Benson Example 23.7

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Let λ be the charge per unit length of the infinite straight line. At a point R away from this infinite line charge, we may infer from parity symmetry that the electric field must point radially away (for $\lambda > 0$) or towards (for $\lambda < 0$) the line: rotating the system by 180° returns the line charge to itself – therefore the electric field cannot have a non-zero component along the line charge.

Refer to Figure 23.19. We will evaluate the electric field in a slightly different manner. Since we already know only the radial component survives, $\vec{E} = E\widehat{r}$, we only need to compute this radial component from each infinitesimal element of charge.

$$\mathrm{d}E = \frac{k\mathrm{d}q}{\ell^2 + R^2}\cos\theta = \frac{k\mathrm{d}q}{\ell^2 + R^2}\frac{R}{\sqrt{\ell^2 + R^2}} \tag{0.0.1}$$

$$= \frac{\lambda\mathrm{d}\ell}{4\pi\varepsilon_0}\frac{R}{(\ell^2 + R^2)^{3/2}}. \tag{0.0.2}$$

The ℓ is a coordinate along the line charge, running from $-\infty$ to 0, which is the point whose perpendicular line intersects the observer at a distance R away, and then to $+\infty$. Note the indefinite integral:

$$\int \frac{\mathrm{d}\ell}{(\ell^2 + R^2)^{3/2}} = \frac{\ell}{R^2\sqrt{\ell^2 + R^2}} + \text{constant}. \tag{0.0.3}$$

Therefore,

$$\int \mathrm{d}E = \int_{-\infty}^{+\infty} \frac{\lambda\mathrm{d}\ell}{4\pi\varepsilon_0}\frac{R}{(\ell^2 + R^2)^{3/2}} \tag{0.0.4}$$

$$= \frac{\lambda}{4\pi\varepsilon_0 R}\left[\frac{\ell}{\sqrt{\ell^2 + R^2}}\right]_{\ell=-\infty}^{\ell=+\infty} \tag{0.0.5}$$

$$= \frac{\lambda}{4\pi\varepsilon_0 R}\left(\lim_{\ell\to+\infty}\frac{\ell}{|\ell|} - \lim_{\ell\to-\infty}\frac{\ell}{|\ell|}\right) = \frac{\lambda}{2\pi\varepsilon_0 R}. \tag{0.0.6}$$

It turns out Coulomb's law in 2 space dimensions does yield an electric field that falls off as 1/(distance), so it is not a coincidence that our result goes as $1/R$. A point charge in 2-dimensions *is* a uniform line charge in 3-dimensions. More generally, Coulomb's law in $D \geq 2$ space dimensions says the magnitude of the electric field of a point charge goes as $1/(\text{distance})^{D-1}$.