Benson Example 22.2

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We will tackle a slightly more general version of Benson Example 2.22. Let the charge on the left be q_1 and the charge on the right be q_2. Let the distance between them be ℓ. We will define q_1 and q_2 to lie on the $x-$axis, with q_2 at $x = 0$. The problem is, where do we place Q such that there will be no net force on it?

The force on Q due to q_1 is

$$\vec{F}_1 = \frac{q_1 Q}{4\pi\varepsilon_0(\ell + d)^2}\widehat{e}_x. \tag{0.0.1}$$

The force on Q due to q_2 is

$$\vec{F}_2 = \frac{q_2 Q}{4\pi\varepsilon_0 d^2}\widehat{e}_x. \tag{0.0.2}$$

The total force is

$$\vec{F}_{\text{tot}} = \vec{F}_1 + \vec{F}_2 \tag{0.0.3}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{q_1}{(\ell + d)^2} + \frac{q_2}{d^2}\right)\widehat{e}_x. \tag{0.0.4}$$

If this is to be zero, we see that q_1 and q_2 has to have opposite signs. That is, one force must be attractive and one must be repulsive. Otherwise it will either act to the left (if both q_1 and q_2 are negative) or to the right (if both q_1 and q_2 are positive).

$$\vec{F}_{\text{tot}} = 0 \qquad \Rightarrow \qquad \frac{q_1}{(\ell + d)^2} = -\frac{q_2}{d^2} \tag{0.0.5}$$

$$-\frac{q_1}{q_2} = \left(\frac{\ell + d}{d}\right)^2 \tag{0.0.6}$$

$$\sqrt{-\frac{q_1}{q_2}} = \pm\frac{\ell + d}{d} \tag{0.0.7}$$

$$\pm d\sqrt{-\frac{q_1}{q_2}} = \ell + d \tag{0.0.8}$$

$$d\left(\pm\sqrt{-\frac{q_1}{q_2}} - 1\right) = \ell. \tag{0.0.9}$$

We have arrived at

$$d = \frac{\ell}{\pm\sqrt{-q_1/q_2} - 1}. \tag{0.0.10}$$

Remember we have assumed q_1 and q_2 are of opposite signs, so $\sqrt{-q_1/q_2}$ is a positive real number.

Case I The first case to consider is $\sqrt{-q_1/q_2} > 1$. Then the solution

$$d_+ \equiv \frac{\ell}{\sqrt{-q_1/q_2} - 1} \tag{0.0.11}$$

is a legitimate one, since it places the charge Q to the right of both q_1 and q_2, with one force attractive and one repulsive – and thereby canceling each other. The other solution

$$d_- \equiv -\frac{\ell}{\sqrt{-q_1/q_2} + 1} \tag{0.0.12}$$

places the Q between q_1 and q_2 (because the magnitude $\ell/(\sqrt{-q_1/q_2} + 1)$ is less than ℓ), where the presence of one repulsive and one attractive force makes the net force always pointing to either left or right – i.e., they can never cancel.

Case II The second case is $0 < \sqrt{-q_1/q_2} < 1$. Then the solution

$$d_+ \equiv -\frac{\ell}{1 - \sqrt{-q_1/q_2}} \tag{0.0.13}$$

is still the legitimate one, but now the magnitude $\ell/(1 - \sqrt{-q_1/q_2})$ is larger than ℓ – and hence the Q lies to the left of q_1 and q_2. The other solution

$$d_- \equiv -\frac{\ell}{\sqrt{-q_1/q_2} + 1} \tag{0.0.14}$$

is still illegitimate because it still lies between q_1 and q_2 since $\ell/(\sqrt{-q_1/q_2} + 1) < \ell$.

Case III The final case is $\sqrt{-q_1/q_2} = 1$, or in other words, $q_1 = -q_2$. Referring back to eq. (0.0.5),

$$\frac{1}{(\ell + d)^2} = \frac{1}{d^2} \tag{0.0.15}$$

$$\frac{1}{\ell + d} = \pm\frac{1}{d} \tag{0.0.16}$$

$$\ell + d = \pm d \tag{0.0.17}$$

Either $\ell = 0$ (for the + sign) or $d = -\ell/2$, which is the midpoint between q_1 and q_2, where the net force points either to the left or to the right. Both are not solutions to our problem. $\square$