Electromagnetism in a Nutshell

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1 Maxwell's Equations & Lorentz Force Law

Classical electromagnetism is the theory of electric and magnetic forces exerted by, respectively, the electric $\vec{E}$ and magnetic $\vec{B}$ vector fields. In a given physical setup, these $\vec{E}$ and $\vec{B}$ fields are generated by the presence of the electric charge density ρ and the electric current $\vec{J}$. The ρ is the electric charge per volume; whereas the $\vec{J}$ is the charge density multiplied by its velocity – i.e., $\vec{J} \cdot \widehat{n}$, for a unit vector $\widehat{n}$, is simply the electric charge per unit time per area crossing the surface perpendicular to $\widehat{n}$. The mathematical theory which describes how the time- and space-dependent behavior of electric $\vec{E}$ and magnetic $\vec{B}$ fields are determined by a given distribution of electric charge density ρ and electric current $\vec{J}$ – is known as Maxwell's equations:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}, \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0; \tag{1.0.1}$$

and

$$\vec{\nabla} \cdot \vec{B} = 0, \qquad \vec{\nabla} \times \vec{B} - \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}. \tag{1.0.2}$$

To understand these formulas one needs to be fluent in multivariable calculus. Briefly, $\vec{\nabla} \cdot \vec{V}$ is the divergence of the vector field $\vec{V}$; in Cartesian coordinates (x, y, z) it reads

$$\vec{\nabla} \cdot \vec{V} = \partial_x V^x + \partial_y V^y + \partial_z V^z, \tag{1.0.3}$$

where $\vec{V} \doteq (V^x, V^y, V^z)$ – the superscripts here are not powers but label the respective components of the vector. And the curl of $\vec{V}$ is

$$\vec{\nabla} \times \vec{V} = \left(\partial_y V^z - \partial_z V^y, \partial_z V^x - \partial_x V^z, \partial_x V^y - \partial_y V^x\right). \tag{1.0.4}$$

It is a fundamental theorem in vector calculus, that to completely determine a vector $\vec{V}$, one needs to specify its divergence and curl. Maxwell's equations in equations (1.0.1) and (1.0.2) are doing precisely that, for a given electric charge/current distribution $(\rho, \vec{J})$. Moreover, another theorem in vector calculus tells us that, if the divergence of a vector field is zero, it must be the curl of something. Hence, we may infer from the divergence-less character of $\vec{B}$ in eq. (1.0.2), there must be some *vector potential* $\vec{A}$ such that

$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{1.0.5}$$

Electric charge The charge density ρ in eq. (1.0.1) can be either positive or negative. In Nature, it turns out there is a fundamental discreteness to it. The elementary particles that comprise the basic constituents turn out to have either zero, $+e$ or $-e$ charge, where

$$e \approx 1.602 \times 10^{-19}\text{C}. \tag{1.0.6}$$

(C is for Coulomb, the SI unit for electric charge.) As you may have heard, matter comes in discrete packets. Electrons, muons and taus have $-e$ charge, for instance; whereas their anti-particles, the positron, anti-muon and anti-tau, have $+e$ charge. (There are also the $W^\pm$ bosons, 2 of the 3 carriers of the Weak Force.) The quarks are fractionally charged $\pm(1/3)e$ and $\pm(2/3)e$, but most likely cannot be found 'free' in Nature – i.e., we will not be able to directly detect these $\pm(1/3)e$ and $\pm(2/3)e$ charges.

In most physical phenomenon, and certainly in our daily experience of the natural world, the primary source of negative charges are electrons – the current in your smart phone, for instance, is due to their motion through its internal circuitry. The positive charges are from the protons, each carrying $+e$ charge, lying within the nuclei of each atom. That macroscopic material is largely electrically neutral (i.e., with zero net charge) is because of the equal number of electrons and protons present.

Local conservation An important property of electric charge is that it cannot be created nor destroyed. Furthermore, this is a local conservation law: electric charge can flow from one place to another, but cannot disappear at $\vec{x}$ and instantaneously reappear at $\vec{x}'$ some finite distance away – even though such a process would preserve the total electric charge present. This local conservation law is not an additional assumption of the theory but built into Maxwell's equations. To understand this, we take the time derivative of the divergence-$\vec{E}$ equation in (1.0.1) and the divergence of the curl-$\vec{B}$ minus $\partial_t \vec{E}$ equation in (1.0.2); using the fact that the divergence of a curl is zero on the latter,

$$\vec{\nabla} \cdot \partial_t \vec{E} = \frac{\partial_t \rho}{\varepsilon_0}, \tag{1.0.7}$$

$$-\frac{\partial \left(\vec{\nabla} \cdot \vec{E}\right)}{\partial t} = \frac{1}{\varepsilon_0} \vec{\nabla} \cdot \vec{J}. \tag{1.0.8}$$

Adding these two equations yields

$$\partial_t \rho = -\vec{\nabla} \cdot \vec{J}. \tag{1.0.9}$$

On the left hand side, we have the time rate-of-change of the total electric charge in an infinitesimal volume. On the right hand side, the divergence of $\vec{J}$ can be viewed – through Gauss' theorem in vector calculus – as the flux of the electric current, namely the rate of flow of electric charge out of (or, into) the infinitesimal volume under consideration. Since the net change in total charge within the volume is equal to the negative of the total flow out of/into the same, and since this holds for any infinitesimal volume, we may thus conclude that Maxwell's equations demand that electric charge is locally conserved.

Linearity & Superposition One important property of electromagnetism is its linearity, which in turn implies the electric and magnetic fields obey superposition: the $\vec{E}$ and $\vec{B}$

engendered by various sources may be calculated separately and simply added to obtain the total. To see this, let there be some charge and current density $(\rho_1, \vec{J}_1)$. They give rise to

$$\vec{\nabla} \cdot \vec{E}_1 = \frac{\rho_1}{\varepsilon_0}, \qquad \vec{\nabla} \times \vec{E}_1 + \frac{\partial \vec{B}_1}{\partial t} = 0; \tag{1.0.10}$$

and

$$\vec{\nabla} \cdot \vec{B}_1 = 0, \qquad \vec{\nabla} \times \vec{B}_1 - \mu_0 \varepsilon_0 \frac{\partial \vec{E}_1}{\partial t} = \mu_0 \vec{J}_1. \tag{1.0.11}$$

If we have instead $(\rho_2, \vec{J}_2)$,

$$\vec{\nabla} \cdot \vec{E}_2 = \frac{\rho_2}{\varepsilon_0}, \qquad \vec{\nabla} \times \vec{E}_2 + \frac{\partial \vec{B}_2}{\partial t} = 0; \tag{1.0.12}$$

and

$$\vec{\nabla} \cdot \vec{B}_2 = 0, \qquad \vec{\nabla} \times \vec{B}_2 - \mu_0 \varepsilon_0 \frac{\partial \vec{E}_2}{\partial t} = \mu_0 \vec{J}_2. \tag{1.0.13}$$

Let us sum the equations in (1.0.10) and (1.0.12); and sum those in equations (1.0.11) and (1.0.13).

$$\vec{\nabla} \cdot \left(\vec{E}_1 + \vec{E}_2\right) = \frac{\rho_1 + \rho_2}{\varepsilon_0}, \tag{1.0.14}$$

$$\vec{\nabla} \times \left(\vec{E}_1 + \vec{E}_2\right) + \frac{\partial \left(\vec{B}_1 + \vec{B}_2\right)}{\partial t} = 0; \tag{1.0.15}$$

and

$$\vec{\nabla} \cdot \left(\vec{B}_1 + \vec{B}_2\right) = 0, \tag{1.0.16}$$

$$\vec{\nabla} \times \left(\vec{B}_1 + \vec{B}_2\right) - \mu_0 \varepsilon_0 \frac{\partial \left(\vec{E}_1 + \vec{E}_2\right)}{\partial t} = \mu_0 \left(\vec{J}_1 + \vec{J}_2\right). \tag{1.0.17}$$

In other words, if we now simultaneously have $(\rho_1, \vec{J}_1)$ and $(\rho_2, \vec{J}_2)$ present in the system, then all we have to do is to solve the $\vec{E}_1$ and $\vec{B}_1$ sourced by $(\rho_1, \vec{J}_1)$; and then solve the $\vec{E}_2$ and $\vec{B}_2$ sourced by $(\rho_2, \vec{J}_2)$. The full electric field is then the superposition $\vec{E} = \vec{E}_1 + \vec{E}_2$ and the magnetic field is $\vec{B} = \vec{B}_1 + \vec{B}_2$. By iterating the argument here, this superposition can be carried out for arbitrary number of isolated systems of electric charges and currents.

Lorentz Force Law For an electrically charged point particle, not necessarily the electron or the proton, and ignoring their quantum behavior; if m and q are its mass and electric charge, its motion $\vec{x}(t)$ obeys the Lorentz force law

$$m \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\vec{v}}{\sqrt{1 - (\vec{v}/c)^2}} \right) = q \left(\vec{E} + \vec{v} \times \vec{B} \right), \tag{1.0.18}$$

$$\vec{v} \equiv \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}, \qquad \vec{v}^2 \equiv \vec{v} \cdot \vec{v}. \tag{1.0.19}$$

In words: the electric force on a point particle proportional to its charge q and parallel to the electric field it is immersed in. Whereas the magnetic force on the same is also proportional to q but perpendicular to both its velocity $\vec{v}$ and the magnetic field $\vec{B}$ it is immersed in; this is the character of the cross product $\vec{v} \times \vec{B}$. Moreover, notice if the charge is not moving, it does not experience any magnetic forces. Finally, the funny $1/\sqrt{1-(\vec{v}/c)^2}$ on the left hand side of eq. (1.0.18) is a consequence of writing $m\vec{a}$ (of Newton's second law) in a manner consistent with Special Relativity.

Electromagnetism: Executive Summary Equations (1.0.1), (1.0.2) and (1.0.18) underlie classical electromagnetic phenomenon. In particular: equations (1.0.1) and (1.0.2) tell us how electric charges and their motion generate electric and magnetic fields; while eq. (1.0.18) tells us how electric point charges move in response to the presence of external electric and magnetic fields.

Speed of Light & Special Relativity The μ_0 and ε_0 that appear in equations (1.0.1) and (1.0.2) are constants due to our choice of SI units. It's important to note, however, that the speed of light is given by

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 299,7992,458 \text{ meters/second.} \tag{1.0.20}$$

It is not a coincidence that the speed of light appears in the constants occurring within Maxwell's electromagnetic equations. In fact, the modern perspective is that electromagnetism is the way it is – to a large extent – because of Special Relativity: when switching from say inertial frame A to inertial frame B, the time and space of A becomes intertwined in such a way that the overall *geometry* (i.e., the notion of distances and time elapsed) remains the same. The requirement that the laws of electromagnetism maintain the same form in both inertial frames A and B, at least in 3 space and 1 time dimension, is intimately tied to the why equations (1.0.1) and (1.0.2) take the form they do.[1]

2 Maxwell Equations: Integral Form

3 Electrostatics & Magnetostatics

The presence of non-zero electric charges at rest generates only electric fields. It is when these same charges move that their electric fields become an admixture of electric and magnetic fields. This is in fact a consequence of Special Relativity! Accelerating electric charges will produce *electromagnetic waves*, where electric and magnetic fields oscillate in a self-sustaining manner that can carry energy-momentum to infinity as they propagate away from their sources – this is nothing but the phenomenon of light itself.

Electrostatics To understand electromagnetism, the usual route is to take time-independent limits. The first case is to assume the electric charges are at rest, so $\vec{J} = 0$. This is the essence of electrostatics. Since nothing is moving it is reasonable to suppose the $\vec{E}$ and $\vec{B}$ themselves are time independent, since they are sourced by ρ. Equations (1.0.1) and (1.0.2) then become

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}, \qquad \vec{\nabla} \times \vec{E} = 0; \tag{3.0.1}$$

[1]The other important ingredient is known as U_1 *gauge invariance*.

and

$$\vec{\nabla} \cdot \vec{B} = 0, \qquad \vec{\nabla} \times \vec{B} = 0. \tag{3.0.2}$$

From eq. (3.0.2) we see that, in the electrostatic case (where charges are not moving), it is mathematically consistent to assume all magnetic fields are zero.

Electric charges at rest ($\vec{J} = 0$) produce only $\vec{E}$ but not $\vec{B}$.

Theorems in vector calculus tell us, if the curl of a vector field is zero, it must be the gradient of a scalar field. In our case, this is nothing but the electric potential U:

$$\vec{E} = -\vec{\nabla} U. \tag{3.0.3}$$

What is the potential at $\vec{x}$ produced by a single point charge at some location $\vec{x}'$? It is

$$U[\text{point charge } q] = \frac{q}{4\pi\varepsilon_0 r}, \tag{3.0.4}$$

where $r = |\vec{x} - \vec{x}'|$ is the radial distance the observer at $\vec{x}$ and the point charge at $\vec{x}'$. By plugging eq. (3.0.4) into (3.0.3), we see the corresponding electric field is

$$\vec{E}[\text{point charge } q] = \frac{q}{4\pi\varepsilon_0 r^2}\widehat{r}; \tag{3.0.5}$$

where $\widehat{r}$ is the unit vector pointing radially away from the point charge at $\vec{x}'$.

Coulomb's law Consider the following the situation. Let us place another point charge q' at the location $\vec{x}$. Referring to the right hand side of the force law in eq. (1.0.18), and utilizing eq. (3.0.5), we may conclude that the force on q' due to q is

$$\vec{F}[q \text{ on } q'] = \frac{q \cdot q'}{4\pi\varepsilon_0 r^2}\widehat{r}. \tag{3.0.6}$$

This is *Coulomb*'s law. The magnitude of the electric force on q' by q falls off as the square of the inverse distance between them. Since the $\widehat{r}$ points from q to q', we see that if $q \cdot q' > 0$ – namely, if both are positive or both are negative (like charges) – then they repel each other. Whereas if $q \cdot q' < 0$ – namely, if one is negative and the other is positive (unlike charges) – they attract each other. You should be able to use eq. (3.0.5) to reason that, the force on q due to q' is equal in magnitude but opposite in direction:

$$\vec{F}[q' \text{ on } q] = -\frac{q \cdot q'}{4\pi\varepsilon_0 r^2}\widehat{r}. \tag{3.0.7}$$

We see from Coulomb's law in eq. (3.0.6) that whether we assign a positive or negative value to a given charge, say q, is actually a human convention. What matters is the product of the charges q and q'. To recap:

Charges with the same sign – i.e., 'like charges' — repel. Whereas charges with opposite signs – i.e., 'unlike charges' – attract.

That is, suppose we define the electron to have negative charge, then the proton must have positive charge because we may experimentally determine that electrons repel each other whereas they are attracted to protons.

For a more general charge distribution ρ in eq. (3.0.1), we have from $\vec{E} = -\vec{\nabla} U$,

$$-\vec{\nabla} \cdot \left(\vec{\nabla} U \right) = \frac{\rho}{\varepsilon_0}. \tag{3.0.8}$$

This is known as Poisson's equation. The solution can be expressed as an integral.

$$\begin{aligned} \vec{E} = -\vec{\nabla} U &= -\vec{\nabla}_{\vec{x}} \int_{\mathbb{R}^3} \frac{\rho(\vec{x}')}{4\pi\varepsilon_0 |\vec{x} - \vec{x}'|} \mathrm{d}^3\vec{x}' \\ &= \int_{\mathbb{R}^3} \widehat{R} \frac{\rho(\vec{x}')}{4\pi\varepsilon_0 |\vec{x} - \vec{x}'|^2} \mathrm{d}^3\vec{x}', \qquad \widehat{R} \equiv \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|}. \end{aligned} \tag{3.0.9}$$

The integral in eq. (3.0.9) can, in turn, be interpreted as the electric field generated by the superposition of point charges, weighted by the density ρ.

Magnetostatics In magnetostatics, we assume there are no net electric charges present: $\rho = 0$. However, we now allow for a steady flow of charge: $\vec{J} \neq \vec{0}$. (Think electrons flowing in a wire – the negative charges are in motion but not the positively charged protons stuck inside the atomic nuclei. The total charge is zero but there is a non-zero $\vec{J}$[electrons].) By steady here, we mean that $\vec{J}$ itself does not depend on time. In this time independent limit, we again assume the electric and magnetic fields are also time independent. Equations (1.0.1) and (1.0.2) thus become

$$\vec{\nabla} \cdot \vec{E} = 0, \qquad \vec{\nabla} \times \vec{E} = 0; \tag{3.0.10}$$

and

$$\vec{\nabla} \cdot \vec{B} = 0, \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}. \tag{3.0.11}$$

From eq. (3.0.10) we see it is consistent to set the electric fields to zero ($\vec{E} = 0$) whenever there is zero net charge and only a steady current $\vec{J}$ present in the system. Furthermore, we may exploit the formula

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A} \right) = \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} \right) - \vec{\nabla} \cdot \vec{\nabla} \vec{A} \tag{3.0.12}$$

to translate the curl equation of eq. (3.0.11) into

$$\vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} \right) - \vec{\nabla} \cdot \vec{\nabla} \vec{A} = \mu_0 \vec{J}. \tag{3.0.13}$$

Notice, in eq. (1.0.5) that $\vec{A}$ and $\vec{A} + \vec{\nabla}\Lambda$ yields the same magnetic field because the curl of a gradient is zero:

$$\vec{\nabla} \times \left(\vec{A} + \vec{\nabla}\Lambda \right) = \vec{\nabla} \times \vec{A} = \vec{B}. \tag{3.0.14}$$

Therefore I will assume that

$$\vec{\nabla} \cdot \vec{A} = 0; \tag{3.0.15}$$

for if it were not true, we may always add a Λ such that

$$\vec{\nabla} \cdot \left(\vec{A} + \vec{\nabla}\Lambda\right) = 0 \qquad \Leftrightarrow \qquad \vec{\nabla} \cdot \vec{A} = -\vec{\nabla}^2\Lambda; \tag{3.0.16}$$

and simply call the combination $\vec{A} + \vec{\nabla}\Lambda$ the 'new' vector potential $\vec{A}$ since both yield the same magnetic field. Using eq. (3.0.15) in eq. (3.0.13), we arrive at vector analog of Poisson's equation in eq. (3.0.8):

$$-\vec{\nabla} \cdot \vec{\nabla}\vec{A} = \mu_0 \vec{J}. \tag{3.0.17}$$

The solution is also analogous to its counterpart in eq. (3.0.9). Recalling that $\vec{B} = \vec{\nabla} \times \vec{A}$,

$$\begin{aligned} \vec{B} &= \frac{\mu_0}{4\pi} \vec{\nabla}_{\vec{x}} \times \int_{\mathbb{R}^3} \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \mathrm{d}^3\vec{x}' \\ &= -\frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \widehat{R} \times \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|^2} \mathrm{d}^3\vec{x}', \qquad \widehat{R} \equiv \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|}. \end{aligned} \tag{3.0.18}$$

Infinitesimally thin wires When the electric current $\vec{J}$ is an infinitesimally thin wire with trajectory $\vec{I}$ (extending in space), the volume integral in eq. collapses to a line integral along it:

$$\begin{aligned} \vec{B} &= -\frac{\mu_0}{4\pi} \int \frac{\widehat{R} \times \vec{I}}{|\vec{x} - \vec{x}'|^2} \mathrm{d}\ell \\ &= \frac{\mu_0}{4\pi} \int \frac{I \mathrm{d}\vec{\ell} \times \widehat{R}}{|\vec{x} - \vec{x}'|^2}, \qquad I \equiv |\vec{I}|; \end{aligned} \tag{3.0.19}$$

where in the first equality $\mathrm{d}\ell$ is the infinitesimal length along the current $\vec{I}$; and in second equality the $\mathrm{d}\vec{\ell}$ is an infinitesimal vector lying parallel to its flow. Eq. (3.0.19) is known as the Biot-Savart law.

4 Solutions to Maxwell's Equations

We now move on to solve Maxwell's equations in full generality. The first step involves taking the curl of the curl equations in (1.0.1) and (1.0.2). Recalling eq. (3.0.12), which holds for any $\vec{A}$,

$$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \vec{\nabla} \cdot \vec{\nabla}\vec{E} + \frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{E}\right) = 0 \tag{4.0.1}$$

$$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{B}\right) - \vec{\nabla} \cdot \vec{\nabla}\vec{B} - \mu_0\varepsilon_0 \frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{E}\right) = \mu_0 \vec{\nabla} \times \vec{J}. \tag{4.0.2}$$

Employing Maxwell's equations (1.0.1) and (3.0.11) to replace the divergence-$\vec{E}$ and divergence-$\vec{B}$ with ρ/ε_0 and 0 respectively; as well as the curl-$\vec{E}$ and curl-$\vec{B}$ respectively with $-\partial_t \vec{B}$ and $\mu_0 \varepsilon_0 \partial_t \vec{E} + \mu_0 \vec{J}$,

$$\vec{\nabla}\left(\frac{\rho}{\varepsilon}\right) - \vec{\nabla} \cdot \vec{\nabla}\vec{E} + \frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}\right) = 0 \tag{4.0.3}$$

$$-\vec{\nabla} \cdot \vec{\nabla}\vec{B} + \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = \mu_0 \vec{\nabla} \times \vec{J}. \tag{4.0.4}$$

Recalling eq. (1.0.20), we now re-arrange these equations to take the forms:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla} \cdot \vec{\nabla}\right)\vec{E} = -\frac{\vec{\nabla}\rho}{\varepsilon_0} - \mu_0 \frac{\partial \vec{J}}{\partial t}, \tag{4.0.5}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla} \cdot \vec{\nabla}\right)\vec{B} = \mu_0 \vec{\nabla} \times \vec{J}. \tag{4.0.6}$$

These are *wave equations* for the electric and magnetic fields.

Vacuum case For simplicity let us first consider the case where there is no electric charges no currents: $\rho = \vec{J} = 0$.

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla} \cdot \vec{\nabla}\right)\vec{E} = 0, \tag{4.0.7}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla} \cdot \vec{\nabla}\right)\vec{B} = 0. \tag{4.0.8}$$

Then we see that both the electric and magnetic fields may be expressed as the real part of the following expressions:

$$\vec{E} \doteq \vec{E}_0 e^{i\vec{k}\cdot\vec{x} - ic|\vec{k}|t}, \tag{4.0.9}$$

$$\vec{B} \doteq \vec{B}_0 e^{i\vec{k}\cdot\vec{x} - ic|\vec{k}|t}; \tag{4.0.10}$$

where the $\vec{E}_0$ and $\vec{B}_0$ are now constant vectors, independent of space and time. Focusing on the plane wave portion of the solution, we see that

$$e^{i\vec{k}\cdot\vec{x} - ic|\vec{k}|t} = \exp\left(i|\vec{k}|\left\{\widehat{k} \cdot \vec{x} - ct\right\}\right), \qquad \widehat{k} \equiv \frac{\vec{k}}{|\vec{k}|}; \tag{4.0.11}$$

teaches us that these are waves traveling along the $\widehat{k}$ direction at the speed of light c.

Inserting these solutions in eq. (4.0.9) and (4.0.10) back into the original Maxwell's equations (1.0.1) and (1.0.2), one obtains

$$\vec{\nabla} \cdot \vec{E} \doteq i\vec{k} \cdot \vec{E}_0 e^{i\vec{k}\cdot\vec{x} - c^2|\vec{k}|t} = 0, \tag{4.0.12}$$

$$\vec{\nabla} \times \vec{E} \doteq i\left(\vec{k} \times \vec{E}_0 - c|\vec{k}|\vec{B}_0\right) e^{i\vec{k}\cdot\vec{x} - c^2|\vec{k}|t} = 0; \tag{4.0.13}$$

and

$$\vec{\nabla} \cdot \vec{B} \doteq i\vec{k} \cdot \vec{B}_0 e^{i\vec{k}\cdot\vec{x}-c^2|\vec{k}|t} = 0, \tag{4.0.14}$$

$$\vec{\nabla} \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \doteq i\left(\vec{k} \times \vec{B}_0 + c^{-1}|\vec{k}|\vec{E}_0\right) e^{i\vec{k}\cdot\vec{x}-c^2|\vec{k}|t} = 0. \tag{4.0.15}$$

From equation (4.0.12) and (4.0.14), we see that both the electric and magnetic fields are perpendicular to the direction of propagation:

$$\widehat{k} \cdot \vec{E}_0 = \widehat{k} \cdot \vec{B}_0 = 0. \tag{4.0.16}$$

Furthermore, equations (4.0.13) and (4.0.15) states the electric and magnetic fields are mutually perpendicular.

$$\widehat{k} \times \vec{E}_0 = c\vec{B}_0 \qquad \Leftrightarrow \qquad \widehat{k} \times \vec{B}_0 = -c^{-1}\vec{E}_0. \tag{4.0.17}$$

To sum: electromagnetic waves in a space without any electric charges nor currents can be expressed as a superposition of the following waves

$$\vec{E} \doteq \vec{E}_0 e^{i\vec{k}\cdot\vec{x}-ic|\vec{k}|t}, \tag{4.0.18}$$

$$\vec{B} \doteq -c\widehat{k} \times \vec{E}_0 e^{i\vec{k}\cdot\vec{x}-ic|\vec{k}|t}. \tag{4.0.19}$$

Non-vacuum case The full solutions to equations (4.0.5) and (4.0.6) read

$$\vec{E}(t,\vec{x}) = \int_{\mathbb{R}^3} \left(\frac{\rho(t_r,\vec{x}')}{\varepsilon_0 R^2}\widehat{R} - \mu_0 \frac{\partial_{t_r}\vec{J}(t_r,\vec{x}')}{R} \right) \frac{\mathrm{d}^3\vec{x}'}{4\pi}, \tag{4.0.20}$$

$$\vec{B}(t,\vec{x}) = -\mu_0 \int_{\mathbb{R}^3} \frac{\widehat{R} \times \vec{J}(t_r,\vec{x}')}{R^2} \frac{\mathrm{d}^3\vec{x}'}{4\pi}, \tag{4.0.21}$$

$$\widehat{R} \equiv \frac{\vec{x}-\vec{x}'}{R}, \qquad R \equiv |\vec{x}-\vec{x}'|; \tag{4.0.22}$$

where the *retarded time* is defined as

$$t_r \equiv t - R/c. \tag{4.0.23}$$

This t_r describes the following fact: the electromagnetic signals generated by $(\rho(\vec{x}'), \vec{J}(\vec{x}'))$ and received at $(t,\vec{x})$, travels at the speed of light from $\vec{x}'$ to $\vec{x}$ – and thereby was emitted at time $t_r = t - R/c$. Notice the solutions in equations (4.0.20) and (4.0.21) reduce to both the electrostatic eq. (3.0.9) and magnetostatic eq. (3.0.18) in the appropriate $(\rho(t,\vec{x}) \to \rho(\vec{x}), \vec{J} = 0)$ and $(\rho = 0, \vec{J}(t,\vec{x}) \to \vec{J}(\vec{x}))$ limits.

5 Last Update: February 27, 2018