\not{\partial} \nu_I \} + \frac{g^2}{M^2} \{ (\bar{\psi}_1 \nu_1) (\bar{\nu}_2 \psi_2) + \text{h.c.} \}. \quad (9.3.1)$$

This is a toy model for beta decay (or muon decay), where M is closely related to the W^\pm and Z boson masses. Muon decay goes as $\mu^- \rightarrow \nu_\mu + W^-$ (virtual) followed by W^- (virtual) $\rightarrow \bar{\nu}_e + e^-$; but the W is heavy enough that, to a good degree of accuracy, its propagator can be replaced with $i\eta_{\mu\nu}/M_W^2$. This leads to an effective 4-fermion interaction.

Beta Decay Assuming $m_1 > m_2$, compute – to leading order in g^2 – the decay rate of the process

$$\psi_1 \rightarrow \nu_1 + \psi_2 + \bar{\nu}_2 \quad (9.3.2)$$

due to the interaction in eq. (5.3.5). Muon and quark decays are special cases of this process. Note that, from the properties of $\{\gamma^\mu\}$ and γ^5 ,

$$(\bar{\psi} \gamma^\mu (c - c' \gamma^5) \nu)^\dagger = \bar{\nu} \gamma^\mu (c - c' \gamma^5) \psi. \quad (9.3.3)$$

Therefore,

$$\mathcal{L}_1 = \frac{g^2}{M^2} \{ \bar{\psi}_1 \nu_1 \bar{\nu}_2 \psi_2 + \bar{\nu}_1 \psi_1 \bar{\psi}_2 \nu_2 \}$$

Let us denote the momentum of ψ_1 as $k^\mu = (m_1, \vec{0})$, i.e., in its rest frame; and let its spin be r . Let the spin-momentum of ψ_2 , $\bar{\nu}_2$ and ν_1 be, respectively, (s'_2, p_2) , (s_2, q_2) and (s_1, q_1) . To contract against the ‘incoming’ $\psi_1(r, k)$, we need the ψ_1 operator. The relevant matrix element is

$$\frac{ig^2}{M^2} \left\langle q_1, s_1; q_2, s_2; p_2, s'_2 \left| \int \bar{\psi}_2 \nu_2 \bar{\nu}_1 \psi_1 \right| k, r \right\rangle. \quad (9.3.4)$$

Hence

$$|\mathcal{M}| = \frac{g^2}{M^2} \left| \bar{u}_{q_1}^{s_1} u_k^r \bar{u}_{p_2}^{s'_2} v_{q_2}^{s_2} \right|. \quad (9.3.5)$$

Let us average over the initial spins and sum over the final ones.

$$\frac{1}{2} \sum_{r, s_1, s_2, s'_2} |\mathcal{M}|^2 = \frac{g^4}{2M^4} \sum_{r, s_1, s_2, s'_2} \bar{u}_{q_1}^{s_1} u_k^r \bar{u}_k^r u_{q_1}^{s_1} \bar{u}_{p_2}^{s'_2} v_{q_2}^{s_2} \bar{v}_{q_2}^{s_2} u_{p_2}^{s'_2} \quad (9.3.6)$$

$$= \frac{g^4}{2M^4} \text{Tr} \left[\not{q}_1 (\not{k} + m_1) \right] \text{Tr} \left[(\not{p}_2 + m_2) \not{q}_2 \right] \quad (9.3.7)$$

$$= \frac{g^4}{2M^4} 16(q_1 \cdot k)(p_2 \cdot q_2) = \frac{8g^4}{M^4} (q_1 \cdot k)(p_2 \cdot q_2). \quad (9.3.8)$$

Note that conservation of momentum tells us

$$(q_2 + p_2)^2 = m_2^2 + 2q_2 \cdot p_2 = (k - q_1)^2 = m_1^2 - 2m_1 q_1, \quad (9.3.9)$$

$$q_2 \cdot p_2 = \frac{m_1^2 - m_2^2}{2} - m_1 q_1. \quad (9.3.10)$$

The decay rate is given by the integral

$$\begin{aligned} \Gamma &= \frac{1}{2m_1} \int \frac{d^3 p_2}{(2\pi)^3 2E_{p_2}} \int \frac{d^3 q_1}{(2\pi)^3 2q_1} \int \frac{d^3 q_2}{(2\pi)^3 2q_2} \frac{1}{2} \sum_{\text{spins}} |\mathcal{M}|^2 \\ &\quad \times (2\pi)^4 \delta(m_1 - q_1 - q_2 - E_{p_2}) \delta^{(3)}(\vec{q}_1 + \vec{q}_2 + \vec{p}_2) \\ &= \int \frac{d^3 p_2}{(2\pi)^3 2E_{p_2}} \int \frac{d^3 q_1}{(2\pi)^3 2q_1} \int \frac{d^3 q_2}{(2\pi)^3 2q_2} (2\pi)^4 \delta(m_1 - q_1 - q_2 - E_{p_2}) \delta^{(3)}(\vec{q}_1 + \vec{q}_2 + \vec{p}_2) \\ &\quad \times \frac{4g^4}{M^4} q_1 \left(\frac{m_1^2 - m_2^2}{2} - m_1 q_1 \right). \end{aligned} \quad (9.3.11)$$

We have $\Gamma = 4(g/M)^4 I$, where

$$\begin{aligned} I &\equiv \frac{1}{2} \int \frac{d^3 p_2}{(2\pi)^3 2E_{p_2}} \int \frac{d^3 q_1}{(2\pi)^3} \int \frac{d^3 q_2}{(2\pi)^3 2q_2} (2\pi)^4 \delta(m_1 - q_1 - q_2 - E_{p_2}) \delta^{(3)}(\vec{q}_1 + \vec{q}_2 + \vec{p}_2) \\ &\quad \times \left(\frac{m_1^2 - m_2^2}{2} - m_1 q_1 \right) \\ &= \frac{1}{2} \int \frac{d^3 p_2}{(2\pi)^3 2E_{p_2}} \int \frac{d^3 q_1}{(2\pi)^3} \frac{1}{2|\vec{q}_1 + \vec{p}_2|} (2\pi) \delta(m_1 - q_1 - q_2 - E_{p_2}) \left(\frac{m_1^2 - m_2^2}{2} - m_1 q_1 \right) \end{aligned} \quad (9.3.12)$$

$$\begin{aligned} &= \frac{1}{2} \int_0^\infty \frac{dp_2 p_2^2}{(2\pi)^2 2E_{p_2}} \int_{-1}^{+1} d\chi \int \frac{d^3 q_1}{(2\pi)^3} \frac{1}{2\sqrt{q_1^2 + p_2^2 - 2q_1 p_2 \chi}} \\ &\quad \times (2\pi) \delta \left(m_1 - q_1 - E_{p_2} - \sqrt{q_1^2 + p_2^2 - 2q_1 p_2 \chi} \right) \left(\frac{m_1^2 - m_2^2}{2} - m_1 q_1 \right) \end{aligned} \quad (9.3.13)$$

Put

$$q_2 \equiv \sqrt{q_1^2 + p_2^2 - 2q_1 p_2 \chi}, \quad (9.3.14)$$

$$dq_2 = -\frac{q_1 p_2}{q_2} d\chi. \quad (9.3.15)$$

$$\begin{aligned} I &= \frac{1}{4} \int_0^\infty \frac{dp_2 p_2^2}{(2\pi)^2 2E_{p_2}} \int \frac{d^3 q_1}{(2\pi)^3} \int_{|q_1 - p_2|}^{q_1 + p_2} \frac{dq_2}{q_1 p_2} \\ &\quad \times (2\pi) \delta(m_1 - q_1 - q_2 - E_{p_2}) \left(\frac{m_1^2 - m_2^2}{2} - m_1 q_1 \right) \end{aligned} \quad (9.3.16)$$

$$\begin{aligned}
&= \frac{\pi}{2} \int_0^\infty \frac{dp_2 p_2^2}{(2\pi) E_{p_2}} \int_0^\infty \frac{dq_1 q_1^2}{(2\pi)^3} \frac{1}{q_1 p_2} \left(\frac{m_1^2 - m_2^2}{2} - m_1 q_1 \right) \\
&\quad \times \Theta(q_1 + p_2 - (m_1 - q_1 - E_{p_2})) \Theta(m_1 - q_1 - E_{p_2} - |q_1 - p_2|) \quad (9.3.17)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \int_0^\infty \frac{dp_2 p_2}{E_{p_2}} \int_0^\infty \frac{dq_1 q_1}{(2\pi)^3} \left(\frac{m_1^2 - m_2^2}{2} - m_1 q_1 \right) \\
&\quad \times \Theta(2q_1 + p_2 - m_1 + E_{p_2}) \Theta(m_1 - q_1 - E_{p_2} - |q_1 - p_2|) \quad (9.3.18)
\end{aligned}$$

In the limit where $m_1 \gg m_2$, let us put $m_2 = 0$ as a leading order approximation.

$$\begin{aligned}
I &= \frac{1}{4} \int_0^\infty dp_2 \int_0^\infty \frac{dq_1 q_1}{(2\pi)^3} \left(\frac{m_1^2}{2} - m_1 q_1 \right) \\
&\quad \times \Theta(2q_1 + 2p_2 - m_1) \Theta(m_1 - q_1 - p_2 - |q_1 - p_2|) \quad (9.3.19)
\end{aligned}$$

Put $k_\pm \equiv q_1 \pm p_2$. Then $dk_+ \wedge dk_- = 2dq_1 \wedge dp_2$.

$$\begin{aligned}
I &= \frac{1}{8} \int \frac{dk_+ dk_-}{(2\pi)^3} \frac{k_+ + k_-}{2} \left(\frac{m_1^2}{2} - m_1 \frac{k_+ + k_-}{2} \right) \\
&\quad \times \Theta\left(k_+ - \frac{m_1}{2}\right) \Theta(m_1 - k_+ - |k_-|) \quad (9.3.20)
\end{aligned}$$

Drawing a plot will show you the relevant regions of integration are

$$\begin{aligned}
I &= \frac{1}{4} \left(\int_0^{m_1/2} \frac{dk_-}{(2\pi)^3} \int_{m_1/2}^{m_1 - k_-} \frac{dk_+}{2} + \int_{-m_1/2}^0 \frac{dk_-}{(2\pi)^3} \int_{m_1/2}^{m_1 + k_-} \frac{dk_+}{2} \right) \\
&\quad \times \frac{k_+ - k_-}{2} \left(\frac{m_1^2}{2} - m_1 \frac{k_+ - k_-}{2} \right) \\
&\approx \frac{m_1^5}{6144\pi^3} (1 + \mathcal{O}((m_2/m_1)^2)) \quad (9.3.21)
\end{aligned}$$

We have arrived at, for $m_1 \gg m_2 \approx 0$, the decay rate

$$\Gamma = \frac{g^4}{M^4} \frac{m_1^5}{1536\pi^3}. \quad (9.3.22)$$

9.4 Appendix

$$\text{Tr} [\mathbb{I}] = 4 \quad (9.4.1)$$

$$\text{Tr} [\text{odd number of } \gamma \text{ matrices}] = 0 \quad (9.4.2)$$

$$\text{Tr} [\gamma^\alpha \gamma^\beta] = 4\eta^{\alpha\beta} \quad (9.4.3)$$

$$\text{Tr} [\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta] = 4(\eta^{\mu\nu} \eta^{\alpha\beta} - \eta^{\mu\alpha} \eta^{\nu\beta} + \eta^{\mu\beta} \eta^{\alpha\nu}) \quad (9.4.4)$$

$$\text{Tr} [\gamma^\alpha \gamma^\beta \gamma^5] = 0 \quad (9.4.5)$$

$$\text{Tr} [\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta \gamma^5] = -4i\epsilon^{\mu\nu\alpha\beta} \quad (9.4.6)$$

10 Acknowledgments

A Cross Sections, Lifetimes, Amplitudes

Cross Section Consider a parallel beam of particles zipping down an accelerator tunnel. If L denotes flux of particles (number of particles per unit time per unit area) across some infinitesimal cross sectional area $d\sigma$, so that the number of particles is $dN = Ld\sigma$, then the differential cross section is defined by the number of particles scattered into a given infinitesimal solid angle $d\Omega$ divided by the luminosity L :

$$\frac{d\sigma}{d\Omega} \equiv \frac{1}{L} \frac{dN}{d\Omega}. \quad (\text{A.0.1})$$

This definition is consistent with how cross section is actually measured: L is set by the parameters of the accelerator setup, whereas $dN/d\Omega$ involves counting particles.

Decay lifetime If we have N number of unstable particles of the same species, the reciprocal of their decay lifetime τ is defined to be the fractional change of its population per unit time:

$$\frac{1}{N} \frac{dN}{dt} = -\frac{1}{\tau} \equiv -\Gamma. \quad (\text{A.0.2})$$

The solution to this equation is

$$N(t) = N_0 \exp(-t/\tau). \quad (\text{A.0.3})$$

In words: the number of particles $N(\tau)$ after time τ has elapsed is $1/e$ of the initial number of particles N_0 .

Amplitudes to Differential Cross Sections & Decay Rates The S matrix for a process $\{p_i\} \rightarrow \{p'_j\}$ is defined by the transition amplitude

$${}_H \langle \{p'_j\}, t \rightarrow \infty | \{p_i\}, t \rightarrow -\infty \rangle_H = {}_S \langle \{p'_j\} | S | \{p_i\} \rangle_S. \quad (\text{A.0.4})$$

This is further split into the identity plus the T -matrix:

$$S \equiv \mathbb{I} + iT, \quad (\text{A.0.5})$$

$$i\mathcal{M} \cdot (2\pi)^4 \delta^{(4)} \left(\sum_i p_i - \sum_j p'_j \right) = {}_S \langle \{p'_j\} | iT | \{p_i\} \rangle_S. \quad (\text{A.0.6})$$

The physical meaning is that the amplitude \mathcal{M} captures the non-trivial part of the interactions; all the relevant dynamics is contained within T .

Given the amplitude \mathcal{M} for a 2 particle to n -particle scattering process, $p_1 + p_2 \rightarrow \{p'_i\}$, the differential cross section is given by

$$d\sigma = |\mathcal{M}|^2 \frac{S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2}} \prod_{i=1}^n \left(\frac{d^3 \vec{p}'_i}{(2\pi)^3 2E'_i} \right) (2\pi)^4 \delta^{(4)} \left(p_1 + p_2 - \sum_{i=1}^n p'_i \right). \quad (\text{A.0.7})$$

In Peskin & Schroeder, the $\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2}$ in the denominator is written as $E_1 E_2 |v_1 - v_2|$, where $|v_1 - v_2|$ is the relative velocity of the 2 incoming particles.

Given the amplitude \mathcal{M} for a particular decay process $1 \rightarrow 2 + 3 + \dots n$, the ‘Golden Rule’ for decay rate of particle 1, which will assume is at rest (i.e., $p_1 = (m_1, \vec{0})$), is given by

$$d\Gamma = |\mathcal{M}|^2 \frac{S}{2m_1} \prod_{i=2}^n \left(\frac{d^3 \vec{p}_i}{(2\pi)^3 2E_i} \right) (2\pi)^4 \delta^{(4)} \left(p_1 - \sum_{i=2}^n p_n \right) \quad (\text{A.0.8})$$

$$S = \prod_{j=1}^N \frac{1}{\ell_j!}. \quad (\text{A.0.9})$$

If there are ℓ identical particles in the final decay product, we need to divide the answer by $\ell!$. The S refers to this – its definition is given by the product all such statistical factors, where ℓ_j is the number of identical particles in the j th group of decay products.

B Last update: November 19, 2022

References