Benson Example 23.2

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We place two charges q_1 and q_2 at, respectively,

$$\vec{r}_1 \equiv \left(0, -\frac{d}{2}\right), \tag{0.0.1}$$

$$\vec{r}_2 \equiv \left(0, +\frac{d}{2}\right); \tag{0.0.2}$$

whereas the observer is at

$$\vec{r} \equiv (x, y). \tag{0.0.3}$$

The vector pointing from q_1 to $\vec{r}$ is

$$\vec{r} - \vec{r}_1 = \left(x, y + \frac{d}{2}\right) \tag{0.0.4}$$

while the vector pointing from q_2 to $\vec{r}$ is

$$\vec{r} - \vec{r}_2 = \left(x, y - \frac{d}{2}\right). \tag{0.0.5}$$

The electric field at (x, y) is therefore

$$\vec{E} = \frac{kq_1}{|\vec{r} - \vec{r}_1|^2} \frac{\vec{r} - \vec{r}_1}{|\vec{r} - \vec{r}_1|} + \frac{kq_2}{|\vec{r} - \vec{r}_2|^2} \frac{\vec{r} - \vec{r}_2}{|\vec{r} - \vec{r}_2|} \tag{0.0.6}$$

$$= \frac{1}{4\pi\varepsilon_0} \left(q_1 \frac{(x, y + d/2)}{(x^2 + (y + d/2)^2)^{3/2}} + q_2 \frac{(x, y - d/2)}{(x^2 + (y - d/2)^2)^{3/2}} \right). \tag{0.0.7}$$

Note: the notation $|\vec{r} - \vec{r}_1|$ (or, $|\vec{r} - \vec{r}_2|$) denotes the distance between $\vec{r}$ and $\vec{r}_1$, which in turn follows from the Pythagorean theorem.

Dipole Let us consider the special case of $q_1 \equiv -Q$ and $q_2 \equiv Q > 0$. The electric field is then

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0} \left(\frac{(x, y - d/2)}{(x^2 + (y - d/2)^2)^{3/2}} - \frac{(x, y + d/2)}{(x^2 + (y + d/2)^2)^{3/2}} \right). \tag{0.0.8}$$

If we further consider the limit where $x^2 + y^2 \gg d^2$, so we may perform a Taylor expansion,

$$\begin{aligned}\vec{E}(x^2 + y^2 \gg d^2) &= \frac{Q}{4\pi\varepsilon_0 (x^2+y^2)^{5/2}} d\left(3xy, 2y^2 - x^2\right) \\ &= \frac{Q}{4\pi\varepsilon_0 r^3} d\left(\frac{3}{2}\sin(2\theta), \frac{1+3\cos(2\theta)}{2}\right); \end{aligned} \tag{0.0.9}$$

where we have defined

$$(x, y) \equiv r(\sin\theta, \cos\theta). \tag{0.0.10}$$

The key point here is that

> If we place two charges of equal magnitudes but opposite signs close to one another, then the electric field they generate goes as $1/r^3$ instead of the $1/r^2$ that a single point charge generates.

The dipole is defined as the charge Q multiplied by the vector joining the negative charge to the positive one. For our case at hand, for instance:

$$\vec{p} \equiv Qd\widehat{e}_y = Qd\,(0, 1)\,. \tag{0.0.11}$$

Dipoles are important because many naturally occurring systems – such as molecules – are overall electrically neutral but are composed of charges with different signs displaced from each other. To first approximation, such systems may be modeled as dipoles.

Immersing a dipole in an external electric field $\vec{E}$, we see that the force exerted on the $+Q$ charge is $Q\vec{E}(0, d/2)$ while that on the $-Q$ charge is $-Q\vec{E}(0, -d/2)$. The total force is therefore

$$\vec{F} = Q\left(\vec{E}(0, d/2) - \vec{E}(0, -d/2)\right) \tag{0.0.12}$$

$$= Q\left(\vec{E}(0,0) + \frac{d}{2}\frac{\partial \vec{E}(0,0)}{\partial y} + \dots\right) - Q\left(\vec{E}(0,0) - \frac{d}{2}\frac{\partial \vec{E}(0,0)}{\partial y} + \dots\right) \tag{0.0.13}$$

$$\approx Qd\partial_y \vec{E}(0,0). \tag{0.0.14}$$

More generally, we may perform a Taylor expansion in along the direction $\vec{d}$ of the dipole:

$$\vec{F} = Q\vec{E}(\vec{x} + \vec{d}) - Q\vec{E}(\vec{x}) = Q\vec{E}(\vec{x}) + Q\vec{d}\cdot\vec{\nabla}\vec{E}(\vec{x}) + \dots - Q\vec{E}(\vec{x}) \tag{0.0.15}$$

$$\approx Q\vec{d}\cdot\vec{\nabla}\vec{E}(\vec{x}) \equiv \vec{p}\cdot\vec{\nabla}\vec{E}(\vec{x}). \tag{0.0.16}$$

That is,

> A dipole immersed in a non-uniform electric field, with $\vec{\nabla}\vec{E} \neq 0$, experiences a net force – given by its dipole moment projected along the gradient of the electric field – despite being electrically neutral.