Coordinates 2D and 3D
Gauss' & Stokes' Theorems

Yi-Zen Chu

1 2 Dimensions

In 2 dimensions, we may use Cartesian coordinates $\vec{r} = (x, y)$ and the associated infinitesimal area

$$\mathrm{d}^2 A = \mathrm{d}x\mathrm{d}y. \tag{1.0.1}$$

The gradient of a scalar V is

$$\vec{\nabla} V = \partial_x V \widehat{x} + \partial_y V \widehat{y}. \tag{1.0.2}$$

We may also employ polar coordinates

$$\vec{r} = (x, y) = \rho (\cos\phi, \sin\phi), \tag{1.0.3}$$

$$\rho \geq 0, \qquad\qquad 0 \leq \phi < 2\pi. \tag{1.0.4}$$

The associated infinitesimal area is

$$\mathrm{d}^2 A = \rho \mathrm{d}\rho \mathrm{d}\phi. \tag{1.0.5}$$

The gradient of a scalar is

$$\vec{\nabla} V = \partial_\rho V \widehat{\rho} + \frac{1}{\rho} \partial_\phi V \widehat{\phi}. \tag{1.0.6}$$

2 3 Dimensions

In 3 dimensions, we may use Cartesian coordinates $\vec{r} = (x, y, z)$ and the associated infinitesimal volume

$$\mathrm{d}^3 V = \mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{2.0.7}$$

The gradient of a scalar V is

$$\vec{\nabla} V = \partial_x V \widehat{x} + \partial_y V \widehat{y} + \partial_z V \widehat{z}. \tag{2.0.8}$$

We may also employ cylindrical coordinates

$$\vec{r} = (x, y, z) = (\rho \cos \phi, \rho \sin \phi, z), \tag{2.0.9}$$

$$\rho \geq 0, \qquad 0 \leq \phi < 2\pi, \qquad z \in \mathbb{R}. \tag{2.0.10}$$

The associated infinitesimal volume is

$$\mathrm{d}^3 V = \rho \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}z. \tag{2.0.11}$$

The outward pointing area element on the curved surface of a cylinder of radius ρ is

$$\mathrm{d}^2 \vec{A} = \widehat{\rho} \mathrm{d}^2 A = \rho \mathrm{d}\phi \mathrm{d}z \widehat{\rho}. \tag{2.0.12}$$

The gradient of a scalar V is

$$\vec{\nabla} V = \partial_\rho V \widehat{\rho} + \frac{1}{\rho} \partial_\phi V \widehat{\phi} + \partial_z V \widehat{z}. \tag{2.0.13}$$

Spherical coordinates are defined as

$$\vec{r} = (x, y, z) = r(\sin \theta \cdot \cos \phi, \sin \theta \cdot \sin \phi, \cos \theta), \tag{2.0.14}$$

$$r \geq 0, \qquad 0 \leq \theta \leq \pi, \qquad 0 \leq \phi < 2\pi. \tag{2.0.15}$$

The associated infinitesimal volume is

$$\mathrm{d}^3 V = r^2 \sin \theta \mathrm{d}r \mathrm{d}\theta \mathrm{d}\phi. \tag{2.0.16}$$

The outward pointing area element on the surface of a sphere of radius r is

$$\mathrm{d}^2 \vec{A} = \widehat{r} \mathrm{d}^2 A = \widehat{r} r^2 \sin \theta \mathrm{d}\theta \mathrm{d}\phi. \tag{2.0.17}$$

The gradient of a scalar V is

$$\vec{\nabla} V = \partial_r V \widehat{r} + \frac{\partial_\theta V}{r} \widehat{\theta} + \frac{\partial_\phi V}{r \sin \theta} \widehat{\phi}. \tag{2.0.18}$$

2.1 Cross Products

The cross product $\vec{a} \times \vec{b}$ between two 3D vectors $\vec{a}$ and $\vec{b}$ returns a vector that is perpendicular to both; the magnitude/length is

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta, \tag{2.1.1}$$

where θ is the angle between the vectors. The direction of $\vec{a} \times \vec{b}$ is determined by the 'right hand rule', which we will shortly elaborate upon.

Let us define the cross product more systematically. Firstly, the cross product is anti-symmetric:

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}. \tag{2.1.2}$$

Secondly, it is linear; for a scalars λ and μ and vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$,

$$\vec{a} \times \left(\lambda\vec{b} + \mu\vec{c}\right) = \lambda\vec{a} \times \vec{b} + \mu\vec{a} \times \vec{c}. \tag{2.1.3}$$

The right hand rule is defined as follows. Let $\widehat{x}$, $\widehat{y}$ and $\widehat{z}$ be the unit vectors along the x, y and z axes. Then, the right hand rule says

$$\widehat{x} \times \widehat{y} = \widehat{z}, \qquad \widehat{y} \times \widehat{z} = \widehat{x}, \qquad \text{and} \qquad \widehat{z} \times \widehat{x} = \widehat{y}. \tag{2.1.4}$$

Example An immediate consequence of eq. (2.1.2) is that the cross product of a vector with itself is zero.

$$\vec{a} \times \vec{a} = -\vec{a} \times \vec{a} \qquad \Rightarrow \qquad 2\vec{a} \times \vec{a} = 0. \tag{2.1.5}$$

Example Even though linearity was defined with respect to the second vector in eq. (2.1.3), note that it is also true that

$$\left(\lambda\vec{b} + \mu\vec{c}\right) \times \vec{a} = \lambda\vec{b} \times \vec{a} + \mu\vec{c} \times \vec{a}. \tag{2.1.6}$$

Can you explain why? (Hint: Use eq. (2.1.2) twice.)
Example Let us work out the cross product of two vectors $\vec{a}$ and $\vec{b}$ using equations (2.1.2), (2.1.3), and (2.1.4).

$$\vec{a} \times \vec{b} = (a_x\widehat{x} + a_y\widehat{y} + a_z\widehat{z}) \times (b_x\widehat{x} + b_y\widehat{y} + b_z\widehat{z}). \tag{2.1.7}$$

By linearity,

$$\vec{a} \times \vec{b} = a_x\widehat{x} \times (b_y\widehat{y} + b_z\widehat{z}) + a_y\widehat{y} \times (b_x\widehat{x} + b_z\widehat{z}) + a_z\widehat{z} \times (b_x\widehat{x} + b_y\widehat{y}) \tag{2.1.8}$$

$$= a_xb_y\widehat{z} - a_xb_z\widehat{y} - a_yb_x\widehat{z} + a_yb_z\widehat{x} + a_zb_x\widehat{y} - a_zb_y\widehat{x} \tag{2.1.9}$$

Collecting the coefficients, we have arrived at

$$\vec{a} \times \vec{b} = (a_yb_z - a_zb_y)\,\widehat{x} + (a_zb_x - a_xb_z)\,\widehat{y} + (a_zb_x - a_xb_z)\,\widehat{z}. \tag{2.1.10}$$

If you have learned how to take the determinant of a matrix, this cross product formula can also be phrased in terms of a 3×3 matrix with columns filled with the components of $\vec{a}$, $\vec{b}$, and $(\widehat{x}, \widehat{y}, \widehat{z})$.

3 Gauss' & Stokes' Theorems

Gauss' and Stokes' theorems are the two mathematical theorems central to electromagnetism.

3.1 Gauss' Theorem

Consider some 3D region $\mathfrak{D}$ in 3D space with a closed boundary $\partial\mathfrak{D}$. By 'closed' here, we mean that there is a clear distinction between 'inside' and 'outside': namely, to get from outside to inside one has to go through the boundary $\partial\mathfrak{D}$.

Gauss' theorem The volume integral of the divergence of some vector field $\vec{V}$ within $\mathfrak{D}$ is equal to its flux through the boundary $\partial\mathfrak{D}$.

$$\int_{\mathfrak{D}} \mathrm{d}^3 V \vec{\nabla} \cdot \vec{V} = \int_{\partial\mathfrak{D}} \mathrm{d}^2 \vec{A} \cdot \vec{V} \tag{3.1.1}$$

Here, $\mathrm{d}^3 V$ is the infinitesimal volume element; for example,

$$\begin{aligned} \mathrm{d}^3 V &= \mathrm{d}x\mathrm{d}y\mathrm{d}z, && \text{(Cartesian)}, && (3.1.2) \\ &= \rho\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}z, && \text{(Cylindrical)}, && (3.1.3) \\ &= r^2 \sin(\theta)\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi, && \text{(Spherical)}. && (3.1.4) \end{aligned}$$

The $\mathrm{d}^2\vec{A}$ is the infinitesimal area element pointing outwards from $\mathfrak{D}$.

Example The integral of the divergence of $\vec{V}$ inside a sphere of radius R is

$$\int_{r \le R} (r^2 \sin\theta \cdot \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi)\vec{\nabla} \cdot \vec{V}(r,\theta,\phi) = R^2 \int_0^\pi \mathrm{d}\theta \sin(\theta) \int_0^{2\pi} \mathrm{d}\phi \left(\widehat{r} \cdot \vec{V}(R,\theta,\phi)\right). \tag{3.1.5}$$

Example Gauss' theorem is key to showing the equivalence between

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \tag{3.1.6}$$

and

$$\int_{\partial\mathfrak{D}} \mathrm{d}^2\vec{A} \cdot \vec{E} = \frac{1}{\varepsilon_0} \left(Q \text{ enclosed within } \partial\mathfrak{D}\right). \tag{3.1.7}$$

Can you explain why?

3.2 Stokes' Theorem

Consider some 2D domain $\mathfrak{D}$ on a 2D surface (embedded in 3D space); and denote the boundary of this 2D domain as $\partial\mathfrak{D}$. By 'closed' here, we mean that there is a clear distinction between 'inside' and 'outside': namely, to get from outside to inside one has to go through the boundary $\partial\mathfrak{D}$. Now pick a unit normal $\widehat{n}$ on this domain $\mathfrak{D}$, so that we may define the directed infinitesimal area as $\mathrm{d}^2\vec{A} = \widehat{n}\mathrm{d}^2 A$.

Stokes' theorem The 2D surface integral of the curl of some vector field $\vec{V}$ within $\mathfrak{D}$ is equal to its line integral along the boundary $\partial\mathfrak{D}$:

$$\int_{\mathfrak{D}} \mathrm{d}^2\vec{A} \cdot \left(\vec{\nabla} \times \vec{V}\right) = \int_{\partial\mathfrak{D}} \mathrm{d}\vec{s} \cdot \vec{V}, \tag{3.2.1}$$

where the line integral on the right hand side is counter-clockwise around $\mathfrak{D}$. That is, when $\widehat{n}$ is pointing towards a hypothetical observer, this observer will perform the line integral on the right hand side in the counter-clockwise fashion.

Example Consider the integral of the curl of $\vec{V}$ on a circle with radius $r \leq R$ on the (x, y) plane in 3D space. Choose the unit normal to be the $z-$direction, i.e., $\widehat{n} = \widehat{z}$; so that $\mathrm{d}\vec{s} = \rho \mathrm{d}\phi \widehat{\phi}$. Then,

$$\int_0^R \rho \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\phi \, \widehat{z} \cdot \left(\vec{\nabla} \times \vec{V} \right) = R \int_0^{2\pi} \mathrm{d}\phi \left(\widehat{\phi} \cdot \vec{V} \right). \tag{3.2.2}$$

Example Stokes' theorem is key to showing the equivalence between

$$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B} \tag{3.2.3}$$

and

$$\int_{\partial \mathfrak{D}} \vec{E} \cdot \mathrm{d}\vec{s} = -\frac{\partial}{\partial t} \int_{\mathfrak{D}} \mathrm{d}^2 \vec{A} \cdot \vec{B}. \tag{3.2.4}$$

Can you explain why?